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FINITE GROUPS WHICH ARE MINIMAL WITH RESPECT TO S-QUASINORMALITY AND SELF-NORMALITY

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Abstract. An $\mathcal{SQNS}$-group G is a group in which every proper subgroup of G is either s-quasinormal or self-normalizing and a minimal non-$\mathcal{SQNS}$-group is a group which is not an $\mathcal{SQNS}$-group but all of whose proper subgroups are $\mathcal{SQNS}$-groups. In this note all the finite minimal non-$\mathcal{SQNS}$-groups are determined.

1. Introduction

Throughout this paper, only finite groups are considered.

Given a group theoretical property $\mathcal{P}$, a group belonging to a class of groups $\mathcal{P}$ is called a $\mathcal{P}$-group and the other groups are called non-$\mathcal{P}$-groups. A minimal non-$\mathcal{P}$-group is a non-$\mathcal{P}$-group all of whose proper subgroups are $\mathcal{P}$-groups. The problem of determining all the finite minimal non-$\mathcal{P}$-groups has been studied by several authors and there are many remarkable examples about the minimal non-$\mathcal{P}$-groups: minimal non-abelian groups (Miller and Moreno [7]), minimal non-nilpotent groups (Schmidt), minimal non-supersolvable groups ([1]) and minimal non-p-nilpotent groups (Itô).

Recall that a subgroup H of a group G is said to be s-quasinormal in G if $HK = KH$ for any Sylow subgroup K of G. Let $\mathcal{SQNS}$ denotes the class of all groups in which every proper subgroup is either s-quasinormal or self-normalizing. Our principal object here is the classification of all the finite minimal non-$\mathcal{SQNS}$-groups. Combining Theorems 3.3 and 3.4 in this paper, we get the following:

Main Theorem. *Let G be a minimal non-$\mathcal{SQNS}$-group. Then G is solvable and is isomorphic to one of the following groups*:

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(1) $G = P \rtimes Q$ *is a minimal non-nilpotent group, and* P *is non-cyclic.*
(2) $G = C_q \rtimes (C_{p^n} \times C_p)$, $\Phi(C_{p^n})C_p = Z(G)$.
(3) $G = C_q \rtimes Q_8$, Q_8 *induces an automorphism of order* 2 *on* C_q.
(4) $G = C_q \rtimes M_{p^{n+1}}$, $C_{M_{p^{n+1}}}(C_q) \cong M_{p^n}$.
(5) $G = C_{q^n} \rtimes C_{p^m}$, $m \geq 2$, $\Phi(\Phi(P)) = Z(G)$.
(6) $G = \langle a, b, c \mid a^{q^m} = b^{q^m} = 1, c^{p^n} = 1, ab = ba, a^c = a^u, b^c = b^v, u \not\equiv v \pmod{q^m}, u \equiv v \pmod{q^{m-1}}, u^p \equiv v^p \equiv 1 \pmod{q^m}\rangle$. *Furthermore,* $u \not\equiv 1 \pmod q$ *and* $v \not\equiv 1 \pmod q$ *if* $m \geq 2$.
(7) $G = \langle x, y_1, y_2, \ldots, y_b \mid x^{p^a} = y_1^q = y_2^q = \cdots = y_b^q, y_i y_j = y_j y_i, i, j = 1, 2, \ldots, b, y_i^x = y_{i+1}, i = 1, 2, \ldots, b-1, y_b^x = y_1^{d_1} y_2^{d_2} \cdots y_b^{d_b}\rangle$, $f(z) = z^b - d_b z^{b-1} - \cdots - d_2 z - d_1$ *is irreducible in* F_q. *Moreover, let*

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ d_1 & d_2 & d_3 & \cdots & d_b \end{bmatrix}.$$

Then there exists $\lambda \not\equiv 1 \pmod q$, *such that* $A^p = \lambda I$ (*where* I *is the identity matrix*) *and* $\lambda^p \equiv 1 \pmod q$.

In the above (2)-(7), p *and* q *are distinct primes and* $p < q$.
(8) $G = (C_q \times C_r) \rtimes C_{p^m}$, *and* $Z(G) = \Phi(C_{p^m}) \times C_r$.
(9) $G = (C_r \rtimes C_{q^m}) \rtimes C_p$, *and* $Z(G) = \Phi(C_{q^m})$ *and* C_p *acts trivially on* C_r.
(10) $G = C_r \rtimes (C_q \times C_p)$, *and* $Z(G) = 1$.

We shall use the established terminology and notation in [2] and [4]. For example, $A \rtimes P$ denotes the semidirect product of A and P; C_n denotes a cyclic group of order n and $\pi(G)$ denotes the set of all prime divisors of $|G|$.

2. Some preliminaries

In this section, we collect some lemmas which will be used in the following.

Lemma 2.1 ([4, 7.2.2]). *Suppose that the Sylow* p*-subgroups of* G *are cyclic, where* p *is the smallest prime divisor of* $|G|$. *Then* G *has a normal* p*-complement.*

Lemma 2.2 ([5]). *Suppose that* p'*-group* H *acts on a* p*-group* G. *Let*

$$\Omega(G) = \begin{cases} \Omega_1(G) & p > 2, \\ \Omega_2(G) & p = 2. \end{cases}$$

If H *acts trivially on* $\Omega(G)$, *then* H *acts trivially on* G *as well.*

Lemma 2.3 (Maschke's Theorem, [4, 8.4.6]). *Suppose that the action of* A *on an abelian group* G *is coprime and* H *is an* A*-invariant direct factor of* G. *Then* H *has an* A*-invariant complement in* G.

Lemma 2.4 ([8]). *If* G *is a minimal non-abelian simple group, then* G *is isomorphic to one of the following simple groups*:

(1) *$PSL(2,p)$, where p is a prime with $p > 3$ and $5 \nmid p^2 - 1$.*
(2) *$PSL(2,2^q)$, where q is a prime.*
(3) *$PSL(2,3^q)$, where q is a prime.*
(4) *$PSL(3,3)$.*
(5) *The Suzuki group $Sz(2^q)$, where q is an odd prime.*

In proving our main theorem, the following result will be frequently used.

Lemma 2.5 ([6, Theorem 7]). *Let G be a finite group. Then all subgroups of G are either s-quasinormal or self-normalizing if and only if either*

(1) *G is nilpotent, or*

(2) *$G = H \rtimes P$, where H is an abelian normal Hall p'-subgroup and $P = \langle x \rangle \in Syl_p(G)$, $\langle x^p \rangle = O_p(G) = Z(G)$, where p is the minimal prime dividing the order of G. Furthermore, x induces a fixed-point-free power automorphism of order p on H.*

Let G be a group. By the proof of [6, Theorem 7], we know that the following statements are equivalent:

1. G is an $\mathcal{SQNS}$-group.
2. Every subgroup of prime power order of G is either s-quasinormal in G or self-normalizing.

We will use above fact freely in our following proof.

3. Minimal non-$\mathcal{SQNS}$-groups

In this section, we classify finite minimal non-$\mathcal{SQNS}$-groups.

Lemma 3.1. *Let G be a minimal non-$\mathcal{SQNS}$-group. Then G is solvable.*

Proof. Suppose that G is not solvable. By Lemma 2.5, every proper subgroup of G is solvable and hence $G/\Phi(G)$ is a minimal simple group. Let H be the 2-complement of $\Phi(G)$. Then $H \trianglelefteq G$ and H is nilpotent since H is an $\mathcal{SQNS}$-group. We have following claims.

(1) $H = 1$.

Suppose that $H \neq 1$. Let $P \in Syl_p(H)$, where p is any prime in $\pi(H)$. Then $P \trianglelefteq G$. Let $S_2 \in Syl_2(G)$ and $K = S_2P$. Then K is a proper subgroup of G, and hence K is an $\mathcal{SQNS}$-group by hypothesis. If K is an $\mathcal{SQNS}$-group as in (2) of Lemma 2.5, then S_2 is cyclic, which concludes that G has normal 2-complement, a contradiction. Hence we may assume that K is nilpotent. But it follows in this case that $S_2 \leq C_G(P) \trianglelefteq G$. Using the simplicity of $G/\Phi(G)$, we conclude that $S_2 \leq C_G(P)\Phi(G)$, which concludes that G is solvable, a contradiction.

(2) Every subgroup of order $2^m p$ (p an odd prime) of $\overline{G} = G/\Phi(G)$ is 2-nilpotent.

Assume that G possesses a subgroup L containing $S_0 = \Phi(G)$ such that L/S_0 is not a 2-nilpotent group of order $2^m p$. Then L contains a minimal non-2-nilpotent subgroup D with order $2^n p$ for some natural number n. Hence

$D = S^*P$ is a minimal non-nilpotent group with a normal Sylow 2-subgroup S^* and $|P| = p$. Since G is non-solvable, D is a proper subgroup of G and so D is an $\mathcal{SQNS}$-group by the hypothesis. Hence D is nilpotent by Lemma 2.5, a contradiction.

(3) Conclusion.

Now, we assert that there is no simple group listed in Lemma 2.4 isomorphic to $\overline{G}$. And then we get that G is solvable. In fact, if $\overline{G}$ is isomorphic to one of $PSL(2,p), PSL(2,3^q)$ and $PSL(3,3)$, then $\overline{G}$ has a subgroup isomorphic to A_4, the alternating group of degree 4, a contradiction to (2). If $\overline{G} \cong PSL(2,2^q)$ or $Sz(2^q)$, then $\overline{G}$ is a Zassenhaus group of odd degree and the stabilizer of a point is a Frobenius group with kernel a 2-group, again a contradiction to (2). Hence $\overline{G}$ cannot be any one of $PSL(2,2^q)$ and $Sz(2^q)$ as well. Thus the proof is completed. □

Lemma 3.2. *Let G be a minimal non-$\mathcal{SQNS}$-group. Then $|\pi(G)| \leq 3$.*

Proof. Suppose that $|\pi(G)| > 3$. Let $\{P_1, P_2, \ldots, P_k, \ldots, P_r\}$, $r > 3$ be a Sylow basis of G, where $P_i \in Syl_{p_i}(G)$, $i = 1, 2, \ldots, r$. Since G is not an $\mathcal{SQNS}$-group, there exists a subgroup $P_i^* \leq P_i$ (for some i) such that P_i^* is neither s-quasinormal nor self-normalizing by Lemma 2.5. By hypothesis P_iP_j is an $\mathcal{SQNS}$-group for each $i \neq j \in \{1, 2, \ldots, r\}$. If P_i is non-cyclic, then by Lemma 2.5, P_i is normal in P_iP_j, which implies that P_i is normal in G and hence P_i^* is a proper subgroup of P_i. Choose a $P_k \in \{P_1, P_2, \ldots, P_k, \ldots, P_r\}$ such that $P_i^*P_k \neq P_kP_i^*$, where $k \neq i, j$. Let $H = P_iP_jP_k$. Then H is an $\mathcal{SQNS}$-group. However, P_i^* is neither s-quasinormal nor self-normalizing in H, a contradiction. Hence we may assume that P_i is cyclic. If $P_i^* \neq P_i$, then by the structure of $\mathcal{SQNS}$-groups, P_i^* is normal in P_iP_j for each $j \neq i$, which implies that P_i^* is normal in G, a contradiction. If $P_i^* = P_i$, then there exists $j \neq k \in \{1, 2, \ldots, r\}$ such that both $P_iP_j = P_j \rtimes P_i$ and $P_iP_k = P_i \times P_k$(or $P_iP_k = P_i \rtimes P_k$) hold. Now $K = P_iP_jP_k$ is a proper subgroup of G and hence an $\mathcal{SQNS}$-group since we assume that $|\pi(G)| > 3$. However, P_i is obvious neither s-quasinormal nor self-normalizing in K, a contradiction. This contradiction shows that $|\pi(G)| \leq 3$. The proof is completed. □

The following theorem classifies all minimal non-$\mathcal{SQNS}$-groups whose order having two prime divisors.

Theorem 3.3. *Let G be a minimal non-$\mathcal{SQNS}$-group with $|\pi(G)| = 2$. Then one of the following holds*:

(1) *$G = P \rtimes Q$ is a minimal non-nilpotent group, and P is non-cyclic.*

(2) *$G = C_q \rtimes (C_{p^n} \times C_p)$, $\Phi(C_{p^n})C_p = Z(G)$.*

(3) *$G = C_q \rtimes Q_8$, Q_8 induces an automorphism of order 2 on C_q.*

(4) *$G = C_q \rtimes M_{p^{n+1}}$, $C_{M_{p^{n+1}}}(C_q) \cong M_{p^n}$.*

(5) *$G = C_{q^n} \rtimes C_{p^m}$, $m \geq 2$, $\Phi(\Phi(P)) = Z(G)$.*

(6) $G = \langle a, b, c \mid a^{q^m} = b^{q^m} = 1, c^{p^n} = 1,\ ab = ba,\ a^c = a^u,\ b^c = b^v,\ u \not\equiv v \pmod{q^m},\ u \equiv v \pmod{q^{m-1}},\ u^p \equiv v^p \equiv 1 \pmod{q^m}\rangle$. *Furthermore,* $u \not\equiv 1 \pmod q$ *and* $v \not\equiv 1 \pmod q$ *if* $m \geq 2$.

(7) $G = \langle x, y_1, y_2, \ldots, y_b \mid x^{p^a} = y_1^q = y_2^q = \cdots = y_b^q, y_iy_j = y_jy_i, i, j = 1, 2, \ldots, b, y_i^x = y_{i+1}, i = 1, 2, \ldots, b-1, y_b^x = y_1^{d_1}y_2^{d_2}\cdots y_b^{d_b}\rangle$, $f(z) = z^b - d_bz^{b-1} - \cdots - d_2z - d_1$ *is irreducible in* F_q. *Moreover, let*

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ d_1 & d_2 & d_3 & \cdots & d_b \end{bmatrix}.$$

Then there exists $\lambda \not\equiv 1 \pmod q$, *such that* $A^p = \lambda I$ (*where* I *is the identity matrix*) *and* $\lambda^p \equiv 1 \pmod q$.

In the above (2)-(7), p *and* q *are distinct primes and* $p < q$.

Proof. Let $G = PQ$, where $P \in Syl_p(G)$ and $Q \in Syl_q(G)$. Without loss of generality, we always assume that $p < q$. Since G is solvable, there is a normal maximal subgroup M of G such that $|G : M| = p$ or q. By our assumption, M is an $\mathcal{SQNS}$-group. Hence we can get that either P or Q must be normal in G.

Case 1. P is normal in G.

By Lemma 2.1, P is not a cyclic subgroup. Let Q_1 and Q_2 be two different maximal subgroups of Q. Then PQ_1 and PQ_2 are all $\mathcal{SQNS}$-groups by hypothesis. By Lemma 2.5, both PQ_1 and PQ_2 are nilpotent, which implies that G is nilpotent, a contradiction. Therefore we get Q is cyclic. Since $\Phi(Q)P$ is an $\mathcal{SQNS}$-group, we have that $\Phi(Q)P = \Phi(Q) \times P$. Thus $\Phi(Q) \leq Z(G)$. Again by Lemma 2.5, we know that Q acts non-trivially on P, but acts trivially on every Q-invariant proper subgroup of P. It is easy to see that, G is a minimal non-nilpotent-group in this case. That is, G is of type (1).

Case 2. Q is normal in G.

(2.1) Suppose in the first place that P is non-cyclic. By Lemma 2.5, we know that P acts non-trivially on Q, but acts trivially on every P-invariant proper subgroup of Q. Applying Hall-Higman-Reduction Theorem, we can get that $\exp(Q) = q$. Assume that $|Q| > q$. Let P_1 and P_2 be two maximal subgroups of P. Then both P_1Q and P_2Q are $\mathcal{SQNS}$-groups. Hence $P_i (i = 1, 2)$ induces a fixed-point-free power automorphism on Q, which implies that the action of P on Q is reducible. Hence $C_Q(P) > 1$. On the other hand, P_1Q or P_2Q must be non-nilpotent by Lemma 2.5. Thus we have that P has at most one non-cyclic maximal subgroup and either $C_Q(P_1) = 1$ or $C_Q(P_2) = 1$, which leads to $C_Q(P) = 1$, a contradiction. Thus we have $|Q| = q$. Since P has at most one non-cyclic maximal subgroup, P is isomorphic to the quaternion group Q_8 or $C_{p^n} \times C_p$ or $P \cong M_{p^{n+1}}$ (see [3, Lemma 2.9]).

If $P \cong C_{p^n} \times C_p$, then $G \cong C_q \rtimes (C_{p^n} \times C_p)$, $\Phi(C_{p^n})C_p = Z(G)$. That is, G is of type (2).

If $P \cong Q_8$, then $G \cong C_q \rtimes Q_8$, Q_8 induces an automorphism of order 2 on C_q. That is, G is of type (3).

If $P \cong M_{p^{n+1}}$, then $G \cong C_q \rtimes M_{p^{n+1}}$, $C_{M_{p^{n+1}}}(C_q) \cong M_{p^n}$. That is, G is of type (4).

(2.2) Suppose that P is cyclic and the action of P on $Q/\Phi(Q)$ is reducible. Let $P = \langle c\rangle$ and $Q = \langle x_1, x_2, \ldots, x_n\rangle$. Since the action of P on $Q/\Phi(Q)$ is reducible, we have that $Q/\Phi(Q)$ is P-completely reducible. If $n \geq 3$, we claim that $C_Q(P) = 1$. Suppose that $C_Q(P) > 1$. Let $x \in C_Q(P)$. Then $P\langle x, x_i\rangle$ is a nilpotent $\mathcal{SQNS}$-group by Lemma 2.5, where $i = 1, 2, \ldots, n$. It implies that PQ is nilpotent, a contradiction. Hence $C_Q(P) = 1$. Since the action of P on $\langle x_i, x_j\rangle$ is invariable, we have $x_ix_j = x_jx_i$ by Lemma 2.5. It implies that Q is abelian and hence $Q = \langle x_1\rangle \times \langle x_2\rangle \times \cdots \times \langle x_n\rangle$. Let $o(x_i) = q^{m_i}$ and let $q^{m_1} = exp(Q)$. Since $C_Q(P) = 1$, $P\langle x_1, x_i\rangle$ is a non-nilpotent $\mathcal{SQNS}$-group. Thus there exists a natural number w satisfying $x^c = x^w$ for each $x \in \langle x_1, x_i\rangle$. Then $w \equiv u_1 \pmod{q^{m_1}}$ and $w \equiv u_i \pmod{q^{m_i}}$ for some natural numbers u_1 and u_i. Thus we get $u_i \equiv u_1 \pmod{q^{m_i}}$. It follows that P induces a fixed-point-free power automorphism of order p on Q and hence G itself is an $\mathcal{SQNS}$-group, a contradiction. Therefore we have $n \leq 2$.

If Q is cyclic, then we have $|P| \geq p^2$ by the structure of $\mathcal{SQNS}$-groups. Hence $G = C_{q^n} \rtimes C_{p^m}$, $m \geq 2$, $\Phi(\Phi(P)) = Z(G)$. That is, G is of type (5).

If Q is non-cyclic and $\Omega_1(Q) = Q$, then Q is an elementary abelian q-group. In this case, $G = \langle a, b, c\rangle, a^q = b^q = c^{p^m} = 1, ab = ba, a^c = a^i, b^c = b^j, i \not\equiv j \pmod{q}, i^p \equiv j^p \equiv 1 \pmod{q}$. That is, G is of type (6).

If Q is non-cyclic and $\Omega_1(Q) \neq Q$, then we can get $C_Q(P) = 1$ by Lemma 2.2. Let $Q = \langle a, b\rangle$. Then $\langle a\Phi(Q)\rangle$ and $\langle b\Phi(Q)\rangle$ are P-invariant. Hence $\langle a\Phi(Q)\rangle$ and $\langle b\Phi(Q)\rangle$ are all abelian by Lemma 2.5, it follows that G is abelian or minimal non-abelian.

If Q is abelian, let $Q = \langle a, b\rangle, a^{q^m} = 1, b^{q^n} = 1$ and $P = \langle c\rangle, a^c = a^u, b^c = b^v$. We claim that $m = n$. Indeed, let $m < n$, and $Q_1 = \langle a, b^q\rangle$. Then $PQ_1 = Q_1 \rtimes P$ is an $\mathcal{SQNS}$-group. Hence there exists a natural number w such that $a^c = a^w, (b^q)^c = (b^p)^w$ by Lemma 2.5. It follows that $u \equiv w \pmod{q^m}$ and $qv \equiv qw \pmod{q^n}$. Thus $v \equiv w \pmod{q^{n-1}}$. Since $m < n$, we get $v \equiv w \pmod{q^m}$ and hence $u \equiv v \pmod{q^m}$. Therefore we have $x^c = x^v$ for every $x \in Q$, which implies that P induces a fixed-point-free power automorphism of order p on Q, a contradiction. Hence $m = n$ and then $G = \langle a, b, c\rangle$, $a^{q^m} = b^{q^m} = 1, c^{p^n} = 1$, $ab = ba$, $a^c = a^u$, $b^c = b^v$. $u \not\equiv v \pmod{q^m}$, $u \equiv v \pmod{q^{m-1}}$, $u^p \equiv v^p \equiv 1 \pmod{q^m}$, $u \not\equiv 1 \pmod{q}$, $v \not\equiv 1 \pmod{q}, m \geq 2$. That is, G is of type (6).

If Q is minimal non-abelian, then by [7], we have $Q = \langle a, b\rangle, a^{q^m} = b^{q^n} = 1, a^b = a^{1+q^{m-1}}, m \geq 2$. Let $P = \langle c\rangle, a^c = b^ia^j, b^c = b^ua^v$. Then $(a^q)^c = b^{qi}a^{qj}, (b^q)^c = b^{qu}a^{qv}$. On the other hand, $q^{n-1}|i$, $q^{m-1}|v$, and $(q, j) = 1$, $(q, u) = 1$. So $(b^ia^j)^{1+q^{m-1}} = b^{i+iq^{m-1}}a^{j+jq^{m-1}}$ and $(b^ia^j)^{b^ua^v} = b^ia^{j+juq^{m-1}}$. Since P induces an automorphism on Q, we get $b^{i+iq^{m-1}}a^{j+jq^{m-1}} = b^ia^{j+juq^{m-1}}$.

Thus $j + jq^{m-1} \equiv j + juq^{m-1} \pmod{q^m}$. Hence $u \equiv 1 \pmod q$, it follows that $(b^{q^{n-1}})^c = b^{q^{n-1}}$, contrary to $C_Q(P) = 1$.

(2.3) Suppose that P is cyclic and the action of P on $Q/\Phi(Q)$ is irreducible.

If Q is cyclic, then G is of type (5).

Suppose that Q is not cyclic. Since the action of P on $Q/\Phi(Q)$ is irreducible, all P-invariant subgroups of Q are contained in $\Phi(Q)$.

If $C_Q(P) \neq 1$, then $C_Q(P)$ is contained in $\Phi(Q)$. Hence $P \leq C_G(\Phi(Q))$, it implies that P acts non-trivially on Q, but acts trivially on every P-invariant subgroup of Q, hence $\exp(Q) = q$. On the other hand, $1 \neq C_Q(P) \leq \Phi(Q)$, so Q is non-abelian, furthermore, $Q \leq C_G(\Phi(P))$. It implies that G is a minimal non-nilpotent group. That is, G is of type (1).

If $C_Q(P) = 1$, we claim that $\Phi(Q) = 1$. Otherwise, $P\Phi(Q)$ is a non-nilpotent $\mathcal{SQNS}$-group. By Lemma 2.5 we have $p|q-1$. On the other hand, if Q is abelian, then $\Omega_1(Q) \neq Q$, hence P acts on $\Omega_1(Q)$ reducibly by Lemma 2.5, furthermore P acts reducibly on Q as well, a contradiction. So Q is non-abelian and $\Phi(P)$ acts trivially on Q. Now let $\overline{P} = P/\Phi(P)$, $\overline{Q} = Q/\Phi(Q)$. Then $\overline{Q} \rtimes \bar{P} = \langle x, y_1, y_2, \dots, y_b \mid x^p = y_1^q = y_2^q = \cdots = y_b^q, y_iy_j = y_jy_i, i, j = 1, 2, \dots, b, y_i^x = y_{i+1}, i = 1, 2, \dots, b-1, y_b^x = y_1^{d_1}y_2^{d_2}\cdots y_b^{d_b}\rangle$, where $f(z) = z^b - d_bz^{b-1} - \cdots - d_2z - d_1$ is irreducible in F_q and $f(z)|z^p - 1$ by [7]. Since $p|q-1$, $z^p - 1|z^{q-1} - 1$. However, $z^{q-1} - 1$ is completely decomposable in F_q, which implies that $f(z)$ is completely decomposable in F_q, a contradiction. Hence our claim holds and so Q is elementary abelian.

If $\Phi(P)$ acts trivially on Q, then G is a minimal non-nilpotent group. That is, G is of type (1).

Suppose that $\Phi(P)$ acts non-trivially on Q. Let $P = \langle x \mid x^{p^a} = 1\rangle$. Then Q is a F_qP-module. By choosing a suitable basis of Q, we have that the representation matrix of x is the following type:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ d_1 & d_2 & d_3 & \cdots & d_b \end{bmatrix}.$$

Thus we obtain that $G = \langle x, y_1, y_2, \dots, y_b \mid x^{p^a} = y_1^q = y_2^q = \cdots = y_b^q, y_iy_j = y_jy_i, i, j = 1, 2, \dots, b, y_i^x = y_{i+1}, i = 1, 2, \dots, b-1, y_b^x = y_1^{d_1}y_2^{d_2}\cdots y_b^{d_b}\rangle$. Since Q is a irreducible F_qP-module, the characteristic polynomial $f(z) = z^b - d_bz^{b-1} - \cdots - d_2z - d_1$ of A is irreducible in F_q. On the other hand, $\Phi(P)Q$ is not nilpotent. Hence $\langle x^p\rangle$ induces a fixed-point-free power automorphism of order p on Q and $\langle x^{p^2}\rangle$ acts trivially on Q by Lemma 2.5. Thus there is a λ such that $\lambda \not\equiv 1 \pmod q$, and $A^p = \lambda I$ (where I is the identity matrix) and $\lambda^p \equiv 1 \pmod q$. That is, G is of type (7).

The proof is completed. □

The following theorem classifies all minimal non-$\mathcal{SQNS}$-groups whose order having three prime divisors.

Theorem 3.4. *Let G be a minimal non-$\mathcal{SQNS}$-group with $|\pi(G)| = 3$. Then one of the following holds*:

(1) $G = (C_q \times C_r) \rtimes C_{p^m}$, *and* $Z(G) = \Phi(C_{p^m}) \times C_r$.

(2) $G = (C_r \rtimes C_{q^m}) \rtimes C_p$, *and* $Z(G) = \Phi(C_{q^m})$ *and* C_p *acts trivially on* C_r.

(3) $G = C_r \rtimes (C_q \times C_p)$, *and* $Z(G) = 1$.

Proof. Since G is solvable, we may assume that $G = PQR$, where $P \in Syl_p(G)$, $Q \in Syl_q(G)$, $R \in Syl_r(G)$. Without loss of generality, we always let p be the smallest prime divisor of $|G|$. If P is non-cyclic, then PQ and PR are all nilpotent by Lemma 2.5. Hence P is normal in G. Let P_1 be a maximal subgroup of P. Then P_1QR is an $\mathcal{SQNS}$-group and hence we get P_1QR is nilpotent by Lemma 2.5, which implies that G itself is nilpotent, a contradiction. Thus we have that P is cyclic. By Lemma 2.1, $QR \trianglelefteq G$. Since QR is an $\mathcal{SQNS}$-group, we have by Lemma 2.5 that RQ is either a nilpotent group or a group of the type (2) in Lemma 2.5.

Case 1. RQ is a nilpotent group.

In this case both Q and R is normal in G. If $PQ = Q \rtimes P$ and $PR = R \rtimes P$, then we have that P induces a fixed-point-free power automorphism of order p on Q and R and $N_G(P) = P$ by Lemma 2.5. Let $z \in R$ is an element of order r. If $\langle z\rangle PQ$ is an $\mathcal{SQNS}$-group, then $\langle z\rangle QP = (\langle z\rangle \times Q) \rtimes P$. Let $P = \langle x\rangle$ and yz be any element of $\langle z\rangle Q$, where $y \in Q$. Then $(yz)^x = (yz)^k$ for a positive integer k. On the other hand, we have $y^x = y^m$ and $z^x = z^n$. Thus $y^m z^n = (yz)^x = (yz)^k = y^m z^k$. Therefore $y^m = y^k$ and $z^n = z^k$, which implies that P induces a fixed-point-free power automorphism of order p on QR, a contradiction. Hence R is of prime order. By the same argument we have Q is of prime order too. Thus $G = (C_q \times C_r) \rtimes C_{p^m}$. However, G is obvious an $\mathcal{SQNS}$-group, a contradiction.

If $PQ = Q \rtimes P$ and $PR = R \times P$(or $PQ = Q \times P$ and $PR = R \rtimes P$), then $N_G(P) = PR$. Let $z \in R$ be an element of order r. If $\langle z\rangle PQ$ is an $\mathcal{SQNS}$-group, then $\langle z\rangle QP = (\langle z\rangle \times Q) \rtimes P$. But in this case we have $N_{\langle z\rangle QP}(P) = P\langle z\rangle > P$, a contradiction. Hence R is of prime order. By the same argument we have Q is of prime order too. Thus $G = (C_q \times C_r) \rtimes C_{p^m}$, and $Z(G) = \Phi(C_{p^m}) \times C_r$. That is, G is of type (1).

Case 2. RQ is a group of the type (2) in Lemma 2.5.

Without loss of generality, we assume that $q < r$. Then $R \not\leq N_G(Q)$. By the same reason as in Case 1, we can get that P and R are both of prime order. If P acts trivially on R, then we have P acts trivially on $\Phi(Q)$ since $P\Phi(Q)R$ is an $\mathcal{SQNS}$-group. Thus $G = (C_r \rtimes C_{q^m}) \rtimes C_p$, and $Z(G) = \Phi(C_{q^m})$. That is, G is of type (2).

If P acts non-trivially on R, then either $\Phi(Q) = 1$ or P acts non-trivially on $\Phi(Q)$. Thus $G = (C_r \rtimes C_{q^m}) \rtimes C_p$, and $Z(G) = 1$. Let $V = C_{q^m} \rtimes C_p$. Then $V/C_V(C_r) \leq \mathrm{Aut}(C_r)$ is a cyclic group. If P acts non-trivially on $\Phi(Q)$, then we can get a contradiction since $V/C_V(C_r)$ is not cyclic. If $\Phi(Q) = 1$, then we obtain that $C_qC_p = C_q \times C_p$. That is, G is of type (3).

Thus our proof is completed. □

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