Bandpass Discrete Prolate Spheroidal Sequences and Its Applications to Signal Representation and Interpolation

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Abstract

In this paper, we propose the bandpass form of discrete prolate spheroidal sequences(DPSS) which have the maximal energy concentration in a given passband and as such are very appropriate to obtain a projection of signals. The basic properties of the bandpass DPSS are also presented. Assuming a signal satisfies the finite time support and the essential band-limitedness conditions with a known center frequency, signal representation and interpolation techniques for band-limited signals using the bandpass DPSS are introduced where the reconstructed signal has minimal out-of-band energy. Simulation results are given to present the usefulness of the bandpass DPSS for efficient representation of band-limited signal.

Keywords : Discrete prolate spheroidal sequences, signal representation, interpolation, bandpass signal

I. Introduction

Slepian[1] identified a set of vectors important to digital signal processing known as the discrete prolate spheroidal sequences(DPSS) which can be viewed as the discretization of the prolate spheroidal wave functions(PSWF). The DPSS have maximum energy concentration within a given time interval and bandwidth. Since then the study of DPSS has been an active area of research such as sampling[2][3] and spectrum estimation[4]. Using DPSS as an orthogonal basis[2][3], it is shown to reduce the sampling rate and reconstruction error. In addition, using the high energy concentration property of DPSS, the DPSS used as multiple window functions[4] are also shown to be perfectly suited to stationary spectrum estimation.

Recently, the continuous bandpass type of the PSWF based on bandpass sampling theorem has been proposed in [5] for maximizing energy concentration on the given pass band. Although the continuous bandpass type of the PSWF has been shown to be efficient representation of carrier frequency type signals over finite intervals, the eigenvectors from the kernel matrix are difficult to generate due to the computation complexity. In [5], a truncated version of the kernel matrix is used for computation of the significant eigenvalues and the corresponding eigenvectors, which causes error in the signal reconstruction. For a discrete signal both in time and frequency domains, the discrete-to-discrete PSWF has been proposed in [6]. The discrete-to-discrete PSWF is also shown to be effective in the area of filter design, multiplexing-modulation, and encryption and suitable for finite time and bandwidth signal processing. The discrete-to-discrete PSWF is one of the discrete bandpass types which will be discussed later. The applications of bandpass type of DPSS are also shown to be very effective in pulse shaping design of ultra wideband[7] and orthogonal frequency division multiplexing[8], where they simply define the bandpass DPSS using the modulation technique.

On the other hand, interpolation and prediction for uniformly sampled signals is an important area in digital signal processing. As indicated in [9] and [10], the accuracy of the existing methods for low pass signals deteriorate significantly when they are applied to bandpass signals with center frequency. Therefore, it is needed to develop an interpolation method specifically for bandpass signals.

In this paper, we introduce new bandpass DPSS which have the maximal energy concentration in a given passband. Some of the basic properties of the bandpass DPSS are also presented. The bandpass DPSS can be obtained from the eigenvalue equation or equivalently

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from modulation of baseband DPSS. In addition, signal representation and interpolation techniques for band-limited signals using the bandpass DPSS are introduced where the reconstructed signal has minimal out-of-band energy.

This paper is organized as follows. In section 2, we briefly review the baseband DPSS, and then define the bandpass DPSS that can be obtained from the eigenvalue equation or directly from modulation of baseband DPSS. We show some of the properties of the bandpass DPSS, which indicates that the bandpass type signal can be efficiently represented. The relationships between the bandpass DPSS and other bandpass forms proposed in [5] and [6] are also discussed in this section. In section 3, as an application of the bandpass DPSS, signal representation and interpolation techniques for band-limited signals using the bandpass DPSS are introduced. The simulation results are also given to present the usefulness of the bandpass DPSS for efficient representation of band-limited signal.

II. Discrete Prolate Spheroidal Sequences

2.1 Baseband DPSS

The discrete prolate spheroidal sequences(DPSS) are parameterized by the time bandwidth product NW and have been defined as the solution to the following eigenvalue problem:

$$\lambda_k \psi_k(m) = \sum_{n=0}^{N-1} \frac{\sin(2\pi W(n-m))}{\pi(n-m)} \psi_k(n) \qquad (1)$$

where $0 \le n,m \le N-1$ and $0 \le W \le 1/2$. The N real eigenfunctions $\psi_k(n)$ are known as DPSS, and the corresponding eigenvalues hold the following relationship: $1 > \lambda_0 > \lambda_1 > \cdots > \lambda_{N-2} > \lambda_{N-1} > 0$. The λ_k decays exponentially from 1 to 0, which indicates the energy concentration in the given bandwidth $[-W, W]$. As a counterpart of baseband DPSS, the periodic DPSS proposed in [11] is used for the application involving the periodic band-limited functions.

2.2. Bandpass DPSS

In this section, we propose a set of functions that are bandpass form of discrete prolate spheroidal sequences, where the functions have the highest energy concentration in a given passband.

Definition of bandpass DPSS

$$\xi_k(n) = e^{-j2\pi W_c n} \psi_k(n) \qquad (2)$$

where the passband is $[0 \le W_c - W, W_c + W \le 1/2]$, and $0 \le k \le K-1$. The passband DPSS maximizes its energy in the specified bandpass frequency range $[W_c - W, W_c + W]$. Instead of solving the eigenvalue equation, the bandpass DPSS can be directly obtained from baseband DPSS multiplied by an exponential carrier $e^{-j2\pi W_c n}$. The properties of the bandpass DPSS depend on the kernel matrix $A(n,m) \equiv e^{-j2\pi W_c(n-m)} \frac{\sin(2\pi W(n-m))}{\pi(n-m)}$. Since the kernel A is Hermitian matrix, i.e., $A = A^H$, the eigenvalues λ_k are real and the eigenvectors $\xi_k(n)$ are orthogonal. The eigenvectors are normalized to be orthonormal such that $\sum_{n=0}^{N-1} \xi_k(n)\xi_l(n) = \delta_{kl}$. Figure 1 shows the eigenvalues λ_k of the kernel matrix A for $N=64$, $W_c = 1/4$, and $W = 1/4$. All the eigenvalues satisfy that $0 < \lambda_k < 1, \forall k$, and most of the eigenvalues are very close to 1 or 0.

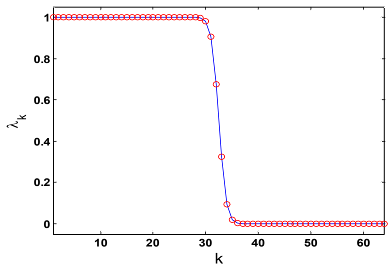
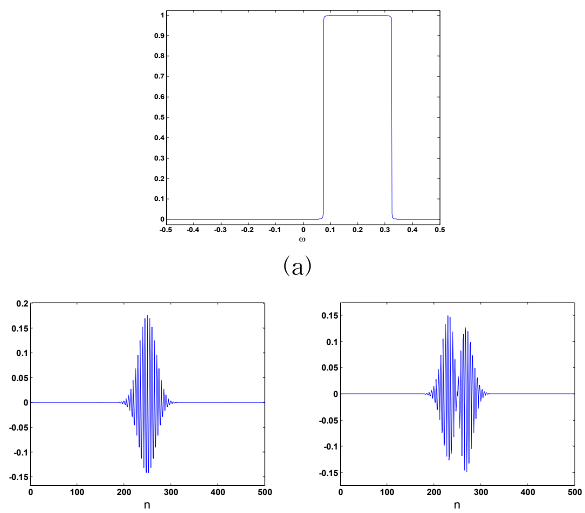


Fig. 1. Eigenvalues λ_k of the kernel matrix A for $N=64$, $W_c = 1/4$, and $W = 1/4$.

Let $\Psi_k(w) = F\{\psi_k(n)\}$ where F denotes Fourier transform. Then $F\{\xi_k(n)\} = \Psi_k(w - W_c), \forall k$. Figure 2 (a) shows the cumulative spectrum of the bandpass DPSS, which is defined by $\sum_{k=0}^{K-1} |F\{\xi_k(n)\}|^2$. It clearly verifies that spectrum of the bandpass DPSS has unity on the given passband $[W_c - W, W_c + W]$ and zero on out of band. As an example, the first and second sequences ($\xi_0(n)$ and $\xi_1(n)$) are depicted in Fig. 2 (b) and (c), respectively.



(a)

(b) (c)

Fig. 2. An example of bandpass DPSS for $W_c = 0.2$ and $W = 1/8$: (a) cumulative spectrum of the bandpass DPSS, (b) $Re\{\xi_0(n)\}$, (c) $Re\{\xi_1(n)\}$ ($Re\{\bullet\}$ denotes real part).

For the signal representation with time width(N) and band width(Ω), the number of DPSS(K) with maximal energy concentration is determined by the time-bandwidth product: $K \simeq 2N\Omega K$ for large N. Suppose that a signal has the bandwidth $\Omega = [-W_c - W, -W_c + W] \cup [W_c - W, W_c + W]$. Then, the required number of sequences for the signal representation should be $K \simeq 2N(W_c + W)$ for baseband DPSS, while $K \simeq 2NW$ for the bandpass DPSS. Therefore, since a bandpass signal has no spectral contents over lower frequency band as well as upper frequency band, the bandpass DPSS will be useful for the efficient representation of the bandpass type signal with small values of K.

2.3 Relationship between bandpass DPSS and other DPSSs

The kernels of baseband and bandpass-type DPSS and corresponding bandwidth are summarized in Table 1.

Table 1. Kernel comparison

	Discrete-time	Bandwidth
Baseband	$\frac{\sin(2\pi W(n-m))}{\pi(n-m)}$ [1] $\frac{\sin\left(\pi(2B+1)\frac{(n-m)}{N}\right)}{N\sin(\pi\frac{(n-m)}{N})}$ [11]	$[-W, W]$
Bandpass	$e^{-j2\pi W_c(n-m)}\frac{\sin(2\pi W(n-m))}{\pi(n-m)}$ $e^{-j2\pi\frac{(m_2+m_1)}{2}\frac{(n-m)}{N}} \times \frac{\sin\left(\pi\left(m_2-m_1+1\right)\frac{(n-m)}{N}\right)}{N\sin(\pi\frac{(n-m)}{N})}$ [6]	$[W_c - W, W_c + W]$ $[m_1, m_2]$

	Continuous-time	Bandwidth
Baseband	$\frac{\sin(2\pi W(t-\tau))}{\pi(t-\tau)}$ [1]	$[-W, W]$
Bandpass	$\cos(2\pi f_0(t-\frac{m}{4B_0}))\times \frac{\sin(2\pi B_0(t-\frac{m}{4B_0}))}{2\pi B_0(t-\frac{m}{4B_0})}$ [5]	$[-f_0-B_0, -f_0+B_0] \cup [f_0-B_0, f_0+B_0]$

As one can see, the discrete-to-discrete PSWF[6] and the periodic DPSS[11] are strongly related for letting $2B = m_2 - m_1$, i.e., the discrete-to-discrete PSWF can be obtained from the periodic DPSS multiplied by an exponential carrier centered at $(m_2 + m_1)/2$. Thus, the discrete-to-discrete PSWF is a bandpass form of the periodic DPSS, while the proposed one is a bandpass form of the DPSS. For $\frac{m_2+m_1}{2N} = W_c$, $\frac{m_2-m_1+1}{2N} = W$, and large N, the kernel of the discrete-to-discrete PSWF can be expressed as

$$e^{-j2\pi W_c(n-m)}\frac{\sin(2\pi W(n-m))}{N\sin(\pi\frac{(n-m)}{N})} \approx e^{-j2\pi W_c(n-m)}\frac{\sin(2\pi W(n-m))}{\pi(n-m)}.$$

Therefore, the discrete-to-discrete PSWF[6] is approximately equivalent to the bandpass DPSS for large N. For a comparison, the continuous-time bandpass PSWF[5] is also listed in Table 1.

III. Applications of Bandpass DPSS

3.1 Signal Representation

We consider an application of the bandpass DPSS for an efficient representation of bandpass type signals. Efficient representation of bandpass signals requires a fixed number K of basis functions $\xi_k(n)$, $0 \le k \le K-1$ which capture most of the energy in the given signal. Since the set of bandpass DPSS $\{\xi_k(n)\}$ is orthonormal, a signal $x(n)$ can be represented in terms of this set.

$$\tilde{x}(n) = \sum_{k=0}^{K-1}\gamma_k \xi_k^*(n) \quad (3)$$

$$\gamma_k = \sum_{n=0}^{N-1} x(n)\xi_k(n) \quad (4)$$

If the signal energy outside the given frequency band ϵ is very small

$$\epsilon = 2\left(\int_0^{W_c - W}|X(w)|^2 dw + \int_{W_c+W}^{1/2}|X(w)|^2 dw\right) \quad (5)$$

an approximate representation of $x(n)$ can be obtained by

$$x(n) \approx \hat{x}(n) = 2Re\{\tilde{x}(n)\} \quad (6)$$

Accuracy of the signal approximation depends on how concentrated the energy is within the given frequency band.

The following two types of bandpass signal are used for test.

Test signal-1:

$$x(n) = \left(\mathrm{sinc}(f_B\frac{n}{N})\right)^2\cos(2\pi f_c\frac{n}{N}+\frac{\pi}{3})$$

Test signal-2:

$$x(n) = 2\cos\left(2\pi f_B \frac{n}{N}\right)\sqrt{2}\cos\left(2\pi f_c \frac{n}{N}\right) - \sin\left(2\pi f_B \frac{n}{N}\right)\sqrt{2}\sin\left(2\pi f_c \frac{n}{N}\right)$$

where $f_B = 10$, $f_c = 10f_B$, and $N = 1000$. Figure 3 and 4 show the representation results of the test signal. In this test, the parameters of the bandpass DPSS are $W_c = 0.1$, $W = 0.02$, and $K = 40$ (see Fig. 3 (b) and 4 (b)). In these examples, the mean absolute error(MAE) are seen to be 0.000 for test signal-1 and 0.092 for test signal-2. Since the DPSS have the properties of optimal energy preservation when both the time and the frequency extensions are finite, for non time-limited signal such as test signal-2, the representation error is relatively large compared with the time-limited and band-limited signal (test signal-1). However, the 40 basis functions are sufficient to represent the test signals with good accuracy.

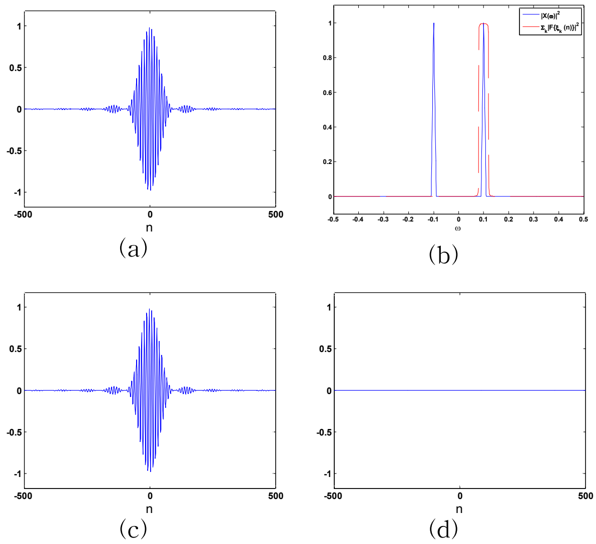
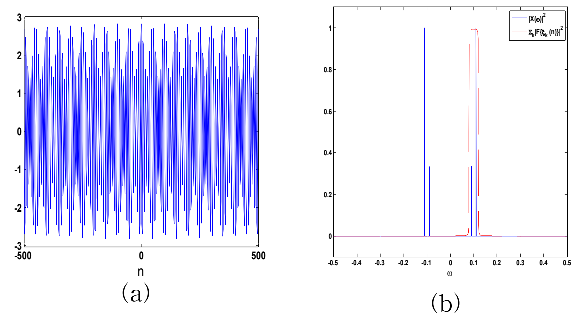


Fig. 3. An example of test signal-1 representation: (a) test signal $x(n)$, (b) cumulative spectrum of the bandpass DPSS and normalized $|X(w)|^2$, (c) $\hat{x}(n)$, (d) error $(x(n) - \hat{x}(n))$ (MAE=0.000).



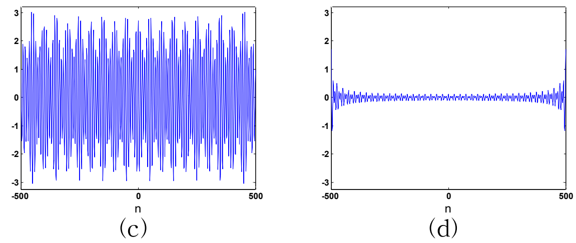


Fig. 4. An example of test signal-2 representation: (a) test signal $x(n)$, (b) cumulative spectrum of the bandpass DPSS and normalized $|X(w)|^2$, (c) $\hat{x}(n)$, (d) error $(x(n) - \hat{x}(n))$ (MAE=0.092).

3.2 Signal Interpolation

Suppose we want to reconstruct a discrete-time signal $x(n)$, $0 \le n \le N-1$ from a set of subsamples $\{x(n_i)\}$, where $0 \le n_0 < n_1 < \cdots < n_{M-1} \le N-1$ and $M < N$. If the signal $x(n)$ belongs to a subspace spanned by K-basis $\phi_k(n)$, where $K \le M < N$, the interpolated signal $\hat{x}(n)$ can be obtained from the following matrix equation:

$$\tilde{x}(n) = [x(n_0)\ x(n_1) \cdots x(n_{M-1})] \times \underbrace{\begin{bmatrix} \phi_0(n_0) & \cdots & \phi_0(n_{M-1}) \\ . & \cdots & . \\ \phi_{K-1}(n_0) & \cdots & \phi_{K-1}(n_{M-1}) \end{bmatrix}^{\dagger}}_{(\hookrightarrow \Phi_{KM})} \begin{bmatrix} \phi_0(n) \\ . \\ \phi_{K-1}(n) \end{bmatrix} \quad (7)$$

$$\hat{x}(n) = \begin{cases} \tilde{x}(n) & \text{for } \phi_k(n) = \psi_k(n) \\ 2Re\{\tilde{x}(n)\} & \text{for } \phi_k(n) = \xi_k(n) \end{cases}$$

Since the matrix Φ_{KM} is not a square matrix, the pseudo-inverse(†) is used for the projection. If the rank of Φ_{KM} is equal to K, there exists a solution in the least square sense.

Uniform Interpolation

In this simulation, we used a speech signal with deterministic passband spectrum. Figure 5 (a) shows the original and uniformly subsampled speech signals, where $N = 1000$ and $M = 200$. The interpolated signals using the baseband and bandpass DPSSs, i.e., $\phi_k(n) = \psi_k(n)$ and $\phi_k(n) = \xi_k(n)$, are shown in Fig. 5 (d) and (e), respectively. Since the small spectral residues in the lower frequency band of the signal produce interpolation errors[10] (see Fig. 5 (f)), the interpolation based on the bandpass DPSS has a better performance in the representation of the bandpass type signal with small values of K (see the interpolation errors shown in Fig. 5 (h) and (i)). In [12], it has been observed that the presence of noise or out-of-band components can produce significant interpolation errors.

Since noises are often present, a noisy subsampled signal is considered. Table 2 shows the MAE performance of the interpolated test signal-1 in the presence of additive Gaussian noise. Figure 6 shows an example of interpolated test signal-1 with SNR=20dB. The interpolation using the bandpass DPSS is shown to provide a better performance for all noise levels. As shown in Fig. 6 (d) and (e), it can be verified that the interpolation error($e(n)=x(n)-\hat{x}(n)$) depends on the bandwidth W of the DPSS, i.e., $e(n) \propto W$. This method also works for low pass signals by setting $W_c=0$ in Eq. (2). The interpolation using bandpass DPSS is robust with respect to noisy samples and can be easily extended to the case of nonuniform sampling[3] and interpolation[13].

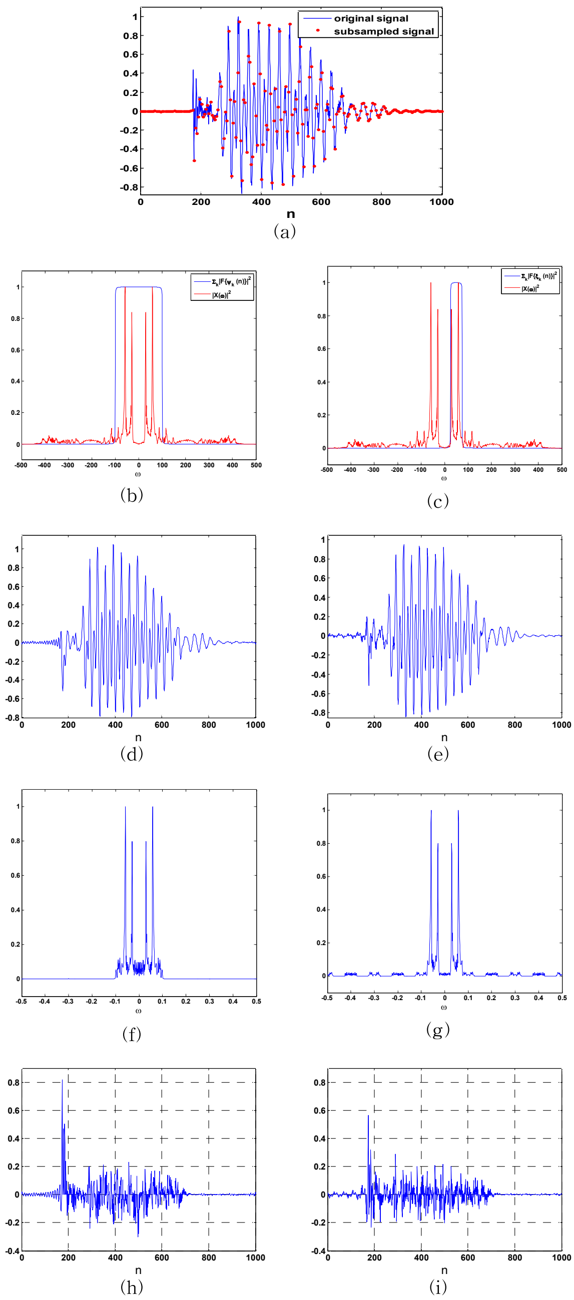


Fig. 5. 1:5 interpolation results: (a) original $x(n)$ and subsampled $x(n_i)$, (b) cumulative spectrum of $\psi_k(n)$ and normalized $|X(w)|^2$, (c) cumulative spectrum of $\xi_k(n)$ and normalized $|X(w)|^2$, (d) $\hat{x}(n)$ by $\psi_k(n)$ with $K=200$, (e) $\hat{x}(n)$ by $\xi_k(n)$ with $K=50$, (f) normalized $|\hat{X}(w)|^2$ for $\psi_k(n)$, (g) normalized $|\hat{X}(w)|^2$ for $\xi_k(n)$, (h) interpolation error for $\psi_k(n)$ (MAE=0.0349), (i) interpolation error for $\xi_k(n)$ (MAE=0.0291).

Table 2. MAE performance of the interpolated test signal-1

Noise condition (dB)	baseband DPSS	bandpass DPSS
40	0.0031	0.0025
30	0.0086	0.0052
20	0.0267	0.0180
10	0.0841	0.0481

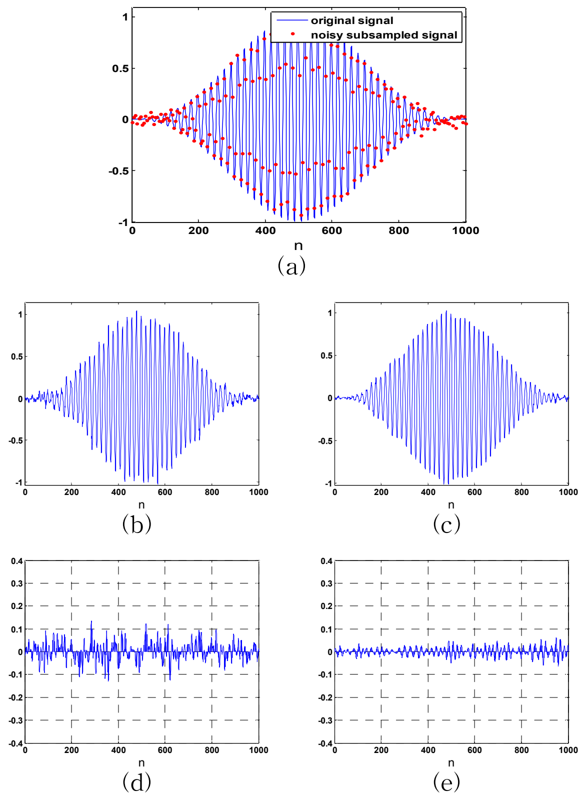


Fig. 6. 1:5 interpolated test signal-1 for SNR=20dB: (a) original $x(n)$ and noisy subsampled $x(n_i)$, (b) $\hat{x}(n)$ by $\psi_k(n)$ with $W=0.1$, $K=200$, (c) $\hat{x}(n)$ by $\xi_k(n)$ with $W=0.02$, $K=40$, (d) interpolation error for $\psi_k(n)$, (e) interpolation error for $\xi_k(n)$.

Missing Data Interpolation

For the interpolation of sampled signals with missing data, the same methodology is applied. The general assumption is that the missing data on the sampled signals are negligible and that the available samples are representatives of the original signal[14]. Figure 7 shows an example of interpolated speech signal of length 1000 with 20 missing data. The MAE performance for a variation of number of missing data is also presented in Fig. 7 (c), where missing interval and position are randomly chosen and 100 experiments are performed. The interpolation using the bandpass DPSS appears to have reasonable performance, while the method by baseband DPSS produces significant errors in the interpolated signal as the missing data increase.

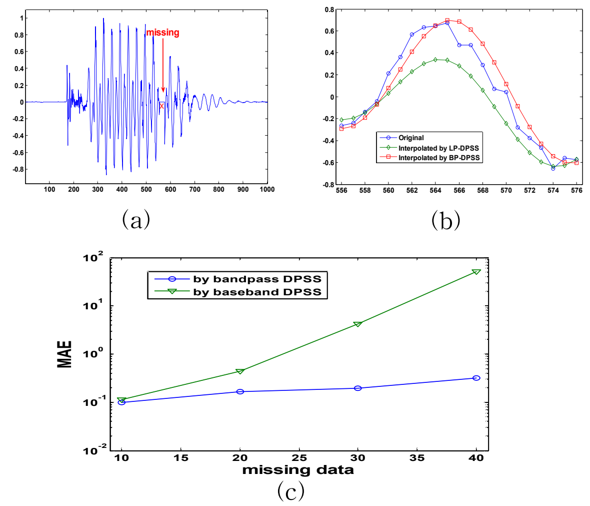


Fig. 7. Interpolated speech signal with 20 missing data: (a) missing signal, (b) the interpolation results by baseband DPSS $K=132$ and bandpass DPSS $K=44$, (c) MAE performance for a variation of number of missing data (100 experiments with different missing position).

IV. Conclusions

In this paper, we introduce the bandpass form of discrete prolate spheroidal sequences(DPSS) which have the maximal energy concentration in a given passband. We show that the bandpass DPSS can be obtained from the eigenvalue equation or directly from modulation of baseband DPSS, and some of the properties of the bandpass DPSS, which indicates that the bandpass type signal can be efficiently represented. New methodology for representation and interpolation of signals using the bandpass DPSS is also presented. The simulation results are shown to demonstrate the usefulness of the bandpass DPSS for an efficient representation of passband type signals. Specially, the interpolation using bandpass DPSS is robust with respect to noisy samples and can be easily extended to the case of nonuniform sampling and interpolation. Future work will focus on the area on nonuniform sampling and interpolation of signals.

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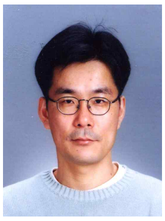

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