Output SNR Analysis of the LPP-Hough Transform

Xiumei Li, Guoqing Yang, and Guangchun Gao

Recently, a new method called the local polynomial periodogram-Hough transform (LHT) was proposed for linear frequency modulated (LFM) signal detection. In this letter, a closed-form expression of the output signal-to-noise ratio is derived for the LHT, showing that the method exhibits a threshold effect for LFM signal detection. Comparisons with the pseudo-Wigner-Hough transform (PWHT) show that the threshold of the LHT is lower (better) than that of the PWHT.

Keywords: Output SNR, Hough transform, linear frequency modulated signal, local polynomial periodogram.

I. Introduction

In many practical applications, such as radar, sonar, and communications, linear frequency modulated (LFM) signals are of great importance. Since an LFM signal can be described as a straight line in the time-frequency domain, with the help of the Hough transform, the task of tracking an LFM signal can be turned into locating the maximum peak in the signal parameter space. The Wigner-Hough transform (WHT) was proposed to detect an LFM signal [1], which is asymptotically efficient and offers a desirable suppression of the cross terms. Moreover, the pseudo-Wigner-Hough transform (PWHT) was also proposed [2] as an estimator for the phase parameters of monocomponent and multicomponent FM signals, with both desirable numerical properties and statistical performance.

However, the Wigner-Ville distribution (WVD) suffers from an inherent noise threshold effect problem [3], and it therefore cannot give satisfactory representation for LFM signals in a significantly noisy environment. As shown in [4], the local polynomial periodogram (LPP) has a much better noise resistance capability than the WVD, and it can obtain a desirable time-frequency representation even in a very low input signal-to-noise ratio (SNR) environment. Therefore, by combining the LPP with the Hough transform, we proposed the LPP-Hough transform (LHT) for detecting LFM signals in very low SNR noise [4].

The output SNR is an important measurement of the method's sensitivity to noise. The closed-form expressions of the output SNR have been found for the WHT and PWHT [1], [2]. It was shown that in the presence of additive white Gaussian noise, the overall mapping of the WHT and the PWHT exhibits a certain threshold effect, and the threshold helps us determine the input SNR value at which the methods can work well. However, the statistical output SNR analysis for the LHT has not been investigated. In this letter, we will focus on the output SNR analysis for the LHT, to show that the LHT also exhibits the threshold effect and that the threshold of the LHT is lower (better) than that of the PWHT.

This letter is organized as follows. Section II provides a brief review of the LHT, the Hough transform, and the local polynomial Fourier transform (LPFT) whose square is the LPP. In section III, output SNR analysis is presented for the LHT, with comparison to that of the PWHT. Finally, the conclusion is drawn in section IV.

II. LHT

The LHT is a combination of the LPP and the Hough transform. In this section, we will briefly review the LPFT and the Hough transform; then, we will provide the definition and algorithm of the LHT.

The LPFT is a generalization of the short-time Fourier transform [5]. The second-order LPFT is a suitable candidate to process LFM signals. Consider an LFM signal expressed as

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$$s(t) = Ae^{j(a_0 t + \frac{b_0}{2} t^2)}, \quad (1)$$

where A is the amplitude, a_0 is the initial frequency, and b_0 is the chirp rate, which shows the instantaneous rate of frequency variations of the signal. The second-order LPFT of the signal $s(t)$ is defined as [5], [6]

$$LPFT(s;t,\omega,\omega_1) = \int_{-\infty}^{\infty} s(t+\tau)h^*(\tau)e^{-j(\omega\tau+\omega_1\tau^2/2)}d\tau, \quad (2)$$

where $h(t)$ is the window function to segment the input signal, and the parameter ω_1, which is proportional to the chirp rate of the signal, can be estimated from the location coordinates of the maximum in the polynomial time frequency transform [7]. Details of the parameter estimation can be found in [7].

The Hough transform is a feature extraction method to detect lines in an image [8]. By using the Hough transform, each point in the time-frequency plane corresponds to a sinusoid in the signal parameter plane. If N points are concentrated along a straight line in the time-frequency plane, they will correspond to N sinusoidal curves intersecting at the same point in the signal parameter plane. The integration along the line produces a maximum, and its coordinates in the signal parameter plane are directly related to the parameters of the line. In this way, the Hough transform turns a difficult global detection problem in the time-frequency plane into a more easily solved local peak detection problem in the signal parameter plane.

Since the LPP can provide better noise resistance capability than the WVD, we applied the Hough transform to the LPP of LFM signals, obtaining a new method known as the LHT for LFM signal detection [4].

The LHT of the signal $s(t)$ is defined as a mapping of the signal from the time-frequency domain into the signal parameter space:

$$\begin{aligned} LHT_s(a,b) &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} s(t+\tau)h^*(\tau)e^{-j[(a+bt)\tau + \frac{\omega_1}{2}\tau^2]} d\tau \right|^2 dt \\ &= \int_{-\infty}^{\infty} LPP_s(t, a+bt)dt; \end{aligned} \quad (3)$$

therefore, the LHT can be interpreted as a line integration of the LPP.

III. Output SNR Performance of LHT

The purpose of this section is to theoretically derive the output SNR for performance evaluation of the LHT.

The discrete form of (3), that is, the discrete LHT of a sequence $s(n)$ for n=0, 1,…, N–1 using the rectangular window, is defined as

$$LHT_s(a,b) = \sum_{n=M}^{N-M-1} \sum_{k=-M}^{M} \sum_{k_1=-M}^{M} s(n+k)s^*(n+k_1) \cdot e^{-j[(a+bn)(k-k_1) + \frac{\omega_1}{2}(k^2-k_1^2)]}, \quad (4)$$

where M is a parameter defining the window length expressed as L=2M+1. Assuming L<<N, we also define the LHT as the summation over the N–L–1 points in the center of the LPP, which leaves out the rising and falling edges of the distribution. It can be easily derived that the maximum value of the LHT of an LFM signal is equal to $(N-2M)(2M+1)^2 A^2$ and located at the coordinate point (a_0, b_0).

When the signal is corrupted by the noise, the corresponding output, that is, $LHT_{s+\eta}(a_0,b_0)$, becomes a random variable and its maximum is at a coordinate point $(a_0+\delta a, b_0+\delta b)$. Following the definition in [1], the output SNR of the LHT is defined as

$$SNR_{\text{out}} = \frac{|LHT_s(a_0,b_0)|^2}{\text{var}\{LHT_{s+\eta}(a_0,b_0)\}}, \quad (5)$$

where $LHT_s(a,b)$ means the LHT of the signal only and $LHT_{s+\eta}(a,b)$ indicates the LHT of the signal corrupted by noise. The noise is assumed to be a stationary complex white Gaussian noise, with a zero mean and a variance of σ_n^2, and independent of the signal. We further assume that the parameter ω_1 is estimated correctly, that is, $\omega_1 = b_0$.

The expectation of $LHT_{s+\eta}(a_0,b_0)$ is

$$\begin{aligned} &E\{LHT_{s+\eta}(a_0,b_0)\} \\ &= \sum_n \sum_k \sum_{k_1} E\{[s(n+k)+\eta(n+k)]\cdot[s^*(n+k_1)+\eta^*(n+k_1)]\} \\ &\quad \cdot e^{-j[(a_0+b_0 n)(k-k_1)+\frac{\omega_1}{2}(k^2-k_1^2)]} \\ &= (N-2M)(2M+1)^2 A^2 + (N-2M)(2M+1)\sigma_n^2, \end{aligned} \quad (6)$$

and its second-order moment is

$$\begin{aligned} &E\{|LHT_{s+\eta}(a_0,b_0)|^2\} \\ &= \sum_n \sum_k \sum_{k_1} \sum_m \sum_l \sum_{l_1} E\{(s_1+\eta_1)(s_2^*+\eta_2^*) \\ &\quad \cdot(s_3^*+\eta_3^*)(s_4+\eta_4)\}e^{-j\phi(n,k,k_1,m,l,l_1)}, \end{aligned} \quad (7)$$

where $s_1 = s(n+k)$, $s_2 = s(n+k_1)$, $s_3 = s(m+l)$, $s_4 = s(m+l_1)$, $\eta_1 = \eta(n+k)$, $\eta_2 = \eta(n+k_1)$, $\eta_3 = \eta(m+l)$, $\eta_4 = \eta(m+l_1)$, and $\phi(n,k,k_1,m,l,l_1) = (a_0+b_0 n)(k-k_1) - (a_0+b_0 m)(l-l_1) + \frac{\omega_1}{2}(k^2-k_1^2-l^2+l_1^2)$.

Following the similar procedure in [1], we obtain

$$\begin{aligned} &E\{|LHT_{s+\eta}(a_0,b_0)|^2\} \\ &= \sum_n \sum_k \sum_{k_1} \sum_m \sum_l \sum_{l_1} [s_1 s_2^* s_3^* s_4 + s_1 s_2^* E\{\eta_3^*\eta_4\} \\ &+ s_1 s_3^* E\{\eta_2^*\eta_4\} + s_2^* s_4 E\{\eta_1\eta_3^*\} + s_3^* s_4 E\{\eta_1\eta_2^*\} \\ &+ E\{\eta_1\eta_2^*\}E\{\eta_3^*\eta_4\} + E\{\eta_1\eta_3^*\}E\{\eta_2^*\eta_4\}] \\ &\cdot e^{-j\phi(n,k,k_1,m,l,l_1)}. \end{aligned} \quad (8)$$

By direct substitution, the terms in the multiple summations satisfy the following properties:

$$
\begin{aligned}
&s_1 s_2^* s_3^* s_4 e^{-j\phi(n,k,k_1,m,l,l_1)} = A^4,\\
&s_1 s_2^* E\{\eta_3^* \eta_4\} e^{-j\phi(n,k,k_1,m,l,l_1)} = A^2\sigma_n^2 \delta(l-l_1),\\
&s_1 s_3^* E\{\eta_2^* \eta_4\} e^{-j\phi(n,k,k_1,m,l,l_1)} = A^2\sigma_n^2 \delta(m+l_1-n-k_1),\\
&s_2^* s_4 E\{\eta_1 \eta_3^*\} e^{-j\phi(n,k,k_1,m,l,l_1)} = A^2\sigma_n^2 \delta(n+k-m-l),\\
&s_3^* s_4 E\{\eta_1 \eta_2^*\} e^{-j\phi(n,k,k_1,m,l,l_1)} = A^2\sigma_n^2 \delta(k-k_1),\\
&s_3^* s_4 E\{\eta_1 \eta_2^*\} e^{-j\phi(n,k,k_1,m,l,l_1)} = A^2\sigma_n^2 \delta(k-k_1),\\
&E\{\eta_1 \eta_2^*\} E\{\eta_3^* \eta_4\} e^{-j\phi(n,k,k_1,m,l,l_1)} = \sigma_n^4 \delta(k-k_1)\delta(l-l_1),\\
&E\{\eta_1 \eta_3^*\} E\{\eta_2^* \eta_4\} e^{-j\phi(n,k,k_1,m,l,l_1)}\\
&\qquad = \sigma_n^4 \delta(n+k-m-l)\delta(m+l_1-n-k_1). \qquad (9)
\end{aligned}
$$

Therefore, the variance of the output is independent of the signal parameters. With the assumption N>4M+1 and inserting the above expressions into (8), we obtain

$$
\begin{aligned}
&E\{|LHT_{s+\eta}(a_0,b_0)|^2\}\\
&= (N-2M)^2(2M+1)^4 A^4\\
&+2A^2\sigma_n^2(N-2M)^2(2M+1)^3\\
&+\frac{2}{3}A^2\sigma_n^2(2M+1)^3[3N(2M+1)-2M(5+8M)]\\
&+\sigma_n^4(N-2M)^2(2M+1)^2\\
&+\frac{1}{6}\sigma_n^4(2M+1)^2[4N(2M+1)-2M(10M+6)]. \qquad (10)
\end{aligned}
$$

From (6) and (10), we obtain the variance as

$$
\begin{aligned}
&\mathrm{var}\{LHT_{s+\eta}(a_0,b_0)\}\\
&=\frac{2}{3}A^2\sigma_n^2(2M+1)^3[3N(2M+1)-2M(5+8M)]\\
&+\frac{1}{6}\sigma_n^4(2M+1)^2[4N(2M+1)-2M(10+6M)]\\
&=\frac{2}{3}A^2\sigma_n^2 L^3[3NL-(L-1)(4L+1)]\\
&+\frac{1}{6}\sigma_n^4 L^2[4NL-(L-1)(5L+1)]. \qquad (11)
\end{aligned}
$$

Based on (5) and $(N-2M)(2M+1)^2 A^2=(N-L+1)L^2 A^2$, which is the maximum of the LHT of the signal only, we obtain the output SNR as a function of the input SNR:

$$
SNR_{\text{out}}^{LHT} = \frac{(N-L+1)^2 L^2 SNR_{\text{in}}^2}{\frac{2}{3}L[3NL-(L-1)(4L+1)]SNR_{\text{in}}+\frac{1}{6}[4NL-(L-1)(5L+1)]}, \qquad (12)
$$

where the input SNR is defined as A^2/σ_n^2. It should be noted that the above closed-form expression is achieved by using the software package Mathematica and is valid under the condition that N>4M+1, that is, N>2L–1.

Based on the relationship between the input and output SNRs given in (12), we can discuss the threshold effect of the output SNR. When L<<N, the denominator of (12) is approximately equal to $2NL^2SNR_{\text{in}}+\frac{2}{3}NL$. For $SNR_{\text{in}} >> 1/(3L)$, the output SNR is approximated as $SNR_{\text{out}} \approx NSNR_{\text{in}}/2$. For $SNR_{\text{in}} << 1/(3L)$, the output SNR degrades rapidly according to $SNR_{\text{out}} \approx 3NLSNR_{\text{in}}^2/2$. Therefore, the threshold on the SNR performance of this method is said to occur at the interception point, $SNR_{\text{in}} = 1/(3L)$, of these two cases considered above.

The output SNR of the PWHT was proposed in [2] as

$$
SNR_{\text{out}}^{PWHT} = \frac{(N-L+1)LSNR_{\text{in}}^2}{(\frac{2}{3}\frac{3NL-(L-1)(4L+1)}{N-L+1})SNR_{\text{in}}+1}. \qquad (13)
$$

Assuming L<<N, the PWHT has an SNR performance threshold at 1/(2L): for $SNR_{\text{in}} >> 1/(2L)$, the output SNR is approximated as $SNR_{\text{out}} \approx NSNR_{\text{in}}/2$; for $SNR_{\text{in}} << 1/(2L)$, the output SNR degrades rapidly according to $SNR_{\text{out}} \approx NL(SNR_{\text{in}}^2)$. For better readability, comparisons of the output SNRs of the PWHT and LHT are listed in Table 1.

Figure 1 shows the output SNR of the LHT for N=512 with different L values. We can clearly observe that as L increases,

Table 1. Comparisons of output SNR of PWHT and LHT.

	Output SNR	Threshold
PWHT	$\frac{(N-L+1)LSNR_{\text{in}}^2}{(\frac{2}{3}\frac{3NL-(L-1)(4L+1)}{N-L+1})SNR_{\text{in}}+1}$	$1/(2L)$: $\begin{cases} SNR_{\text{out}} \approx NSNR_{\text{in}}/2, & \text{when } SNR_{\text{in}} >> 1/(2L) \\ SNR_{\text{out}} \approx NLSNR_{\text{in}}^2, & \text{when } SNR_{\text{in}} << 1/(2L) \end{cases}$
LHT	$\frac{(N-L+1)^2 L^2 SNR_{\text{in}}^2}{\frac{2}{3}L[3NL-(L-1)(4L+1)]SNR_{\text{in}}+\frac{1}{6}[4NL-(L-1)(5L+1)]}$	$1/(3L)$: $\begin{cases} SNR_{\text{out}} \approx NSNR_{\text{in}}/2, & \text{when } SNR_{\text{in}} >> 1/(3L) \\ SNR_{\text{out}} \approx 3NLSNR_{\text{in}}^2/2, & \text{when } SNR_{\text{in}} << 1/(3L) \end{cases}$

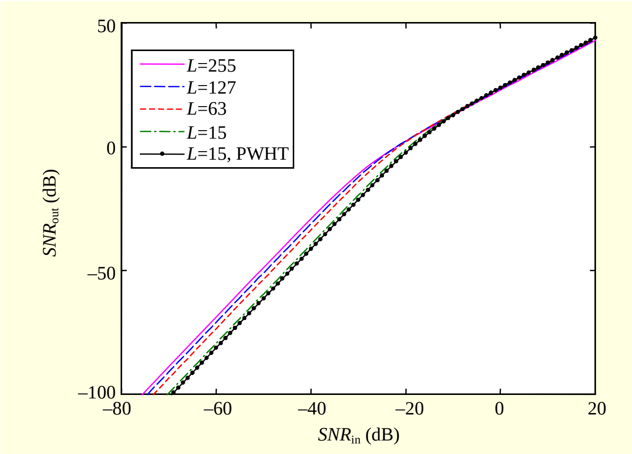

Fig. 1. Output SNR of LHT vs. input SNR for *N*=512. Output SNR of PWHT for *L*=15 is given for comparison.

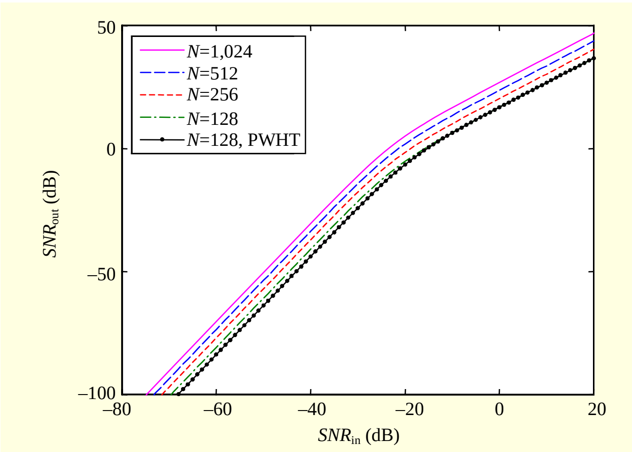

Fig. 2. Output SNR of LHT vs. input SNR for *L*=63, with *N*=128, 256, 512, and 1,024. Output SNR of PWHT for *N*=128 is given for comparison.

the threshold becomes lower (better) and the performance below the threshold becomes better. Figure 2 shows the output SNR of the LHT for *L*=63 for different *N* values. It can be seen that as *N* increases, the performance of the output SNR becomes better and the threshold becomes lower (better). The output SNR of the PWHT for *L*=15 and the output SNR of the PWHT for *N*=128 are shown in Fig. 1 and Fig. 2, respectively, for comparison. It can be seen that the output SNR performance of the LHT above the input SNR threshold is the same as that of the PWHT, and the output SNR performance of the LHT below the threshold is better than that of the PWHT. Moreover, the threshold of the LHT is slightly lower (better) and is able to achieve a better performance than the PWHT.

IV. Conclusion

In this letter, the output SNR of the LHT was derived and showed that the LHT exhibits the input SNR threshold effect. Compared with the PWHT, the LHT has a lower (better) SNR threshold. When the SNR is above the threshold, the LHT achieves the same output SNR performance as the PWHT. When the SNR is lower than the threshold, the LHT achieves better output SNR performance than the PWHT.

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