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CONVERGENCE ANALYSIS ON GIBOU-MIN METHOD FOR THE SCALAR FIELD IN HODGE-HELMHOLTZ DECOMPOSITION

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ABSTRACT. The Hodge-Helmholtz decomposition splits a vector field into the unique sum of a divergence-free vector field (solenoidal part) and a gradient field (irrotational part). In a bounded domain, a boundary condition needs to be supplied to the decomposition. The decomposition with the non-penetration boundary condition is equivalent to solving the Poisson equation with the Neumann boundary condition. The Gibou-Min method is an application of the Poisson solver by Purvis and Burkhalter to the decomposition.

Using the L^2-orthogonality between the error vector and the consistency, the convergence for approximating the divergence-free vector field was recently proved to be $O(h^{1.5})$ with step size h. In this work, we analyze the convergence of the irrotattional in the decomposition. To the end, we introduce a discrete version of the Poincare inequality, which leads to a proof of the $O(h)$ convergence for the scalar variable of the gradient field in a domain with general intersection property.

1. INTRODUCTION

The Hodge-Helmholtz decomposition theorem [6] states that any smooth vector field U^* can be decomposed into the sum of a gradient field ∇p and a divergence-free vector field U. The decomposition is unique and orthogonal in L^2. The Hodge projection of a vector field is defined as the divergence-free component in its Hodge-Helmholtz decomposition.

One of the main applications of the decomposition is the incompressible fluid flow, whose phenomenon is represented by the Navier-Stokes equations. Consisting of the conservation equation of momentum and the state equation of divergence-free condition, the equations can be described by a convection-diffusion equation with the Hodge projection applied at every

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moment. Chorin's seminal approximation [3] for the fluid flow first solves the convection-diffusion equation in a usual manner, and then applies the Hodge projection. Other successful fluid solvers such as Kim-Moin's [7], Bell *et al.*'s [1], Gauge method [8] are in the same direction as Chorin's.

The Hodge-Helmholtz decomposition $U^* = U + \nabla p$ in a domain Ω can be implemented through the Poisson equation $-\Delta p = -\nabla \cdot U^*$. In a bounded domain, the equation needs to be supplied with boundary condition. There are two types fluid boundary conditions. One is the non-penetration boundary condition, $U \cdot n = 0$ on $\Gamma = \partial\Omega$, and the other is the free boundary condition, $p = \sigma\kappa$ on Γ [10]. The free boundary condition is, in other words, the Dirichlet boundary condition of the Poisson equation, and the non-penetration boundary condition corresponds to the Neumann boundary condition, $\frac{\partial p}{\partial n} = U^* \cdot n$ on Γ.

To approximate the Poisson equation, we consider the standard finite volume method. A standard finite difference/volume method for the Poisson equation with the Neumann boundary condition was introduced by Purvis and Burkhalter [11]. Though implemented in uniform grid, the method can handle arbitrarily shaped domains. It is a simple modification of the standard five-point finite difference method, and it constitutes a five-banded sparse linear system that is diagonally dominant, symmetric and positive semi-definite. Due to these nice properties, the linear system can be efficiently solved by the Conjugate Gradient method with various efficient ILU preconditioners.

The Gibou-Min method [4, 9] is an application of the Purvis-Burkhalter method on the Hodge-Helmholtz decomposition. In implementing the Hodge decomposition, the Neumann boundary condition takes the divergence form $\frac{\partial p}{\partial n} = \nabla \cdot U^*$.

Using the orthogonality condition between the error $U - U^h$ and the consistency of the method, the method was proved in [13] to provide 1.5 order of accuracy in approximating the divergence-free vector field U of the Hodge projection.

In this work, we estimate the convergence of the pressure p given in the Hogde-Helmholtz decomposition. Using the orthogonality, we obtain the estimate $\left\|Gp - Gp^h\right\|_{L^2} = O\left(h^{1.5}\right)$ for the gradient of the pressure error. On introducing a discrete version of the Poincare inequality, we derive $\left\|p - p^h\right\| = O\left(h_{min}^{-0.5} \cdot h^2\right)$ for h_{min} the smallest distance from grid nodes inside to the boundary. Our estimate reads that for many domains with $h_{min} = O(h^2)$, for instance, domains with general intersection property introduced in [12], the pressure convergence is $O(h)$. According to our numerical tests, even though the estimate $\left\|U - U^h\right\|_{L^2} = O\left(h^{1.5}\right)$ is tight, however, the estimate does not meet the observed order $\left\|p - p^h\right\| = O\left(h^2\right)$. We put it off to future research to improve the estimate.

2. NUMERICAL METHOD

In this section, we briefly review the Gibou-Min method [4, 9] for the Hodge decomposition with the non-penetration boundary condition. Given a vector field U^* in a bounded and connected domain Ω, the following Poisson equation is solved for scalar p with the Neumann boundary condition.

$$\begin{cases} -\Delta p &= -\nabla\cdot U^* \quad \text{in} \quad \Omega \\ \frac{\partial p}{\partial n} &= U^*\cdot n \qquad \text{on} \quad \Gamma \end{cases} \tag{2.1}$$

Then a vector field U, which is defined as $U = U^* - \nabla p$, is the desired Hodge projection of U^* that satisfies the divergence-free condition $\nabla\cdot U = 0$ in Ω, and the non-penetration boundary condition $U\cdot n = 0$ on Γ. The Gibou-Min method samples the vector fields and scalar field on the Marker-and-Cell (MAC) staggered grid [5]. Let $h\mathbb{Z}^2$ denote the uniform grid in $\mathbb{R}^2$ with step size h. For each grid node $(x_i, y_j)\in h\mathbb{Z}^2$, C_{ij} denotes the rectangular control volume centered at the node, and its four edges are denoted by $E_{i\pm\frac{1}{2},j}$ and $E_{ij\pm\frac{1}{2}}$ as follows.

$$\begin{array}{lcccc} C_{ij} & := & [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] & \times & [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}] \\ E_{i\pm\frac{1}{2}j} & := & x_{i\pm\frac{1}{2}} & \times & [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}] \\ E_{ij\pm\frac{1}{2}} & := & [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] & \times & y_{j\pm\frac{1}{2}} \end{array}$$

Based on the MAC configuration, we define the node set and the edge sets.

Definition 2.1 (Node and edge sets)**.** *By $\Omega^h := \left\{(x_i, y_j)\in h\mathbb{Z}^2 | C_{ij}\cap\Omega\neq\emptyset\right\}$ we denote the set of nodes whose control volumes intersecting the domain. In the same way, we define the edge sets by $E^h_x := \left\{(x_{i+\frac{1}{2}}, y_j) | E_{i+\frac{1}{2},j}\cap\Omega\neq\emptyset\right\}$ and $E^h_y := \left\{(x_i, y_{j+\frac{1}{2}}) | E_{i,j+\frac{1}{2}}\cap\Omega\neq\emptyset\right\}$, and then $E^h := E^h_x\cup E^h_y$.*

By the standard central finite differences, a discrete gradient operator is defined.

Definition 2.2 (Discrete gradient)**.** *Given $p:\Omega^h\to\mathbb{R}$, its gradient $Gp: E^h\to\mathbb{R}$ is defined as*

$$\begin{aligned} (G^x p)_{i+\frac{1}{2},j} &= \frac{p_{i+1,j} - p_{ij}}{h} \\ (G^y p)_{i,j+\frac{1}{2}} &= \frac{p_{ij+1} - p_{ij}}{h}. \end{aligned}$$

Whenever $E_{i+\frac{1}{2},j}\cap\Omega\neq\emptyset$, $C_{ij}\cap\Omega\neq\emptyset$ and $C_{i+1,j}\cap\Omega\neq\emptyset$, since $E_{i+\frac{1}{2},j}\subset C_{ij}, C_{i+1,j}$. Hence the above definition is well posed for $G^x p$, and so is for $G^y p$. Discrete gradient was simply defined by the finite differences, however discrete gradient can not be defined so. For each $(x_i, y_j)\in\Omega^h$, its four neighboring edges may not be in E^h, since $C_{ij}\cap\Omega\neq\emptyset$ neither imply $E_{i\pm\frac{1}{2},j}\cap\Omega\neq\emptyset$ nor $E_{i,j\pm\frac{1}{2}}\cap\Omega\neq\emptyset$. A proper definition comes from the following identity.

$$\begin{aligned} \int_{C_{ij}\cap\Omega} \nabla\cdot U\,dx &= \int_{\partial(C_{ij}\cap\Omega)} U\cdot\vec{n}\,ds \\ 0 &= \int_{\partial C_{ij}\cap\Omega} U\cdot\vec{n}\,ds + \int_{C_{ij}\cap\Gamma} U\cdot\vec{n}\,ds \end{aligned} \tag{2.2}$$

With the non-penetration boundary condition $U \cdot \vec{n} = 0$, the identity represents an integral value of the divergence by a line integral over the fraction of edges. To measure the fraction, the following Heaviside functions are defined on the edge set.

Definition 2.3 (Heaviside function)**.** *For each edge,*

$$H_{i+\frac{1}{2},j} = \frac{\textit{length}\left(E_{i+\frac{1}{2},j} \cap \Omega\right)}{\textit{length}\left(E_{i+\frac{1}{2},j}\right)}, \textit{ and } H_{i,j+\frac{1}{2}} = \frac{\textit{length}\left(E_{i,j+\frac{1}{2}} \cap \Omega\right)}{\textit{length}\left(E_{i,j+\frac{1}{2}}\right)}.$$

Note that $H_{i+\frac{1}{2},j}$, $H_{i,j+\frac{1}{2}} \in [0,1]$. Its value 1 implies that the edge is totally inside the domain, and value 0 implies completely outside. Using the Heaviside function, now we define discrete divergence operator.

Definition 2.4 (Discrete divergence)**.** *Given $U = (u,v) : E^h \to \mathbb{R}$, its discrete divergence $DU : \Omega^h \to \mathbb{R}$ is defined as*

$$\begin{aligned} (DU)_{ij} &= \left(u_{i+\frac{1}{2},j} H_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j} H_{i-\frac{1}{2},j}\right) \cdot h \\ &+ \left(v_{i,j+\frac{1}{2}} H_{i,j+\frac{1}{2}} - v_{i,j-\frac{1}{2}} H_{i,j-\frac{1}{2}}\right) \cdot h. \end{aligned}$$

Note that the calculation of the discrete divergence involves the vector field only in E^h. The edges not in E^h, whose Heaviside function values are zero, are ignored in the calculation.

Given a vector field $U^* : \Omega \cup \Gamma \to \mathbb{R}^2$, we define a discrete vector field $U^* = (U_x^*, U_y^*)$ on E^h as

$$(U_x^*)_{i+\frac{1}{2},j} := \frac{1}{H_{i+\frac{1}{2},j} h} \int_{E^h_{i+\frac{1}{2},j} \cap \Omega} U_x^*(x_{i+\frac{1}{2}}, y) dy$$

and

$$(U_y^*)_{i,j+\frac{1}{2}} := \frac{1}{H_{i,j+\frac{1}{2}} h} \int_{E^h_{i,j+\frac{1}{2}} \cap \Omega} U_y^*(x, y_{j+\frac{1}{2}}) dy$$

With the vector field $U^* : E^h \to \mathbb{R}^2$, the Gibou-Min method computes a vector field $U^h : E^h \to \mathbb{R}$ and a scalar field $p^h : \Omega^h \to \mathbb{R}$ such that $DU^h = 0$ in Ω^h and $U^* = U^h + Gp^h$ in Ω^h. Substituting U^h with $U^* - Gp^h$ in $DU^h = 0$, we have the equation for p^h,

$$-DGp^h = -DU^* \text{ in } \Omega^h. \tag{2.3}$$

After p^h is obtained by solving the above linear system, the solenoidal vector field $U^h = U^* - Gp^h$ is calculated.

3. CONVERGENCE ANALYSIS FOR THE HODGE PROJECTION U^h

From now on, we consider the convergence for the Gibou-Min method. Let $L^h := DG$ denote its associated linear operator, then it maps a discrete function $p^h : \Omega^h \to \mathbb{R}$ to another function $L^h p^h : \Omega^h \to \mathbb{R}$ such that

$$\begin{aligned} \left(L^h p^h\right)_{ij} \;=\;& H_{i+\frac{1}{2},j}\left(p^h_{i+1,j} - p^h_{ij}\right) - H_{i-\frac{1}{2},j}\left(p^h_{ij} - p^h_{i-1j}\right) \\ &+ H_{ij+\frac{1}{2}}\left(p^h_{ij+1} - p^h_{ij}\right) - H_{ij-\frac{1}{2}}\left(p^h_{ij} - p^h_{ij-1}\right) \end{aligned}, \tag{3.1}$$

for each $(x_i, y_j) \in \Omega^h$.

Most of lemmas and theorems in this section will be just stated without proofs, which we refer to [13] for details, for our main theme of this work is to introduce the convergence of the gradient in the Hodge-Helmholtz decomposition.

In this setting, we have that $Ker\left(L^h\right) = span\left\{1_{\Omega^h}\right\}$ and for a vector field $U^* : E^h \to \mathbb{R}$, $-L^h p^h = -DU^*$ has a unique solution $p^h \in \{1_{\Omega^h}\}^\perp$.

To analyze the convergence for the scheme, we introduce two inner products defined on E^h and Ω^h.

Definition 3.1. *Let E^h and Ω^h be the sets of edges and grid nodes, respectively.*

(i) *(Inner product between vector fields) Given two vector fields U^1, $U^2 : E^h \to \mathbb{R}$, their inner product is defined as*

$$\left\langle U^1, U^2 \right\rangle_{E^h} := h^2 \sum_{i,j} H_{i+\frac{1}{2},j} u^1_{i+\frac{1}{2},j} u^2_{i+\frac{1}{2},j} + h^2 \sum_{i,j} H_{i,j+\frac{1}{2}} v^1_{i,j+\frac{1}{2}} v^2_{i,j+\frac{1}{2}}$$

(ii) *(Inner product between scalar fields) Given two discrete functions p^1, $p^2 : \Omega^h \to \mathbb{R}$, their inner product is defined as*

$$\left\langle p^1, p^2 \right\rangle_{\Omega^h} := h^2 \sum_{i,j} p^1_{i,j} \cdot p^2_{i,j}$$

With the two inner product spaces, we can see in the following lemma that G is the adjoint operator of $-\frac{1}{h^2}D$.

Lemma 3.2 (Integration-by-parts)**.** *Let G and D be the discrete gradient and divergence operators, respectively. Then for any discrete fuction p on Ω^h and vector field U on E^h, we have*

$$\langle Gp, U \rangle_{E^h} = -\left\langle p, \frac{1}{h^2} DU \right\rangle_{\Omega^h}.$$

The integration-by-parts leads to a discrete version of the Helmholtz decomposition.

Theorem 3.3. *Given vector field $U^* : E^h \to \mathbb{R}$, there exists a unique $p^h \in \{1_{\Omega^h}\}^\perp$ such that $DGp = DU^*$. Therefore, the decomposition*

$$U^* = U^h + Gp^h \quad \text{with } DGp^h = DU^*$$

is unique. Furthermore, the decomposition is orthogonal, i.e, $\langle U^h, Gp^h \rangle_{\Omega^h} = 0$.

Now, we are ready to analyze the Gibou-Min method. In the remainder of this work, by $p : \Omega \to \mathbb{R}$ and $U = (u, v) : \Omega \to \mathbb{R}^2$, we denote the analytic solutions of the Helmholtz decomposition $U^* = U + \nabla p$ for the given vector field $U^* = (u^*, v^*) : \Omega \to \mathbb{R}^2$. Also let $p^h : \Omega^h \to \mathbb{R}$ and $U^h : E^h \to \mathbb{R}$ denote the numerical solutions for the Gibou-Min method for the given $U^* : E^h \to \mathbb{R}$.

Definition 3.4. *The convergence error* $e^h : \Omega^h \to \mathbb{R}$ *and consistency error* $c^h : \Omega^h \to \mathbb{R}$ *are defined as* $e^h := p - p^h$ *and* $c^h := L^h\left(p - p^h\right)$.

The consistency error c^h is given as a divergence of some vector field, $c^h = \frac{1}{h} D d^h$ with $d^h : E^h \to \mathbb{R}$ defined as

$$\begin{aligned} d_{i+\frac{1}{2},j} =& \frac{1}{H_{i+\frac{1}{2},j}} \int_{E_{i+\frac{1}{2},j} \cap \Omega} \left[\frac{p_{i+1,j} - p_{ij}}{h} - u^*(x_{i+\frac{1}{2}}, y_j) \right] dy \\ &+ \frac{1}{H_{i+\frac{1}{2},j}} \int_{E_{i+\frac{1}{2},j} \cap \Omega} \left[u^*(x_{i+\frac{1}{2}}, y_j) - \frac{\partial p}{\partial x}\left(x_{i+\frac{1}{2}}, y_j\right) \right] dy, \end{aligned}$$

and $d_{i,j+\frac{1}{2}}$ is defined in the same manner. We can estimate the vector d^h for which $c^h = \frac{1}{h} D d^h$. For each i and j, the Taylor series expansion shows

$$d_{i+\frac{1}{2},j} = \begin{cases} O(h^3), & \text{if } H_{i+\frac{1}{2},j} = 1, \\ O(h^2), & \text{if } 0 < H_{i+\frac{1}{2},j} < 1 \end{cases} \tag{3.2}$$

and we have the same result for $d_{i,j+\frac{1}{2}}$.

Theorem 3.5. *Given a smooth vector field* U^*, *let* U *be its analytic Hodge projection and* U^h *the numerical approximation from the Gibou-Min method,*

$$U^* = U + \nabla p \quad \textit{and} \quad U^* = U^h + Gp^h \quad (p^h \in \{1_{\Omega^h}\}^\perp \textit{ and } L^h p^h = DU^*). \tag{3.3}$$

Then we have

(i) $\left\|U - U^h\right\| = O\left(h^{1.5}\right)$.

(ii) $\left\|G\left(e^h\right)\right\| = \left\|Gp - Gp^h\right\| = O\left(h^{1.5}\right)$.

Proof. From the decompositions (3.3), we have $U - U^h = (U^* - \nabla p) - (U^* - Gp^h)$. Since G is the standard central finite difference operator, $\nabla p - Gp = O(h^2)$. Hence, the estimate (i) follows from (ii) and it suffices to show $\left\|Gp - Gp^h\right\| = O\left(h^{1.5}\right)$. Lemma 3.2 shows

$$\left\langle \frac{1}{h} d^h - G(p - p^h), G(p - p^h) \right\rangle_{E^h} = -\frac{1}{h^2} \left\langle \frac{1}{h} D d^h - L^h(p - p^h), p - p^h \right\rangle_{\Omega^h} = 0 \tag{3.4}$$

Here we used the facts that $DG = L^h$ and $\frac{1}{h} D d^h = c^h = L^h e^h$. Equation (3.4) means that $\frac{1}{h} d^h - G(p - p^h)$ is orthogonal to $G(p - p^h)$, which implies

$$\left\| \frac{1}{h} d^h \right\| = \left\| \frac{1}{h} d^h - G(p - p^h) \right\|_{E^h}^2 + \left\| G(p - p^h) \right\| \geq \left\| G(p - p^h) \right\|.$$

On the other hand, the pointwise estimate of d^h given in (3.2) gives

$$\begin{aligned}\langle \frac{1}{h}d^h, \frac{1}{h}d^h\rangle_{E^h} &= \sum_{ij} H_{i+\frac{1}{2},j}\left(d_{i+\frac{1}{2},j}\right)^2 + \sum_{ij} H_{i,j+\frac{1}{2}}\left(d_{i,j+\frac{1}{2}}\right)^2 \\ &= \sum_{H_{i+\frac{1}{2},j},H_{i,j+\frac{1}{2}}=1} O(h^6) + \sum_{0<H_{i+\frac{1}{2},j},H_{i,j+\frac{1}{2}}<1} O(h^4) \\ &= O(h^6)O(h^{-2}) + O(h^4)O(h^{-1}) = O(h^3).\end{aligned}$$

Here, we used the fact that the number of inside edges, $H_{i+\frac{1}{2},j} = 1$ and $H_{i,j+\frac{1}{2}} = 1$, grows quadratically so that it becomes $O(h^{-2})$, and that of edges near the boundary is $O(h^{-1})$. Consequently, we have $\left\|Gp - Gp^h\right\| = O\left(h^{1.5}\right)$, which completes the proof. □

4. CONVERGENCE ANALYSIS FOR PRESSURE p

In order to estimate the convergence error using the gradient estimation, we need the Poincare-Friedrichs inequality for piecewise constant functions as follows. Let $D \in \mathbb{R}^2$ be a bounded and connected polygonal domain and $\mathcal{T}$ a simplicial triangulation of D. By $\mathcal{E}^i(\mathcal{T})$, we denote the set of the interior edges of $\mathcal{T}$. For an interior edge e shared by two triangles T_1 and T_2 in $\mathcal{T}$, we define a jump $[[w]]$ across e as

$$[[w]] = w_1 n_1 + w_2 n_2$$

where n_j is the outer normal unit vector of T_j and $w_j = wj_{T_j}$ for $j = 1, 2$. Then we have the Poincare-Friedrichs inequality for piecewise constant functions with respect to $\mathcal{T}$ ([2, Lemma 10.6.6]).

Lemma 4.1. *There exists a constant $C > 0$ depending only on the minimum angle of $\mathcal{T}$ such that*

$$\|c\|_{L^2(D)} \leq C\left[\left|\int_D c dx\right| + \left(\sum_{e\in\mathcal{E}^i(\mathcal{T})} |e|^{-1}\|[[c]]\|^2_{L^2(e)}\right)^{1/2}\right]$$

for any piecewise constant function c with respect to $\mathcal{T}$.

Theorem 4.2. *Let $u : \Omega^h \to \mathbb{R}$ be a discrete function with*

$$\int_{\Omega^h} u(P)dP = \sum_{P\in\Omega^h} u(P)vol(\Omega_P) = 0, \qquad (\Omega_P = C_P \cap \Omega).$$

Then there exists a constant C independent of the step size h such that

$$C\frac{h_{min}}{h}\|u\|^2_{L^2(\Omega^h)} \leq \sum_{Q\in E_h}\left(\frac{u(Q^+) - u(Q^-)}{h}\right)^2 H_Q h^2. \tag{4.1}$$

Proof. For each (x_i, y_j), we take a control volume $D_{ij} \subset [x_i - \frac{h}{2}, x_i + \frac{h}{2}] \times [y_j - \frac{h}{2}, y_j + \frac{h}{2}]$ as D_{ij} is a union of triangles such that $vol(D_{ij}) = vol(\Omega_{ij})$ and

$$D_{ij} \cap D_{i+1,j} = L_{i+\frac{1}{2},j}, \quad D_{ij} \cap D_{i,j+1} = L_{i,j+\frac{1}{2}}$$

and any angle θ of the triangles is bounded as

$$\theta_1 \leq \theta \leq \theta_2$$

where θ_1 and θ_2 are independent of h. In this setting, we can see that for every edge e, either $e \cap E_Q = e$ for some $Q \in E^h$ or $|e \cap E_Q| = 0$ for all $Q \in E^h$. From this setting, the number of triangles in D_{ij} sharing the edge $L_{i+\frac{1}{2},j}$ is $O((hH_{i+\frac{1}{2},j})/h_{min})$ where $\left|L_{i+\frac{1}{2},j}\right| = hH_{i+\frac{1}{2},j}$ and $h_{min} = \min\{\left|L_{i\pm\frac{1}{2},j}\right|, \left|L_{i,j\pm\frac{1}{2}}\right|\}$.

Now, we define a piecewise constant function u_c as $u_c = u_{ij}$ on D_{ij}. Then we have

$$\int_{\cup D_{ij}} |u_c(x)|^2 dx = \int_{\Omega^h} |u(P)|^2 dP \quad \text{and} \quad \int_{\cup D_{ij}} u_c(x) dx = \int_{\Omega^h} u(P) dP = 0.$$

We note that $\|[[u_c]]\|_{L^2(e)} = 0$ if $|e \cap L_{i+\frac{1}{2},j}| = 0$ and $\|[[u_c]]\|^2_{L^2(e)} = |e|(u_{i+,j} - u_{ij})^2$ if $|e \cap L_{i+\frac{1}{2},j}| = |e|$. Applying u_c to Lemma 4.1, we verify that there exists a constant C independent of h such that

$$\begin{aligned}\int_{\cup D_{ij}} |u_c(x)|^2 dx &\leq C \sum_{Q \in E_h} (u_c(Q^+) - u_c(Q^-))^2 \frac{H_Q h}{h_{min}} \\ &= C \frac{h}{h_{min}} \sum_{Q \in E_h} \left(\frac{u(Q^+) - u(Q^-)}{h} \right)^2 H_Q h^2\end{aligned}$$

which completes the proof. □

We note that from the argument used for the proof of Theorem 4.2, we can see that we have an equality with $h/h_{min} = 1$ in the case when there are two positive constants C_1 and C_2 independent of h such that

$$C_1 \leq \frac{H_{i\pm\frac{1}{2},j}}{H_{i\pm\frac{1}{2},j} + H_{i,j\pm\frac{1}{2}}} \leq C_2, \quad \text{for all } (x_i, y_j) \in \Omega^h.$$

Since $p + c$ is also an analytic solution of equation (2.1) for an analytic solution p and a constant c, we may assume that

$$\sum_{P \in \Omega^h} \left(p - p^h\right)(P) vol(\Omega_P) = 0$$

for the numerical solution p_h.

Theorem 4.3 (Convergence of pressure)**.** *Let p be an analytic solution to* (2.1) *and let p^h be the numerical solution to* (3.1) *such that*

$$\sum_{P\in\Omega^h}\left(p-p^h\right)(P)vol(\Omega_P)=0.$$

Then we have

$$\|p-p^h\|_{L^2(\Omega^h)}\leq\frac{O(h^2)}{\sqrt{h_{min}}}$$

with $h_{min}=\min\{\left|L_{i\pm\frac{1}{2},j}\right|,\left|L_{i,j\pm\frac{1}{2}}\right|\}$.

Proof. Applying the convergence error $e^h=p-p^h$ to Theorem 4.2, we have

$$\|e^h\|^2_{L^2(\Omega^h)}\leq C\frac{h}{h_{min}}\sum_{Q\in E_h}\left(\frac{e^h(Q^+)-e^h(Q^-)}{h}\right)^2H_Qh^2=C\frac{h}{h_{min}}\|Ge^h\|^2.$$

On the other hand, we showed $\|Ge^h\|^2\leq O(h^3)$ in Theorem 3.5 (ii). Consequently, we have the convergence accuracy as

$$\|e^h\|^2_{L^2(\Omega^h)}\leq\frac{O(h^4)}{h_{min}}$$

which shows the theorem. □

We observed in [12] that for many domains, however, we have $h_{min}=O(h^2)$ as h tends to zero.

Definition 4.4. *Let $\Omega\subset\mathbb{R}^2$ be a bounded domain. We say that Ω has the general intersection property if the cumulative distribution function $p(\nu)$ defined by*

$$p(\nu):=|\{(x_i,y_j)\in\Omega_h: dist\left((x_i,y_j),\Gamma_h\right)\leq\nu\}| \tag{4.2}$$

is almost linear, i.e, $p(\nu)=O(h^{-2}\nu)$.

Many domains with smooth boundary as well as rectangular and circular shapes have the general intersection property. Note that when the domain Ω has the property, the set $\Omega_h^{\tau*}$ becomes empty as h tends to zero so that the threshold treatment works nothing.

Theorem 4.5. *Let Ω be a bounded open domain with smooth boundary. Assume that Ω has the general intersection property. Then, for sufficiently small h, we have*

$$\|e^h(h)\|_{L^2(\Omega^h)}=\|p-p^h\|_{L^2(\Omega^h)}=O(h).$$

Proof. Assume that Ω has the general intersection property. Then, we have $h_{min}=O(h^2)$ as h tends to zero because $p(h^\alpha)=\emptyset(h^{\alpha-2})<1$ for any $\alpha>1$. In this case, Theorem 4.3 implies

$$\|p-p^h\|_{L^2(\Omega^h)}\leq\frac{O(h^2)}{\sqrt{h_{min}}}=O(h)$$

and it shows the theorem. □

5. NUMERICAL TEST

5.1. Two dimensional example. In $\Omega = \{(x,y)|x^2+y^2<1\}$, we take a vector field $U = (u,v)$ with $u(x,y) = -2xy + \frac{xy}{\sqrt{x^2+y^2}}$ and $v(x,y) = 3x^2+y^2 - \frac{2x^2+y^2}{\sqrt{x^2+y^2}}$, and choose a scalar variable $p(x,y) = e^{x-y}$. Note that $U \cdot \vec{n} = 0$ on $\partial\Omega$ and $\nabla \cdot U = 0$ in Ω. The run of the Gibou-Min method on $U^* = U + \nabla p$ is reported in Table 1.

TABLE 1. Convergence order

grid	$\|U-U^h\|_{L^2}$	order	$\|p-p^h\|_{L^2}$	order
40^2	6.67×10^{-3}		1.33×10^{-3}	
80^2	2.48×10^{-3}	1.42	2.49×10^{-4}	2.41
160^2	8.14×10^{-4}	1.60	6.59×10^{-5}	1.92
320^2	3.05×10^{-4}	1.41	1.32×10^{-5}	2.31
640^2	1.01×10^{-4}	1.58	3.73×10^{-6}	1.82

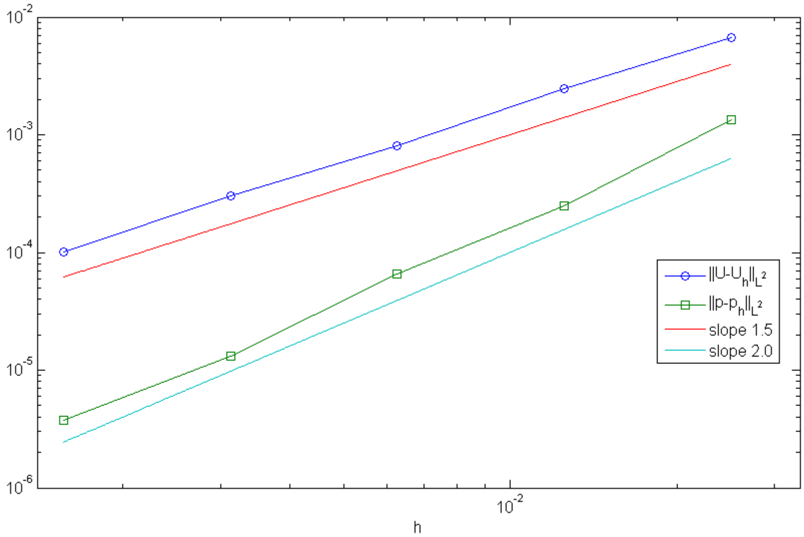

FIGURE 1. Convergence order

5.2. Three dimensional example. In $\Omega = \{(x,y,z)|x^2+y^2+z^2<1\}$, we take a vector field $U = (x^2z+3y^2z, -2xyz, -x^3-xy^2)$ and a scalar variable $p(x,y,z) = e^{x-y+z}$. Note that $U \cdot \vec{n} = 0$ on $\partial\Omega$ and $\nabla \cdot U = 0$ in Ω. The run of the Gibou-Min method on $U^* = U + \nabla p$ is reported in Table 2.

TABLE 2. Convergence order

grid	$\left\|U-U^h\right\|_{L^2}$	order	$\left\|p-p^h\right\|_{L^2}$	order
20^3	1.11×10^{-2}		3.47×10^{-3}	
40^3	3.89×10^{-3}	1.51	7.21×10^{-4}	2.26
80^3	1.31×10^{-3}	1.57	1.53×10^{-4}	2.23
160^3	4.43×10^{-4}	1.56	3.50×10^{-5}	2.12

6. CONCLUSION

In this work, we performed convergence analysis for the Gibou-Min method that calculates the Hodge-Helmholtz decomposition. Using the L^2-orthogonality between the error vector $U-U^h$ and the consistency d^h, we proved the estimate $\left\|U-U^h\right\|_{L^2}=O\left(h^{1.5}\right)$ and $\left\|Gp-Gp^h\right\|_{L^2}=O\left(h^{1.5}\right)$ for the gradient of the pressure error, as well. We then introduced a discrete version of the Poincare inequality, which led us to the result $\left\|p-p^h\right\|=O\left(h_{min}^{-0.5}h^2\right)$ with $h_{min}=\min\{\left|L_{i\pm\frac{1}{2},j}\right|,\left|L_{i,j\pm\frac{1}{2}}\right|\}$. Our estimate reads that for many domains with $h_{min}=O(h^2)$, for instance, domains with general intersection property, the pressure convergence is $O(h)$. According to our numerical tests, even though the estimate $\left\|U-U^h\right\|_{L^2}=O\left(h^{1.5}\right)$ is tight, however, the estimate does not meet the observed order $\left\|p-p^h\right\|=O\left(h^2\right)$. We put it off to future research to improve the estimate.

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