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# STATISTICAL CONVERGENCE FOR GENERAL BETA OPERATORS

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Abstract. In this paper, we consider general Beta operators, which is a general sequence of integral type operators including Beta function. We study the King type Beta operators which preserves the third test function $x^2$. We obtain some approximation properties, which include rate of convergence and statistical convergence. Finally, we show how to reach best estimation by these operators.

## 1. Introduction

Three classical operators $L_n$ (Bernstein operators, Szász-Mirakjan operators and Baskakov operators) preserve $e_i(x) = x^i (i = 0, 1)$, i.e., $L_n(e_0; x) = e_0(x)$ and $L_n(e_1; x) = e_1(x), n \in \mathbb{N}$. For each of these operators, $L_n(e_2; x) \neq e_2(x) = x^2$. In the year 2003, J. P. King [10] presented a non-trivial sequence of positive linear operators $V_n : C[0, 1] \to C[0, 1]$, given as follows:

$$V_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} (r_n^*(x))^k (1 - r_n^*(x))^{n-k} f\left(\frac{k}{n}\right), 0 \leq x \leq 1,$$

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where $r_n^*(x) : [0,1] \to [0,1]$, are defined by

$$r_n^*(x) = \begin{cases} x^2, & \text{n=1}, \\ -\frac{1}{2(n-1)} + \sqrt{\frac{n}{n-1}x^2 + \frac{1}{4(n-1)^2}}, & \text{n=2,3,....} \end{cases}$$

This sequence preserves the test functions $e_0, e_2$ and $V_n(f,x) = r_n^*(x)$ holds. Replacing $r_n^*(x)$ by $e_1$, then we obtain classical Bernstein operators.

Beta operators were introduced by Lupaş [11] and further modified and studied by Khan [9], Upreti [15], Divis [5] and others.

The Beta approximation $\beta_n(f)$ to a function $f : [0,1] \to \mathbb{R}$ is the operator:

$$\beta_n(f;x) = \frac{1}{B(nx, n(1-x))} \int_0^1 t^{nx-1}(1-t)^{n(1-x)-1} f(t)dt \tag{1.1}$$

where $B(u,v)$ is the well-known beta probability density function

$$B(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}dt;\ u,v > 0,$$

with the support $(0,1)$ such that $t$ denotes a value of the random variable $T$, where $n \in \mathbb{N},\ x \in (0,1)$ and $f$ is any real measurable, Lebesgue integrable function defined on $[0,1]$. When $x = 0$ or $x = 1$, then $\beta_n(f,x) = f(x)$ for all $n$.

Now the following Lemmas follow from [16], for the operators $\beta_n$ mentioned by (1.1).

Lemma 1.1 ( [16]). *Let* $e_i(x) = x^i,\ i = 0,1,2$. *Then, for each* $0 < x < 1$ *and* $n \in \mathbb{N}$, *we have*

(*i*) $\beta_n(e_0;x) = 1$,
(*ii*) $\beta_n(e_1;x) = x$,
(*iii*) $\beta_n(e_2;x) = \frac{x(1+nx)}{n+1}$.

Lemma 1.2 ( [5]). *For each* $0 < x < 1$ *and* $n \in \mathbb{N}$ *and* $\varphi_x(t) = t - x$, *we have* $\beta_n(\varphi_x^2;x) = \frac{x(1-x)}{n+1}$.

The aim of this article is to construct a general Beta type operators including the King type Beta operators which preserves the third test function $x^2$. We study some approximation properties, which include rate of convergence and statistical convergence. Finally, we show how to reach best estimation by these operators than the original Beta operators $\beta_n(f,x)$. Note that rate of convergence and statistical convergence of many other approximation operators are available in literatures(See [1], [2], [4], [6], [7], [8], [12], [13], [14]).

## 2. King Type Beta operators

Let $\{\alpha_n(x)\}$ be a sequence of real-valued continuous functions defined on $[0,1]$ with $0 < \alpha_n(x) < 1$. Now consider a sequence of positive linear operators:

(2.1)

$$\hat{\beta}_n(f,x) = \frac{1}{B(n\alpha_n(x), n(1-\alpha_n(x)))} \int_0^1 t^{n\alpha_n(x)-1}(1-t)^{n(1-\alpha_n(x))-1} f(t)dt,$$

where $x \in [0,1]$, $f \in [0,1]$ and $n \in \mathbb{N}$(set of natural numbers). If $\alpha_n(x)$ is replaced by $e_1$, then we obtain original beta operators (1.1). Note that

Lemma 2.1. *For each $0 \le x \le 1$ and $n \in \mathbb{N}$ and $\varphi_x(t) := t - x$, we have*

(i) $\hat{\beta}_n(e_0;x) = 1$,

(ii) $\hat{\beta}_n(e_1;x) = \alpha_n(x)$,

(iii) $\hat{\beta}_n(e_2;x) = \dfrac{\alpha_n(x)(1+n\alpha_n(x))}{n+1}$,

(iv) $\hat{\beta}_n(\varphi_x^2;x) = (\alpha_n(x)-x)^2 + \dfrac{\alpha_n(x)(1-\alpha_n(x))}{n+1}$.

Now, if we replace $\alpha_n(x)$ by

$$\alpha_n^*(x) = \frac{-1+\sqrt{1+4n(n+1)x^2}}{2n}, \quad x \in [0,1] \text{ and } n \in \mathbb{N},$$

then the operators $\hat{\beta}_n$ defined in (2.1) reduce to the operators

(2.2)

$$\beta_n^*(f;x) = \frac{1}{B(n\alpha_n^*(x), n(1-\alpha_n^*(x)))} \int_0^1 t^{n\alpha_n^*(x)-1}(1-t)^{n(1-\alpha_n^*(x))-1} f(t)dt.$$

These operators are the King type Beta operators. Furthermore, the following Lemma hold:

Lemma 2.2. *The operators defined by (2.2) verify the following identities*

(i) $\beta_n^*(e_0;x) = 1$,

(ii) $\beta_n^*(e_1;x) = \frac{-1+\sqrt{1+4n(n+1)x^2}}{2n}$,
(iii) $\beta_n^*(e_2;x) = x^2$.

LEMMA 2.3. *For each $0 \le x \le 1$ and $n \in \mathbb{N}$ and $\varphi_x(t) = t - x$, we have*

(i) $\beta_n^*(\varphi_x;x) = \frac{\sqrt{1+4n(n+1)x^2}-(1+2nx)}{2n}$,
(ii) $\beta_n^*(\varphi_x^2;x) = \frac{(1+2nx)x - x\sqrt{1+4n(n+1)x^2}}{n}$.

## 3. Rate of Convergence

In this section we study the rate of convergence of the operators $\hat{\beta}_n\,(f;x)$ to $f(x)$ by means of the modulus of continuity and Peetre's $K$-functional . For $f \in C[a,b]$, the modulus of continuity of $f$, denoted by $\omega\,(f;\delta)$, is defined to be

$$\omega\,(f;\delta) = \sup_{|y-x|<\delta, x,y\in[a,b]} |f(y) - f(x)|\,.$$

It is known that for any $\delta > 0$ and $x, y \in [a,b]$, we have

$$|f(y) - f(x)| \le \omega\,(f;\delta)\left(\frac{|y-x|}{\delta} + 1\right).$$

THEOREM 3.1. *For every $f \in C[0,1]$ and $0 \le x \le 1$, we have*

$$\left|\hat{\beta}_n\,(f;x) - f(x)\right| \le 2\omega\,(f, \delta_{n,x})$$

*where* $\delta_{n,x} := \sqrt{(\alpha_n(x) - x)^2 + \frac{\alpha_n(x)(1-\alpha_n(x))}{n+1}}$ *and $\omega\,(f, \delta_{n,x})$ is the modulus of continuity of $f$.*

*Proof.* Let $f \in C[0,1]$ and $x \in [0,1]$. Since $\hat{\beta}_n(e_0,x) = e_0(x)$, from Cauchy-Schwarz inequality for linear positive operators, we obtain for every $\delta > 0$ and $n \in \mathbb{N}$, that

$$\left|\hat{\beta}_n(f;x) - f(x)\right| \le \left[\hat{\beta}_n(e_0;x) + \frac{1}{\delta_{n,x}}\left(\hat{\beta}_n\left((e_1 - x)^2;x\right)\right)^{1/2}\right]\omega\,(f, \delta_{n,x})\,.$$

Choosing $\delta_{n,x} = \sqrt{\hat{\beta}_n\left((e_1 - x)^2;x\right)} = \sqrt{(\alpha_n(x) - x)^2 + \frac{\alpha_n(x)(1-\alpha_n(x))}{n+1}}$, we obtain

$$\left|\hat{\beta}_n(f;x)-f(x)\right|\leq 2\omega\left(f,\delta_{n,x}\right).$$

□

For the King type Beta operators we have the following Corollary at once:

Corollary 3.2. *For every $f\in C[0,1]$ and $0\leq x\leq 1$, we have*

$$\left|\beta_n^*\left(f;x\right)-f(x)\right|\leq 2\omega\left(f,\delta_{n,x}\right)$$

*where* $\delta_{n,x}=\sqrt{\frac{(1+2nx)x-x\sqrt{1+4n(n+1)x^2}}{n}}$.

Now we give the rate of convergence for the operators $\hat{\beta}_n\left(f;x\right)$ by using the Peetre's $K$-functional in the space $C^2[0,1]$. We recall some definitions and notations. The classical Peetre's $K-$functional of a function $f\in C[0,1]$ is defined by

$$K\left(f,\delta\right)=\inf\left\{\left\|f-g\right\|_{C[0,1]}+\delta\left\|g''\right\|_{C[0,1]}:g\in C^2[0,1]\right\},\quad\delta>0$$

where $C^2[0,1]=\{g\in C[0,1]:g',g''\in C^2[0,1]\}$.
and the norm

$$\|f\|_{C^2[0,1]}=\|f\|_{C[0,1]}+\|f'\|_{C[0,1]}+\|f''\|_{C[0,1]}.$$

Theorem 3.3. *For each $f\in C[0,1]$*

$$\left|\hat{\beta}_n\left(f;x\right)-f(x)\right|$$
$$\leq K\left(f;\left(\left|\alpha_n(x)-x\right|+\left|\left(\alpha_n(x)-x\right)^2+\frac{\alpha_n(x)(1-\alpha_n(x))}{n+1}\right|\right)\right).$$

*Proof.* Applying Taylor expansion to the function $g\in C^2[0,1]$, we get

$$\hat{\beta}_n(g,x)-g(x)=g'(x)\hat{\beta}_n((e_1-x),x)+\frac{1}{2}\hat{\beta}_n\left(g''(\xi)(e_1-x)^2,x\right);\xi\in(t,x).$$

Hence

$$\left|\hat{\beta}_n\left(g;x\right)-g(x)\right|$$
$$\leq\|g'\|_{C[0,1]}\left|\hat{\beta}_n((e_1-x),x)\right|+\|g''\|_{C[0,1]}\left|\hat{\beta}_n((e_1-x)^2,x)\right|$$
$$=\|g'\|_{C[0,1]}\left|\alpha_n(x)-x\right|+\|g''\|_{C[0,1]}\left|\left(\alpha_n(x)-x\right)^2+\frac{\alpha_n(x)(1-\alpha_n(x))}{n+1}\right|.$$

For each $f \in C[0,1]$, we can write

$$\begin{aligned}
&\left|\hat{\beta}_n(f,x) - f(x)\right| \\
&\leq \left|\hat{\beta}_n(f,x) - \hat{\beta}_n(g,x)\right| + \left|\hat{\beta}_n(g,x) - g(x)\right| + |g - f| \\
&\leq 2\|g-f\|_{C[0,1]} + \left|\hat{\beta}_n(g;x) - g(x)\right| \\
&\leq 2\|g-f\|_{C[0,1]} \\
&\qquad + \left(|\alpha_n(x) - x| + \left|(\alpha_n(x)-x)^2 + \frac{\alpha_n(x)(1-\alpha_n(x))}{n+1}\right|\right)\|g''\|_{C[0,1]} \\
&\leq 2\Bigg(\|g-f\|_{C[0,1]} + |\alpha_n(x) - x| \\
&\qquad + \left|(\alpha_n(x)-x)^2 + \frac{\alpha_n(x)(1-\alpha_n(x))}{n+1}\right|\|g''\|_{C[0,1]}\Bigg)
\end{aligned}$$

Taking infimum over $g \in C^2[0,1]$, we get

$$\begin{aligned}
&\left|\hat{\beta}_n(f,x) - f(x)\right| \\
&\qquad \leq K\left(f; \left(|\alpha_n(x) - x| + \left|(\alpha_n(x)-x)^2 + \frac{\alpha_n(x)(1-\alpha_n(x))}{n+1}\right|\right)\right).
\end{aligned}$$

□

For the King type Beta operators we immediately have the following Corollary:

Corollary 3.4. *For each* $f \in C[0,1]$

$$\left|\hat{\beta}_n(f;x) - f(x)\right| \leq K\left(f;\gamma_{n,x}\right),$$

*where* $\gamma_{n,x} = \frac{1}{2n}(2x-1)\left(2nx - \sqrt{4n^2x^2 + 4nx^2 + 1} + 1\right)$.

## 4. Statistical convergence

In this part of the paper, we use concept of statistical convergence and study the Korovkin type approximation theorem for the operators $\hat{\beta}_n$. Before we present the main results, we shall recall some notation on the statistical convergence.

Let $M$ be any subset of $\mathbb{N}$. The density of $M$ is defined by

$$\delta(M) = \lim_n \frac{1}{n} \sum_{j=1}^{n} \chi_M(j)$$

provided the limit exists, where $\chi_M$ is the characteristic function of $M$. A sequence $x = (x_k)$ is said to be statistical convergence to the number $l$,

$$\delta \{k \in \mathbb{N}\colon |x_k - l| \geq \varepsilon\} = 0$$

for every $\varepsilon > 0$ or equivalently there exists a subset $K \subseteq \mathbb{N}$ with $\delta(K) = 1$ and $n_0(\varepsilon)$ such that $k > n_0$ and $k \in K$ imply that $|x_k - l| < \varepsilon$. We write

$$st - \lim_n x_k = l$$

Assume that for each $x \in [0,1]$,$(\alpha_n(x))_{n \in \mathbb{N}}$ is a sequence in $(0,1)$ satisfying

$$st - \lim_n \alpha_n(x) = x. \tag{4.1}$$

Then we have

$$st - \lim_n |x - \alpha_n(x)| = 0, \tag{4.2}$$

and

$$st - \lim_n \left| \frac{\alpha_n(x)(1 - \alpha_n(x))}{n+1} \right| = 0. \tag{4.3}$$

Such a sequence $(\alpha_n(x))_{n \in \mathbb{N}}$ can be constructed as follows. Choose

$$\alpha_n(x) = \begin{cases} 2 & , \text{if } n = m^2 \ (m \in \mathbb{N}) \\ \alpha_n^*(x) & , \text{ otherwise} \end{cases}$$

where

$$\alpha_n^*(x) = \frac{-1 + \sqrt{1 + 4n(n+1)x^2}}{2n}, \quad x \in [0,1] \text{ and } n \in \mathbb{N}.$$

It is clear that (4.1) is satisfied.

Theorem 4.1. *For each $x \in [0,1]$ and for every $f \in C[0,1]$, we have*

$$st - \lim_n \left| \hat{\beta}_n(f;x) - f(x) \right| = 0.$$

*Proof.* For a given $r > 0$ choose $\varepsilon > 0$ such that $\varepsilon < r$. Now define the sets:

$$U := \left\{n : \delta_{n,x}^2 \geq r\right\},$$
$$U_1 := \left\{n : |x - \alpha_n(x)| \geq \sqrt{\frac{r-\varepsilon}{2}}\right\},$$
$$U_2 := \left\{n : \left|\frac{\alpha_n(x)(1-\alpha_n(x))}{n+1}\right| \geq \frac{r-\varepsilon}{2}\right\},$$

where $\delta_{n,x} := \sqrt{(\alpha_n(x) - x)^2 + \dfrac{\alpha_n(x)(1-\alpha_n(x))}{n+1}}$. Then it follows that $U \subseteq U_1 \cup U_2$, which gives

$$\sum_{j=1}^{n} \chi_U(j) \leq \sum_{j=1}^{n} \chi_{U_1}(j) + \sum_{j=1}^{n} \chi_{U_2}(j) \tag{4.4}$$

Multiplying both sides of (4.4) by $\dfrac{1}{n}$ and letting $n \to \infty$, we get using (4.2) and (4.3) that

$$\lim_{n\to\infty} \sum_{j=1}^{n} \chi_U(j) = 0.$$

This guarantees that $st-\lim_n \delta_{n,x}^2 = 0$ which implies $st-\lim_n \omega(f, \delta_{n,x}) = 0$ .Using Theorem 3.1 completes the proof. □

Remark 4.2. If we choose the sequence $(\alpha_n(x))_{n\in\mathbb{N}}$ as in (4.1), then our statistical approximation result (Theorem 4.1) works; however its classical version does not work since

$$\alpha_n(x) \nrightarrow x$$

in the usual sense.

## 5. Best Error Estimation

Let $\psi_x$ be the first central moment function defined by $\psi_x(y) = y - x$. In order to get a better error estimation on a subinterval $I$ of $[0, 1]$, in the approximation by means of the operators $\beta_n$ , we are aimed to find a functional sequence $(s_n)$, $s_n : I \to A$, satisfying

$$\delta_{n,x}^* := \sqrt{\hat{\beta}_n(\psi_x^2; u_n(x))} \leq \sqrt{\beta_n(\psi_x^2; x)} =: \delta_{n,x} \quad \text{for } x \in I. \tag{5.1}$$

By Lemmas 1.2 and 2.1 (d), (5.1) takes the form

$$(5.2)\quad \frac{n}{n+1}s_n^2(x) + \left(\frac{1}{n+1} - 2x\right) s_n(x) - (\frac{n}{n+1} - 2)x^2 - \frac{1}{n+1}x \le 0.$$

Let

$$\Delta_n(x) := \left(\frac{1}{n+1} - 2x\right)^2 + 4\frac{n}{n+1}\left\{(\frac{n}{n+1} - 2)x^2 + \frac{1}{n+1}x\right\}.$$

Then it is clear that

$$(5.3)\qquad \Delta_n(x) \ge 0$$

and

$$(5.4)\qquad x + \frac{x}{n} - \frac{1}{2n} \in [0,1]$$

hold for every $x \in I = [\frac{1}{4}, \frac{3}{4}]$ and for every $n \ge 1$. Therefore, from (5.2), (5.3) and (5.4), we get

$$\frac{2x - \frac{1}{n+1} - \sqrt{\Delta_n(x)}}{2\frac{n}{n+1}} \le s_n(x) \le \frac{2x - \frac{1}{n+1} + \sqrt{\Delta_n(x)}}{2\frac{n}{n+1}}.$$

Then $s_n(x)$ takes its minimum when

$$s_n(x) := x + \frac{x}{n} - \frac{1}{2n}.$$

Therefore, for all $x \in [\frac{1}{4}, \frac{3}{4}]$, we define a new Beta type operator by

$$\begin{aligned}\beta_n^s(f;x) = {} & \beta_n(f; s_n(x)) \\ = {} & \frac{1}{B\left(ns_n(x), n(1-s_n(x))\right)} \int_0^1 t^{ns_n(x)-1}(1-t)^{n(1-s_n(x))-1} f(t)dt.\end{aligned}$$

Then, for all $x \in [\frac{1}{4}, \frac{3}{4}]$ and $n \ge 1$, we have

$$\beta_n^s(\psi_x^2; x) = \frac{x(1-x)}{n} - \frac{1}{4n(n+1)} \le \frac{x(1-x)}{n+1} = \beta_n(\psi_x^2; x)$$

which shows that the operators $\beta_n^s(f;x)$ provides the better estimation than the operators $\beta_n(f;x)$.

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