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HOW TO PREPARE FOR RETIREMENT? OPTIMAL SAVING, LABOR SUPPLY, AND INVESTMENT STRATEGY

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ABSTRACT. In this paper we study consumption-labor supply decision of an agent who prepares for retirement at a known time in the future. The agent is assumed to have a preference which is represented by the von Neumann-Morgenstern utility function in which the felicity function has constant relative risk aversion over the composite of consumption and leisure. The composite is obtained by the Cobb-Douglas function. A general problem has been studied by Bodie *et al.* (2004). We contribute to the literature by deriving the Slutsky equations and conducting comparative statics. In particular, we show that wealth effect can exhibit an interesting property depending upon the time until retirement, as the interest rate increases.

1. INTRODUCTION

A mathematical method to derive the optimal consumption and investment in one-period model has been studied by Markowitz(1952) [1]. His mean-variance portfolio selection model was simple, and hence had strong impact on the financial market although it ignores many things in the real market. Merton(1969) [2] and Samuelson(1969) [3] have improved Makowitz's model to the many period life cycle model and shown the explicit solution of the optimal intertemporal consumption and portfolio. Merton and Samuelson used the stochastic dynamic programming to solve their problem.

On the above quantitative models, the lifetime consumption-leisure choice problem has been developed by Bodie *et al.* (1992) [5]. And Bodie *et al.* (2004) [7] have improved the consumption-leisure/labor decision problem with habit formation, stochastic opportunity set, stochastic wages and labor supply flexibility.

The model that implicitly selects the time to retire has been studied by Choi *et al.* (2008) [8]. They used the constant elasticity of substitution (CES) function combined with a CRRA utility.

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We study the optimal consumption and leisure choice of an individual who has a CRRA utility function with retirement in continuous time model. Our model is one-case of Bodie *et al.* (2004) [7] with no habit formation and constant wages.

We assume that an individual knows his/her retirement time. And we contribute to derive the Slutsky equation which helps explain the causes of economic factors. The substitution effect and income effect of the optimal consumption and labor is calculated in the market where the risky asset is not tradable. The method to derive the Slutsky equation has been shown by Grandville(1989) [4].

Section 2 presents a model for continuous-time optimal consumption, leisure/labor, and portfolio selection problem where the tradable asset is the only risk-free asset. Our agent is assumed to have a composition of CRRA utility and Cobb-Douglas type utility showing substitutability of consumption and leisure. We use the Lagrangian method to calculate the optimal choice values.

In Section 3, by using the component of shadow price, we describe an explicit form of the solution and the Slutsky equation to decompose the effects of the interest rate change into the income effect and substitution effect. We use the property of the dual value function to calculate the Slutcky equation.

Section 4 introduces a model with stock investment. Section 5 gives a closed form solution of the model in Section 4.

2. THE MODEL

We consider a consumption and portfolio selection problem when an agent prepares for retirement. We assume that the agent has a time separable von Neumann-Morgenstern utility, so that he/she will try to maximize his/her utility given initial wealth amount, $X(0)$, as follows:

$$V(X(0)) = \max_{\{c(t),\ell(t)\}_0^T, X(T)} \Big[\int_0^T e^{-\rho t} u(c(t), \ell(t)) dt + e^{-\rho T} U(X(T)) \Big] \tag{2.1}$$

where $\mathbf{c}(t) \triangleq \{c(s) | \, t \le s \le T\}$, $\boldsymbol{\ell}(t) \triangleq \{\ell(s) | \, t \le s \le T\}$ denote the agent's consumption stream and leisure choice, respectively.

We assume that there is a tradable asset in the form of bond. Then, the wealth evolution equation is given by

$$dX(t) = \Big(rX(t) - c(t) + w(t)\big(\overline{L} - \ell(t)\big) \Big) dt \tag{2.2}$$

where $\overline{L} - \ell_t$ is interpreted as the labor that the agent supplies, and $w(t)$ is the wage rate of the labor supply.

3. MAIN RESULTS

Lemma 3.1. *For any given initial wealth level, X_0, at time 0, the wealth process (2.2) satisfies the following equation:*

$$X_0 = e^{-rT}X(T) + \int_0^T e^{-rt}\Big(c(t) - w(t)\big(\overline{L} - \ell(t)\big)\Big)dt. \tag{3.1}$$

Proof. We can change equation (3.1) into the following form:

$$e^{-rt}dX(t) - re^{-rt}X(t)dt = e^{-rt}\Big(- c(t) + w(t)\big(\overline{L} - \ell(t)\big)\Big)dt$$

where LHS equals to the derivative of $e^{-rt}X(t)$. Thus, if we take a integral between t and T, we get

$$e^{-rt}X(t) = e^{-rT}X(T) + \int_t^T e^{-rs}\Big(c(s) - w(s)\big(\overline{L} - \ell(s)\big)\Big)ds.$$

□

Now, we assume that the form of utility function and the bequest function are all of the constant relative risk aversion (CRRA) type:

$$u(c,\ell) = v\big(c^\alpha \ell^{1-\alpha}\big), \quad U(X(T)) = Kv(X(T)),$$

where $v(x) = \frac{x^{1-\gamma}}{1-\gamma}$. Using Lemma 3.1, we get the following result.

Proposition 3.2. *We obtain the optimal consumption, c_t^*, and the optimal labor choices, ℓ_t^*, the optimal level of terminal wealth, $X^*(T)$, to the optimization problem (2.1) as follows:*

$$c^*(t) = A\big(\lambda e^{(\rho-r)t}\big)^{-1/\gamma} w(t)^{-\frac{(1-\alpha)(1-\gamma)}{\gamma}}, \tag{3.2}$$

$$\ell^*(t) = \frac{1-\alpha}{\alpha w(t)}c^*(t), \tag{3.3}$$

$$X^*(T) = \Big(\frac{\lambda e^{(\rho-r)T}}{K}\Big)^{-1/\gamma}, \tag{3.4}$$

where

$$A = \Big(\alpha\Big(\frac{1-\alpha}{\alpha}\Big)^{(1-\alpha)(1-\gamma)}\Big)^{\frac{1}{\gamma}}. \tag{3.5}$$

The Lagrange multiplier to the budget constraint, λ, can be obtained by plugging $c^*(t)$, $\ell^*(t)$, and $X^*(T)$ into (3.1).

If $w(t)$ is given with $w(t) = w_0 e^{gt}$, then the Lagrange multiplier, λ^*, to the budget constraint is given by the following equation:

$$(\lambda^*)^{\frac{1}{\gamma}} = \frac{\Big(\alpha^\alpha\big(\frac{1-\alpha}{w_0}\big)^{1-\alpha}\Big)^{\frac{1-\gamma}{\gamma}} \frac{1}{R+G}\big(1 - e^{-(R+G)T}\big) + K^{\frac{1}{\gamma}}e^{-RT}}{X_0 + \frac{w_0\overline{L}}{r-g}\big(1 - e^{-(r-g)T}\big)}, \tag{3.6}$$

where

$$R \triangleq r + \frac{\rho - r}{\gamma}, \ \ and \ \ G \triangleq \frac{(1-\gamma)(1-\alpha)}{\gamma} g. \tag{3.7}$$

Proof. The first order conditions (FOCs) in the Lagrangian are as follows:

$$\begin{cases} u_c(c(t), \ell(t)) = \lambda e^{(\rho - r)t}, \\ u_\ell(c(t), \ell(t)) = \lambda w(t) e^{(\rho - r)t}, \\ V'(X(T)) = \lambda e^{(\rho - r)T}. \end{cases}$$

By the assumption, $u_c(c, \ell) = v'(c^\alpha \ell^{1-\alpha}) \alpha c^{\alpha - 1} \ell^{1-\alpha}$ and $u_\ell(c, \ell) = v'(c^\alpha \ell^{1-\alpha})(1-\alpha) c^\alpha \ell^{-\alpha}$. Since the first two conditions in FOCs imply $u_c(c(t), \ell(t)) = w(t) u_\ell(c(t), \ell(t))$, we get

$$\ell(t) = \frac{1-\alpha}{\alpha w(t)} c(t),$$

which confirms (3.3). Substituting this into $u_c(\cdot)$ gives us

$$\begin{aligned} u_c\big(c(t), \ell(c(t))\big) &= \alpha \Big[\Big(\frac{1-\alpha}{\alpha w(t)} \Big)^{1-\alpha} c(t) \Big]^{-\gamma} \Big(\frac{1-\alpha}{\alpha w(t)} \Big)^{(1-\alpha)} \\ &= \alpha \Big(\frac{1-\alpha}{\alpha w(t)} \Big)^{(1-\alpha)(1-\gamma)} (c(t))^{-\gamma} \\ &= w(t)^{-(1-\alpha)(1-\gamma)} \Big(\frac{c(t)}{A} \Big)^{-\gamma} \quad \text{by (3.5).} \end{aligned}$$

and hence, we get the optimal level of consumption: equation (3.2). Similarly, from the third equation in the FOCs, we get the optimal level of terminal wealth: equation (3.4). □

Let us denote λ^* as the following for convenience:

$$\lambda^* \triangleq \Big(\frac{\Lambda_1}{\Lambda_2 + \Lambda_3} \Big)^{-\gamma} \tag{3.8}$$

for $\Lambda_1 \triangleq X_0 + \frac{w_0 \overline{L}}{r-g}(1 - e^{-(r-g)T})$, $\Lambda_2 \triangleq \Big(\alpha^\alpha \big(\frac{1-\alpha}{w_0} \big)^{1-\alpha} \Big)^{\frac{1-\gamma}{\gamma}} \frac{1}{R+G} \big(1 - e^{-(R+G)T} \big)$, and $\Lambda_3 \triangleq K^{\frac{1}{\gamma}} e^{-RT}$.

Remark. (i) Λ_1 can be interpreted as the lifetime income since Λ_1 equals to $X_0 + \int_0^T e^{-rt} w(t) \big(\overline{L} - \ell^*(t) \big) dt + \int_0^T e^{-rt} w(t) \ell^*(t) dt$.

(ii) Λ_2 is associated with the cost of consumption and leisure before retirement. ($\Lambda_2 = \int_0^T e^{-rt} \cdot (\lambda^*)^{1/\gamma} \cdot \big(c^*(t) + w(t) \ell^*(t) \big) dt$)

(iii) Λ_3 is associated with the cost of wealth at retirement. ($\Lambda_3 = e^{-rT} \cdot (\lambda^*)^{1/\gamma} \cdot W^*(T)$)

Lemma 3.3. *The value function can be represented as the lifetime income times shadow price divided by* $1 - \gamma$*:*

$$V(X_0) = \frac{\Lambda_1}{1-\gamma} \lambda.$$

Proof. The proof is derived by substituting (3.2), (3.3), (3.4), and (3.8) into (2.1). □

Now, consider an expenditure problem which is dual to the our primal problem:

$$E = \min_{\{c(t),\ell(t)\}_0^T, X(T)} \Big[e^{-rT}X(T) + \int_0^T e^{-rt}\Big(c(t) - \omega(t)\big(\overline{L} - \ell(t)\big)\Big)dt\Big]$$

with constraint

$$\int_0^T e^{-\rho t}u(c(t),\ell(t))dt + e^{-\rho T}V(W(T)) \geq \overline{U}$$

where $\overline{U}$ is a given utility. By using lemma 3.3 and the property of duality, we can derive the expenditure function as

$$\begin{aligned} E = X_0 &= V^{-1}(\overline{U}) \\ &= \big((1-\gamma)\cdot\overline{U}\big)^{\frac{1}{1-\gamma}} \cdot (\Lambda_2 + \Lambda_3)^{-\frac{\gamma}{1-\gamma}} - \frac{\omega_0\overline{L}}{r-g}\big(1 - e^{-(r-g)T}\big). \end{aligned} \tag{3.9}$$

Then, the Hicksian version of consumption should equal to

$$c_H(t) = A\big(\lambda_H e^{(\rho-r)t}\big)^{-1/\gamma} w(t)^{-\frac{(1-\alpha)(1-\gamma)}{\gamma}} \tag{3.10}$$

for $\lambda_H \triangleq \left(\frac{\Lambda_H}{\Lambda_2+\Lambda_3}\right)^{-\gamma} = \left(\frac{E + w_0\overline{L}\int_0^T e^{-(r-g)t}dt}{\Lambda_2+\Lambda_3}\right)^{-\gamma}$. By using equations (3.2) and (3.10), we get both the substitution and income effect w.r.t. (with respect to) the interest rate denoted by r.

Proposition 3.4. *(i) The substitution effect (SE) to the consumption:*

$$-c^*(0) \cdot \frac{1}{1-\gamma} \cdot \frac{\Lambda_2' + \Lambda_3'}{\Lambda_2 + \Lambda_3}.$$

(ii) The income effect (IE) to the consumption:

$$c^*(0) \cdot \left(\frac{\Lambda_1'}{\Lambda_1} + \frac{\gamma}{1-\gamma} \cdot \frac{\Lambda_2' + \Lambda_3'}{\Lambda_2 + \Lambda_3}\right).$$

Proof. Derivatives of optimal consumption and shadow price w.r.t interest rate are as follows:

$$\frac{\partial c^*(0)}{\partial r} = -\frac{1}{\gamma} \cdot \frac{c^*(0)}{\lambda} \cdot \frac{\partial \lambda}{\partial r} \quad \& \quad \frac{\partial \lambda}{\partial r} = -\gamma\lambda^{\frac{1+\gamma}{\gamma}} \cdot \frac{(\Lambda_1'\Lambda_2 - \Lambda_1\Lambda_2') + (\Lambda_1'\Lambda_3 - \Lambda_1\Lambda_3')}{(\Lambda_2+\Lambda_3)^2}.$$

Substituting $\frac{\partial\lambda}{\partial r}$ into $\frac{\partial c^*(0)}{\partial r}$, using the definition of λ, (3.8), we calculate $\frac{\partial c^*(0)}{\partial r}$:

$$c^*(0) \cdot \Big(\frac{\Lambda_1'}{\Lambda_1} - \frac{\Lambda_2' + \Lambda_3'}{\Lambda_2 + \Lambda_3}\Big).$$

In the similar way to deriving $\frac{\partial c^*(0)}{\partial r}$, the substitution effect of $c^*(0)$ is as follows:

$$\frac{\partial c_H}{\partial r} = c^*(0) \cdot \Big(\frac{{\Lambda_H}'}{\Lambda_H} - \frac{\Lambda_2' + \Lambda_3'}{\Lambda_2 + \Lambda_3}\Big) \quad (\because c^*(0) = c_H(0)).$$

By using definition of Λ_H, Λ_H can be written as the following

$$\left((1-\gamma)\cdot\overline{U}\right)^{\frac{1}{1-\gamma}}\cdot(\Lambda_2+\Lambda_3)^{-\frac{\gamma}{1-\gamma}}.$$

Since Λ_2 and Λ_3 are function of r, we get the proposition (3.4)-(i).

IE can be derived by using the Slutsky equation:

$$-\frac{\partial c^*}{\partial X}\cdot\frac{\partial E}{\partial r}=\frac{\partial c^*(0)}{\partial r}-\frac{\partial c_H}{\partial r}.$$

□

We introduce the elasticity of substitution between the cost of total consumption, which is the cost of consumption and leisure for lifetime, and the lifetime income w.r.t. the interest rate as

$$\frac{(\Lambda_2+\Lambda_3)'/(\Lambda_2+\Lambda_3)}{\Lambda_1'/\Lambda_1}:ESCIR.$$

And note that IE can be rewritten as $-SE-\frac{(\lambda^*)'\cdot c^*(0)}{\gamma\lambda^*}$.

Remark. The following statements are the effects of the interest rate to the consumption for $\gamma>1$; it is obvious that the SE is less than 0 because both Λ_2' and Λ_3' are less than 0.

(i) If the ESCIR is in $(0,\frac{\gamma-1}{\gamma})$, then the IE and the total effect(IE+SE) are less than 0.
(ii) If the ESCIR is in $[\frac{\gamma-1}{\gamma},1]$, then the IE is larger than 0 but the total effect is less than 0.
(iii) If the ESCIR is in $(1,\infty)$, then the IE and the total effect are larger than 0.

In a similar manner, we can also derive the substitution and income effects of labor and terminal wealth w.r.t. the interest rate:

	Substitution Effect	Income Effect
Labor $(\overline{L}-\ell^*(0))$	$\ell^*(0)\cdot\frac{1}{1-\gamma}\cdot\frac{\Lambda_2'+\Lambda_3'}{\Lambda_2+\Lambda_3}>0$	$-\ell^*(0)\cdot\left(\frac{\Lambda_1'}{\Lambda_1}+\frac{\gamma}{1-\gamma}\cdot\frac{\Lambda_2'+\Lambda_3'}{\Lambda_2+\Lambda_3}\right)\lesseqgtr 0$
Terminal Wealth $(X^*(T))$	$X^*(T)\cdot\left(\frac{T}{\gamma}-\frac{1}{1-\gamma}\cdot\frac{\Lambda_2'+\Lambda_3'}{\Lambda_2+\Lambda_3}\right)\lesseqgtr 0$	$X^*(T)\cdot\left(\frac{\Lambda_1'}{\Lambda_1}+\frac{\gamma}{1-\gamma}\cdot\frac{\Lambda_2'+\Lambda_3'}{\Lambda_2+\Lambda_3}\right)\lesseqgtr 0$

The SE and IE to the labor have the opposite sign to consumption. Additionally, the sign of SE to the terminal wealth (or income) depends on how much time remains from now. When an agent faces retirement, the SE to the terminal wealth has a negative sign. However, if he/she has plenty of time until his retirement, then his/her terminal wealth will increase.

The substitution and wealth effects w.r.t. the growth rate of wage are the following:

	Substitution Effect	Income Effect
Consumption $(c^*(0))$	$-c^*(0)\cdot\frac{1}{1-\gamma}\cdot\frac{(\Lambda_2)_g}{\Lambda_2+\Lambda_3}$	$c^*(0)\cdot\left[\frac{(\Lambda_1)_g}{\Lambda_1}+\frac{\gamma}{1-\gamma}\cdot\frac{(\Lambda_2)_g}{\Lambda_2+\Lambda_3}\right]$
Labor $(\overline{L}-\ell^*(0))$	$\ell^*(0)\cdot\frac{1}{1-\gamma}\cdot\frac{(\Lambda_2)_g}{\Lambda_2+\Lambda_3}$	$-\ell^*(0)\cdot\left[\frac{(\Lambda_1)_g}{\Lambda_1}+\frac{\gamma}{1-\gamma}\cdot\frac{(\Lambda_2)_g}{\Lambda_2+\Lambda_3}\right]$
Terminal Wealth $(X^*(T))$	$-X^*(T)\cdot\frac{1}{1-\gamma}\cdot\frac{(\Lambda_2)_g}{\Lambda_2+\Lambda_3}$	$X^*(T)\cdot\left[\frac{(\Lambda_1)_g}{\Lambda_1}+\frac{\gamma}{1-\gamma}\cdot\frac{(\Lambda_2)_g}{\Lambda_2+\Lambda_3}\right]$

The substitution and wealth effects w.r.t. the initial wage:

	Substitution Effect	Income Effect
Consumption $(c^*(0))$	$-c^*(0)\cdot\frac{1-\gamma}{\gamma}\cdot\frac{1-\alpha}{\omega_0}$	$c^*(0)\cdot\frac{(\Lambda_1)_{\omega_0}}{\Lambda_1}$
Labor $(\overline{L}-\ell^*(0))$	$\ell^*(0)\cdot\frac{1-\gamma}{\gamma}\cdot\frac{1-\alpha}{\omega_0}$	$-\ell^*(0)\cdot\frac{(\Lambda_1)_{\omega_0}}{\Lambda_1}$
Terminal Wealth $(X^*(T))$	0	$X^*(T)\cdot\frac{(\Lambda_1)_{\omega_0}}{\Lambda_1}$

4. The Model With Stock Investment

Now, we assume that there are two tradable assets: bond and stock. So, an agent has the following expected utility:

$$V(X(0)) = \max_{\{\mathbf{c}(t),\boldsymbol{\ell}(t),\mathbf{\Pi}(t)\}_{t=0}^{T},X(T)} E_0\Big[\int_0^T e^{-\rho t}u(c(t),\ell(t))dt + e^{-\rho T}U(X(T))\Big], \tag{4.1}$$

where $\mathbf{c}(t) \triangleq \{c(s)|\, t\le s\le T\}$, $\boldsymbol{\ell}(t) \triangleq \{\ell(s)|\, t\le s\le T\}$, $\mathbf{\Pi}(t) \triangleq \{\Pi(s)|\, t\le s\le T\}$ denote the agent's consumption stream, leisure choice and investment decision, respectively.

The price of bond, $S_0(t)$, is given by $dS_0(t) = rS_0(t)dt$. We assume the price of stock, $S_1(t)$, follows a geometric Brownian motion, with drift, μ, and volatility, σ:

$$dS_1(t) = \mu S_1(t)dt + \sigma S_1(t)dB(t),$$

where $B(t)$ denotes the standard Brownian motion. Then, the wealth amount of the agent, $X(t)$, at time t is given by

$$dX(t) = \big(rX(t) + (\mu-r)\Pi(t) + w(t)(\overline{L}-\ell(t)) - c(t)\big)dt + \sigma\Pi(t)dB(t). \tag{4.2}$$

5. The Martingale Method And Results

Let us denote $\theta \triangleq \frac{\mu-r}{\sigma}$ and define H by

$$H(t) \triangleq e^{-\left[\left(r+\frac{1}{2}\theta^2\right)t+\theta B(t)\right]}.$$

Then, we have the following result.

Lemma 5.1. *For any given initial wealth level, X_0, at time 0, the wealth process (4.2) satisfies the following equation:*

$$X_0 = E_0\Big[\int_0^T H(t)\big(c(t) - w(t)(\overline{L}-\ell(t))\big)dt + H(T)X(T)\Big]. \tag{5.1}$$

Proof. By Itô's theorem, we know that $dH = -H(rdt + \theta dB)$. We apply Itô's theorem to $H(t)X(t)$ again, and substitute the above dH and dX in (4.2), so that we have

$$\begin{aligned} d(HX) &= XdH + HdX + dH\cdot dX \\ &= -H\big(c - w(\overline{L}-\ell)\big)dt - H(\sigma\Pi - X\theta)dB. \end{aligned}$$

Changing this into the integral form between t and T, we get

$$H(t)X(t) = H(T)X(T) + \int_t^T H(s)\big(c(s) - w(s)(\overline{L} - \ell(s))\big)ds + \int_t^T H(s)\big(\theta X(s) - \sigma\Pi(s)\big)dB(s).$$

Since the last term in the above equation is martingale and $H(0) = 1$, we get the desired result. $\square$

Now, we assume that the form of utility function and the bequest function are all of constant relative risk aversion (CRRA) type:

$$u(c, \ell) = v\big(c^\alpha \ell^{1-\alpha}\big), \quad U(X(T)) = Kv(X(T)),$$

where $v(x) = \frac{x^{1-\gamma}}{1-\gamma}$. By using Lemma 5.1, we get the following result.

Proposition 5.2. *We obtain the optimal consumption, c_t^*, and the optimal labor choices, ℓ_t^*, the optimal level of terminal wealth, $X^*(T)$, to the optimization problem (4.1) as follows:*

$$c^*(t) = A\big(\lambda e^{\rho t} H(t)\big)^{-1/\gamma} w(t)^{-\frac{(1-\alpha)(1-\gamma)}{\gamma}}, \tag{5.2}$$

$$\ell^*(t) = \frac{1-\alpha}{\alpha w(t)} c^*(t), \tag{5.3}$$

$$X^*(T) = \Big(\frac{\lambda e^{\rho T} H(T)}{K}\Big)^{-1/\gamma}. \tag{5.4}$$

The Lagrange multiplier to the budget constraint, λ, can be obtained by plugging $c^*(t)$, $\ell^*(t)$, and $X^*(T)$ into (5.1).

Proof. The proof is similar to proposition (3.2). $\square$

Remark. Now, consider the optimized wealth process $X^*(t, B(t))$ for all $0 \le t \le T$:

$$X^*(t, B(t)) \triangleq E_t\Big[\int_t^T \frac{H(s)}{H(t)}\Big(c^*(s) - w(s)\big(\overline{L} - \ell^*(s)\big)\Big)ds + \frac{H(T)}{H(t)} X^*(T)\Big].$$

By (5.3), we have

$$X^*(t) = E_t\Big[\int_t^T \frac{H(s)}{H(t)}\Big(\frac{c^*(s)}{\alpha} - w(s)\overline{L}\Big)ds + \frac{H(T)}{H(t)} X^*(T)\Big]. \tag{5.5}$$

Comparing this to equation (4.2), we can derive the optimal cash amount invested in the risky asset, $\Pi^*(t)$, as follows:

$$\Pi^*(t) = \frac{1}{\sigma} \frac{\partial X^*(t, B(t))}{\partial B(t)}. \tag{5.6}$$

To get an explicit form of the solution, we assume that the wage rate grows with rate g from the initial value $w(0) = w_0$: $w(t) = w_0 e^{gt}$. Then, we get the following result.

Proposition 5.3. *If $w(t)$ is given with $w(t) = w_0 e^{gt}$, then the Lagrange multiplier, $\hat{\lambda}^*$, to the budget constraint is given by the following equation:*

$$\hat{\lambda}^* = \left(\frac{\Lambda_1}{\hat{\Lambda}_2 + \hat{\Lambda}_3} \right)^{-\gamma}, \tag{5.7}$$

where[1]

$$\hat{R} \triangleq r + \frac{\rho - r}{\gamma} - \frac{1-\gamma}{2\gamma^2}\theta^2,$$
$$\hat{\Lambda}_2 \triangleq \left(\alpha^{\alpha} \big(\frac{1-\alpha}{w_0}\big)^{1-\alpha} \right)^{\frac{1-\gamma}{\gamma}} \frac{1}{\hat{R}+G}\big(1 - e^{-(\hat{R}+G)T}\big),$$
$$\hat{\Lambda}_3 \triangleq K^{\frac{1}{\gamma}} e^{-\hat{R}T}.$$

And the optimal cash amount invested in the risky asset is given with

$$\Pi^*(t) = \frac{1}{\sigma} \cdot \frac{\theta}{\gamma} \Big(X^*(t) + E_t \Big[\overline{L} \int_t^T \frac{H(s)}{H(t)} w(s) ds \Big] \Big). \tag{5.8}$$

Proof. The proof is in the appendix. □

Since Lemma 3.3 also satisfies for the model of stock investment

$$V(x) = \frac{\Lambda_1}{1-\gamma} \hat{\lambda} \quad \text{for } X_0 = x,$$

we derive the similar result to the previous model. By using proposition 5.2, we derive the substitution effect and income effect which is the same as proposition 3.4 in the market with the risky asset.

6. Conclusion

In this paper, we have studied the optimal consumption and portfolio selection problem when an agent chooses his/her labor supply. We have classified the IE, SE, and their sum depending on the size of ESCIR:

(i) the signs of IE and SE for the consumption and labor have been shown by the interval of size of ESCIR.
(ii) the signs of IE and SE for the terminal wealth depends on 2 factors, which are the size of ESCIR and the length of remaining time to the retirement.

It is interesting to conduct the comparative static analysis of variables in the future as Grandville(1989) [4] did.

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[1] $\hat{R}$ in equation (5.7) is corresponding to the constant, ν, in Merton(1969) [2].

APPENDIX

Proof of Proposition 5.3. equation (5.5) at time 0 gives an alternative form of the budget constraint:

$$X_0 = \frac{1}{\alpha} E_0\Big[\int_0^T H(t)c^*(t)dt\Big] - \overline{L} E_0\Big[\int_0^T H(t)w(t)dt\Big] + E_0[H(T)X(T)].$$

Note that $B(T) - B(t)$ is normally distributed ($\sim N(0, T-t)$) and independent of $B(t)$ which implies

$$E_t\big[e^{\mu T + \theta B(T)}\big] = e^{\mu T + \theta B(t) + \frac{\theta^2}{2}(T-t)}, \quad \text{for all } 0 \le t \le T. \tag{A.1}$$

Applying (A.1) to the last term in the above equation for budget constraint gives us

$$\begin{aligned} E_0[H(T)X^*(T)] &= \Big(\frac{\lambda e^{\rho T}}{K}\Big)^{-\frac{1}{\gamma}} E_0\Big[H^{1-\frac{1}{\gamma}}\Big] \\ &= \Big(\frac{\lambda e^{\rho T}}{K}\Big)^{-\frac{1}{\gamma}} E_0\Big[e^{\frac{1-\gamma}{\gamma}\big[\big(r+\frac{1}{2}\theta^2\big)T + \theta B(T)\big]}\Big] \\ &= \Big(\frac{\lambda}{K}\Big)^{-\frac{1}{\gamma}} e^{-RT}. \end{aligned}$$

We now use the assumption that $w(t) = w_0 e^{gt}$ to calculate the first and the second term. Applying (A.1) and Fubini's theorem to the second term, we have

$$\begin{aligned} E_0\Big[\int_0^T H(t)w(t)dt\Big] &= w_0 \int_0^T E_0\big[H(t)e^{gt}\big]dt \\ &= w_0 \int_0^T e^{-(r-g)t}dt \\ &= \frac{w_0}{r-g}\big(1 - e^{-(r-g)T}\big). \end{aligned}$$

Similarly,

$$\begin{aligned} E_0\Big[\int_0^T H(t)c^*(t)dt\Big] &= A\lambda^{-1/\gamma} \int_0^T E_0\Big[e^{-\frac{\rho}{\gamma}t} H(t)^{\frac{\gamma-1}{\gamma}} w(t)^{-\frac{(1-\alpha)(1-\gamma)}{\gamma}}\Big]dt \\ &= A\lambda^{-1/\gamma} w_0^{-\frac{(1-\alpha)(1-\gamma)}{\gamma}} \int_0^T e^{-(\hat{R}+G)t}dt \\ &= A\lambda^{-1/\gamma} w_0^{-\frac{(1-\alpha)(1-\gamma)}{\gamma}} \frac{1}{\hat{R}+G}\Big(1 - e^{-(\hat{R}+G)T}\Big). \end{aligned}$$

Substituting the above results in the budget constraints, we have

$$X_0 = \frac{1}{\alpha} A\lambda^{-1/\gamma} w_0^{-\frac{(1-\alpha)(1-\gamma)}{\gamma}} \frac{1}{\hat{R}+G}\big(1 - e^{-(\hat{R}+G)T}\big) - \overline{L}\frac{w_0}{r-g}\big(1 - e^{-(r-g)T}\big) + \Big(\frac{\lambda}{K}\Big)^{-\frac{1}{\gamma}} e^{-\hat{R}T},$$

or

$$X_0 + \frac{w_0\overline{L}}{r-g}\big(1 - e^{-(r-g)T}\big) = \lambda^{-1/\gamma}\Big[\Big(\alpha^\alpha\big(\frac{1-\alpha}{w_0}\big)^{1-\alpha}\Big)^{\frac{1-\gamma}{\gamma}} \frac{1}{\hat{R}+G}\big(1 - e^{-(\hat{R}+G)T}\big) + K^{\frac{1}{\gamma}} e^{-\hat{R}T}\Big].$$

This is equivalent to equation (5.7).

The calculations conditioned at time t is nearly the same as those at time 0, so that we have

$$\begin{aligned}E_t[H(T)X^*(T)] &= \Big(\frac{\lambda e^{\rho T}}{K}\Big)^{-\frac{1}{\gamma}} E_t\Big[H^{1-\frac{1}{\gamma}}\Big]\\ &= \Big(\frac{\lambda}{K}\Big)^{-\frac{1}{\gamma}} e^{-RT-\frac{1}{2}(\frac{1-\gamma}{\gamma})^2\theta^2 t+\frac{1-\gamma}{\gamma}\theta B(t)}.\end{aligned}$$

We also have

$$\begin{aligned}E_t\Big[\int_t^T H(s)w(s)ds\Big] &= w_0\int_t^T E_t\big[H(s)e^{gs}\big]ds\\ &= w_0 e^{-\theta B(t)-\frac{\theta^2}{2}t}\int_t^T e^{-(r-g)s}ds\\ &= \frac{w_0}{r-g}e^{-\theta B(t)-\frac{\theta^2}{2}t}\big(e^{-(r-g)t}-e^{-(r-g)T}\big),\end{aligned}$$

and

$$\begin{aligned}E_t\Big[\int_t^T H(s)c^*(s)ds\Big] &= A\lambda^{-1/\gamma}\int_t^T E_t\Big[e^{-\frac{\rho}{\gamma}s}H(s)^{\frac{\gamma-1}{\gamma}}w(s)^{-\frac{(1-\alpha)(1-\gamma)}{\gamma}}\Big]ds\\ &= A\lambda^{-1/\gamma}w_0^{-\frac{(1-\alpha)(1-\gamma)}{\gamma}}e^{-\frac{1}{2}(\frac{1-\gamma}{\gamma})^2\theta^2 t+\frac{1-\gamma}{\gamma}\theta B(t)}\int_t^T e^{-(\hat{R}+G)s}ds\\ &= A\lambda^{-1/\gamma}w_0^{-\frac{(1-\alpha)(1-\gamma)}{\gamma}}e^{-\frac{1}{2}(\frac{1-\gamma}{\gamma})^2\theta^2 t+\frac{1-\gamma}{\gamma}\theta B(t)}\frac{e^{-(\hat{R}+G)t}-e^{-(\hat{R}+G)T}}{\hat{R}+G}.\end{aligned}$$

By differentiating the terms in (5.5) partially in $B(t)$, we have

$$\begin{aligned}\frac{\partial E_t[H(T)X^*(T)]}{\partial B(t)} &= \frac{1-\gamma}{\gamma}\theta E_t[H(T)X^*(T)]\\ \frac{\partial E_t\Big[\int_t^T H(s)w(s)ds\Big]}{\partial B(t)} &= -\theta E_t\Big[\int_t^T H(s)w(s)ds\Big]\\ \frac{\partial E_t\Big[\int_t^T H(s)c^*(s)ds\Big]}{\partial B(t)} &= \frac{1-\gamma}{\gamma}\theta E_t\Big[\int_t^T H(s)c^*(s)ds\Big],\end{aligned}$$

which give us

$$\begin{aligned}\frac{\partial(HX^*)}{\partial B} &= \frac{1-\gamma}{\gamma}\theta E_t[H(T)X^*(T)]\\ &= \frac{\theta}{\gamma}\Big(\overline{L}E_t\Big[\int_t^T H(s)w(s)ds\Big]+(1-\gamma)HX^*\Big).\end{aligned}$$

Since $\frac{\partial H}{\partial B} = -\theta H$ and $\Pi^*(t) = \frac{1}{\sigma}\frac{\partial X^*}{\partial B}(t)$, we have

$$\Pi^*(t) = \frac{1}{\sigma}\Big(\theta X^*(t) + \frac{1}{H(t)}\frac{\partial (HX^*)}{\partial B}(t)\Big).$$

By canceling $\theta X^*(t)$ in the right-hand side of the above equation, we finally have

$$\Pi^*(t) = \frac{1}{\sigma}\frac{\theta}{\gamma}\frac{1}{H(t)}\Big(\overline{L}E_t\Big[\int_t^T H(s)w(s)ds\Big] + H(t)X^*(t)\Big),$$

which is similar to equation (5.8). □

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