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CONICS ON A GENERAL HYPERSURFACE IN COMPLEX PROJECTIVE SPACES

Dongsoo Shin

Abstract. In this paper we consider the existence of smooth conics on a general hypersurface of degree d in $\mathbb{P}^n$.

1. Introduction

Let X be a general hypersurface of degree d in a complex projective space $\mathbb{P}^n$. Recall that $R_e(X)$ is the open subscheme of the Hilbert scheme $\mathrm{Hilb}^{et+1}(X)$ which parameterizes smooth rational curves of degree e lying on X. Recently J. Harris, M. Roth, and J. Starr obtained the following general result for $d < \frac{n+1}{2}$:

Theorem (Harris–Roth–Starr [2]). *If $n > 2$ and $d < \frac{n+1}{2}$, then $R_e(X)$ is an integral, local complete intersection scheme of dimension $(n+1-d)e + (n-4)$ for every $e \geq 1$.*

However the upper bound of $d < \frac{n+1}{2}$ in the above theorem is not optimal for a certain degree e. For instance, we have a sharp bound for the degree d in case of $e = 1$:

Theorem (Barth–Van de Ven [1]). *If $d > 2n-3$, then $R_1(X)$ is empty. If $d \leq 2n-3$, then $R_1(X)$ is smooth of dimension $2n-3-d$.*

In this paper we will investigate the nonemptyness and smoothness of $R_2(X)$.

Theorem 1.1. *Let X be a general hypersurface of degree d in $\mathbb{P}^n$.*

(a) *If $2d > 3n-2$, then $R_2(X)$ is empty.*
(b) *If $2d \leq 3n-2$, then $R_2(X)$ is smooth and of dimension $3n-2d-2$.*

2. Proof

This section is devoted to the proof of Theorem 1.1.

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Let $\mathbb{G}(2,n)$ be the Grassmannian of 2-dimensional subspaces in $\mathbb{P}^n$ and $U \to \mathbb{G}(2,n)$ the universal bundle. Since every conic is contained in a unique $\mathbb{P}^2$ in $\mathbb{P}^n$, it follows that

$$\mathrm{Hilb}^{2t+1}(\mathbb{P}^n) \cong \mathbb{P}(\mathrm{sym}^2 U^*).$$

Hence $R_2(\mathbb{P}^n)$ is open in $\mathbb{P}(\mathrm{sym}^2 U^*)$. Let H_d be the parameter space of hypersurfaces of degree d in $\mathbb{P}^n$, i.e., $H_d = \mathbb{P}^N$, $N = \binom{n+d}{d} - 1$. Define the incidence scheme

$$J_d = \{(C,F) \in R_2(\mathbb{P}^n) \times H_d : C \subset F\}$$

and consider two projections $p_R : J_d \to R_2(\mathbb{P}^n)$ and $p_H : J_d \to H_d$. Then we have

$$R_2(X) \cong p_H^{-1}(X).$$

To prove Theorem 1.1, we modify the proof in Kollár [5, Theorem 4.3] which considers lines on a general hypersurface. The idea is a comparison of the dimensions of J_d and the parameter space H_d. A similar technique is used in Katz [4] which shows that there are 609250 conics on a general quintic threefold.

We begin with the following lemma.

Lemma 2.1. *The incidence scheme J_d is irreducible, nonsingular, and of codimension $2d+1$ in $R_2(\mathbb{P}^n) \times H_d$.*

Proof. Since we have

$$\mathrm{Hilb}^{2t+1}(\mathbb{P}^n) \cong \mathbb{P}(\mathrm{sym}^2 U^*),$$

it follows that $\mathrm{Hilb}^{2t+1}(\mathbb{P}^n)$ is irreducible because $\mathbb{P}(\mathrm{sym}^2 U^*)$ is a vector bundle on the irreducible variety $\mathbb{G}(2,n)$. Therefore $R_2(\mathbb{P}^n)$ is irreducible because $R_2(\mathbb{P}^n)$ is open in $\mathbb{P}(\mathrm{sym}^2 U^*)$.

Note that

$$p_R^{-1}(C) = \ker(\alpha : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(C, \mathcal{O}_C(d))).$$

Since C is 2-regular, it follows that α is surjective. Furthermore we have

$$h^0(C, \mathcal{O}_C(d)) = 2d + 1 + h^1(C, \mathcal{O}_C(d)) = 2d+1.$$

Therefore we have

$$\mathrm{codim}(p_R^{-1}(C), C \times H_d) = 2d+1$$

for all $C \in R_2(\mathbb{P}^n)$. Thus J_d is irreducible and

$$\mathrm{codim}(J_d, R_2(\mathbb{P}^n) \times H_d) = 2d+1.$$

Further $p_R : J_d \to R_2(\mathbb{P}^n)$ is smooth by Hartshorne [3, III, Ex.10.2]. Therefore J_d is nonsingular. □

Assume that $2d \leq 3n-2$. Let $C \subset \mathbb{P}^n$ be a conic. One can choose coordinates (x_i) on $\mathbb{P}^n$ such that

$$C \subset \mathbb{P}^2 = \langle x_3 = x_4 = \cdots = x_n = 0 \rangle \subset \mathbb{P}^n$$

and hence

$$C = \langle Q(x_0, x_1, x_2) = 0, x_3 = x_4 = \cdots = x_n = 0\rangle,$$

where $Q(x_0, x_1, x_2)$ is a degree 2 polynomial of x_0, x_1, x_2. If a hypersurface $X \subset \mathbb{P}^n$ of degree d contains C, then the equation of X can be written as

$$Qf + \sum_{i=3}^{n} x_i f_i = 0,$$

where $\deg f = d-2$, $\deg f_i = d-1$. Here and afterward, f and f_i are considered as functions on C.

Lemma 2.2 (Notation as above). (a) *The hypersurface X is singular at $p \in C$ if and only if*

$$f(p) = f_3(p) = \cdots = f_n(p) = 0.$$

(b) *If X is smooth along the conic C, then the projection $p_H : J_d \to H_d$ is smooth at (C, X) if and only if*

$$H^0(C, \mathcal{O}_C(d)) = fH^0(C, \mathcal{O}_C(2)) + \sum_{i=3}^{n} f_i H^0(C, \mathcal{O}_C(1)).$$

Proof. For $i = 0, 1, 2$,

$$\begin{aligned} \left.\frac{\partial X}{\partial x_i}\right|_p &= \frac{\partial Q}{\partial x_i}(p)f(p) + Q(p)\frac{\partial f}{\partial x_i}(p) + x_3(p)\frac{\partial f_3}{\partial x_i}(p) + \cdots + x_n(p)\frac{\partial f_n}{\partial x_i}(p) \\ &= \frac{\partial Q}{\partial x_i}(p)f(p) = 0. \end{aligned}$$

However C is smooth at p. Hence not all $\frac{\partial Q}{\partial x_0}(p)$, $\frac{\partial Q}{\partial x_1}(p)$, $\frac{\partial Q}{\partial x_2}(p)$ are zero. Therefore $f(p) = 0$. For $i = 3, \ldots, n$,

$$\begin{aligned} \left.\frac{\partial X}{\partial x_i}\right|_p &= Q(p)\frac{\partial f}{\partial x_i}(p) + x_3\frac{\partial f_3}{\partial x_i}(p) + \cdots + f_i(p) + x_i\frac{\partial f_i}{\partial x_i}(p) + \cdots + x_n\frac{\partial f_n}{\partial x_i}(p) \\ &= f_i(p) = 0. \end{aligned}$$

Therefore we get the first result.

Assume that X is smooth along C. The projection p_H is smooth at (C, X) if and only if $\ker dp_H(C, X)$ has the expected dimension at (C, X), i.e., dp_H is surjective at (C, X). However $R_2(X) \cong p_H^{-1}(X)$. Hence $\ker dp_H(C, X)$ is the Zariski tangent space to $R_2(X)$ at C, i.e.,

$$\ker dp_H(C, X) \cong H^0(C, \mathcal{N}_{C/X}).$$

Hence p_H is smooth at (C, X) if and only if

$$h^0(C, \mathcal{N}_{C/X}) = \dim J_d - \dim H_d = 3n - 2d - 2.$$

Consider the exact sequence:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{N}_{C/X} & \longrightarrow & \mathcal{N}_{C/\mathbb{P}^n} & \longrightarrow & \mathcal{N}_{X/\mathbb{P}^n}\big|_C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & \mathcal{N}_{C/X} & \longrightarrow & \mathcal{O}_C(2) \bigoplus \mathcal{O}_C(1)^{\oplus(n-2)} & \longrightarrow & \mathcal{O}_C(d) & \longrightarrow & 0
\end{array}$$

It follows that

$$h^0(C, \mathcal{N}_{C/X}) = 3n - 2d - 2 + h^1(C, \mathcal{N}_{C/X}).$$

Therefore p_H is smooth at (C, X) if and only if $H^1(C, \mathcal{N}_{C/X}) = 0$, which means the restriction map

$$H^0(C, \mathcal{O}_C(2) \oplus \mathcal{O}_C(1)^{\oplus(n-2)}) \to H^0(C, \mathcal{O}_C(d))$$

is surjective. Therefore the projection $p_H : J_d \to H_d$ is smooth at (C, X) if and only if

$$H^0(C, \mathcal{O}_C(d)) = fH^0(C, \mathcal{O}_C(2)) + \sum_{i=3}^{n} f_i H^0(C, \mathcal{O}_C(1)). \qquad \square$$

Fix an isomorphism $\varphi : C \to \mathbb{P}^1$. Then we have

$$H^0(C, \mathcal{O}_C(k)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2k)).$$

Let

$$\begin{aligned}
m_1 &: H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) \times H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2d-2)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2d)), \\
m_2 &: H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4)) \times H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2d-4)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2d))
\end{aligned}$$

be the multiplication maps. If $V \subset H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2d))$ is a subspace, then set

$$\begin{aligned}
m_1^{-1}(V) &= \{g \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2d-2)) : m(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) \times \{g\}) \subset V\}, \\
m_2^{-1}(V) &= \{g \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2d-4)) : m(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4)) \times \{g\}) \subset V\}.
\end{aligned}$$

Lemma 2.3. *Let $V \subset H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2d))$ be a hyperplane. Then either*

(a) *$V = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2d)(-p))$ for some $p \in \mathbb{P}^1$ and*

$$m_i^{-1}(V) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2d-2i)(-p))$$

for each i; or

(b) *there is no such p, and $m_1^{-1}(V) \subset H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2d-2))$ has codimension 3 and $m_2^{-1}(V) \subset H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2d-4))$ has codimension 5.*

Proof. We identify $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k))$ with the vector space of polynomials of degree k. Let $\sum_{i=0}^{2d} u_i x^i$ denote a general polynomial of degree $2d$ and $\sum_{i=0}^{2d-2} v_i x^i$

a general polynomial of degree $2d-2$. If V is given by a linear equation $\sum_{i=0}^{2d} c_i u_i = 0$, then $m_1^{-1}(V)$ is given by the three equations

$$\sum_{i=0}^{2d-2} c_i v_i = 0, \quad \sum_{i=0}^{2d-2} c_{i+1} v_i = 0, \quad \sum_{i=0}^{2d-2} c_{i+2} v_i = 0.$$

If these three equations are linearly dependent, then there is a point $p=(s,t) \in \mathbb{P}^1$ such that $sc_i = tc_{i+1}$ for $0 \le i \le 2d-2$. Equivalently,

$$(c_0, \ldots, c_d) = \text{constant} \cdot (t^d, t^{d-1}s, \ldots, s^d).$$

This means that $V = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2d)(-p))$ for some $p \in \mathbb{P}^1$. Otherwise $m_1^{-1}(V)$ has codimension 3. With similar method we can prove the other statement for $m_2^{-1}(V)$. □

Proposition 2.4. *Let $J_d^0 \subset J_d$ be the open subset consisting of those pairs (C,X) such that X is smooth along C. Let $Z^0 \subset J_d^0$ be the closed subset of those pairs (C,X) such that p_H is not smooth at (C,X). Then we have*

$$\operatorname{codim}(Z^0, J_d^0) \ge 3n - 2d - 1.$$

Proof. It is enough to show that

$$\operatorname{codim}(Z^0 \cap p_R^{-1}(C), p_R^{-1}(C)) \ge 3n - 2d - 1.$$

Note that

$$J_d^0 \cap p_R^{-1}(C) \cong \{(f, f_3, \ldots, f_n) : f, f_3, \cdots, f_n \text{ have no common zero}\},$$

$$Z^0 \cap p_R^{-1}(C) \cong \{(f, f_3, \ldots, f_n) : \ fH^0(C, \mathcal{O}(2)) + \sum_{i=3}^{n} f_i H^0(C, \mathcal{O}(1)) \subsetneq H^0(C, \mathcal{O}(d))\}$$

Let V be a hyperplane contained in $H^0(C, \mathcal{O}_C(d))$. Define

$$Z_V^0 = \{(f, f_3, \ldots, f_n) : fH^0(C, \mathcal{O}_C(2)) + \sum_{i=3}^{n} f_i H^0(C, \mathcal{O}(1)) \subset V\}$$

in $H^0(C, \mathcal{O}_C(d-2)) \oplus H^0(C, \mathcal{O}_C(d-1))^{\oplus(n-2)}$. Then we have

$$Z^0 \cap p_R^{-1}(C) = \bigcup_V Z_V^0.$$

Since $\dim \mathbb{P}H^0(C, \mathcal{O}(d)) = 2d$, it follows that

$$\dim(Z^0 \cap p_R^{-1}(C)) - \dim Z_V^0 = 2d.$$

However we have

$$\dim(J_d^0 \cap p_R^{-1}(C)) = \dim p_R^{-1}(C);$$

hence it is enough to show that

$$\operatorname{codim}(Z_V^0, J_d^0 \cap p_R^{-1}(C)) \ge 3n - 1.$$

Note that

$$\begin{aligned} Z_V^0 &= \{f \in H^0(C, \mathcal{O}_C(d-2)) : fH^0(C, \mathcal{O}_C(2)) \subset V\} \\ &\bigoplus \bigoplus_{i=3}^{n} \{f_i \in H^0(C, \mathcal{O}_C(d-1)) : f_i H^0(C, \mathcal{O}_C(1)) \subset V\} \\ &\cong m_2^{-1}(V) \oplus m_1^{-1}(V)^{\oplus(n-2)}. \end{aligned}$$

Therefore it follows by Lemma 2.3 that

$$\operatorname{codim}(Z_V^0, J_d^0 \cap p_R^{-1}(C)) \geq 5 + 3(n-2) = 3n - 1. \qquad \square$$

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. If $2d > 3n - 2$, then we have

$$\dim J_d < \dim H_d$$

by Lemma 2.1; hence it follows that

$$\operatorname{codim}(p_H(J_d), H_d) \geq 1.$$

Therefore $R_2(X)$ is empty for general X.

Suppose that $2d \leq 3n - 2$. Then it is not difficult to show that the map $p_H : J_d \to H_d$ is dominant by counting parameters: A conic C in $\mathbb{P}^n$ is given by $n + 1$ homogeneous forms $\alpha_i(u, v)(i = 0, \ldots, n)$ of degree 2 in two variables. Taking into account the ambiguity arising from the $\mathbb{P}GL_2$ action of $\mathbb{P}^1$, a conic C depends on $3n - 1$ parameters. If a conic C is contained in a hypersurface X in $\mathbb{P}^n$ defined by a homogeneous polynomial $h(x_0, \ldots, x_n)$ of degree d, then

$$h(\alpha_0(u, v), \ldots, \alpha_n(u, v)) = 0.$$

The left hand side of the above equation is a polynomial of degree $2d$ in u and v. Its $2d + 1$ coefficients are constantly zero. Therefore if X is general, then these equations may impose independent conditions to the $3n - 1$ parameters of the conic C. Hence the dimension of the solutions is $3n - 2d - 2$. That is, a general hypersurface of degree d in $\mathbb{P}^n$ contains a smooth conic. Therefore the map p_H is dominant as asserted.

Let $H_d^0 \subset H_d$ be the open subset parameterizing smooth hypersurfaces. If $X \in H^0$, then $p_H^{-1}(X)$ is smooth if and only if $x \notin p_H(Z^0)$. By Proposition 2.4 the codimension of Z^0 is greater than the generic fiber dimension of p_H. This shows that $R_2(X)$ is smooth for general X. $\square$

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Department of Mathematics
Chungnam National University
Daejeon 305-764, Korea
and
School of Mathematics
Korea Institute for Advanced Study
Seoul 130-722, Korea
E-mail address: dsshin@cnu.ac.kr