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# ON THE SIGNED TOTAL DOMINATION NUMBER OF GENERALIZED PETERSEN GRAPHS $P(n, 2)$

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Abstract. Let $G = (V, E)$ be a graph. A function $f : V \to \{-1, +1\}$ defined on the vertices of $G$ is a signed total dominating function if the sum of its function values over any open neighborhood is at least one. The signed total domination number of $G$, $\gamma_t^s(G)$, is the minimum weight of a signed total dominating function of $G$. In this paper, we study the signed total domination number of generalized Petersen graphs $P(n, 2)$ and prove that for any integer $n \geq 6$, $\gamma_t^s(P(n, 2)) = 2\lfloor \frac{n}{3} \rfloor + 2t$, where $t \equiv n (\bmod\ 3)$ and $0 \leq t \leq 2$.

## 1. Introduction

For notation and graph theory terminology we in general follow [5]. Specially, let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $v$ be a vertex in $V(G)$. The *open neighborhood* of $v$ is $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$. If the graph $G$ is clear from the text, we simply write $V$, $E$ and $N(v)$ rather than $V(G)$, $E(G)$ and $N_G(v)$. If $X \subseteq V$, then $\langle X \rangle$ is the subgraph induced by $X$. The distance $d(x, y)$ between two vertices $x$ and $y$ in $G$ is the length of the shortest path from $x$ to $y$. For a real-valued function $f : V \to R$, the weight of $f$ is $w(f) = \sum_{v \in V} f(v)$, and for $X \subseteq V$ we define $f(X) = \sum_{v \in X} f(v)$.

Let $G = (V, E)$ be a graph without isolated vertices. A set $S \subseteq V$ is a *total dominating set* if every vertex in $V$ is adjacent to a vertex in $S$. The *total domination number* of $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a total domination set. Total domination in graph theory is well studied in graph theory. The literature on this subject has been surveyed and detailed in the two books by Haynes et al. ([4], [5]).

A *signed total dominating function*, abbreviated as STDF, of a graph $G$ is defined in [11] as a function $f : V \to \{-1, +1\}$ such that $f(N(v)) \geq 1$ for every $v \in V$. The *signed total domination number* of $G$, denoted by $\gamma_t^s(G)$, is the minimum cardinality of a weight of STDF for $G$. A STDF $f$ of cardinality $\gamma_t^s(G)$ we call a $\gamma_t^s$-function of $G$.

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The *generalized Petersen graph* $P(n,k)$ is the graph with vertex set $V = U \cup W$, where $U = \{u_i \mid 0 \leq i \leq n-1\}$ and $W = \{w_i \mid 0 \leq i \leq n-1\}$, and edge set $E = \{u_iu_{i+1}, u_iw_i, w_iw_{i+k} \mid 0 \leq i \leq n-1,$ subscripts modulo $n\}$.

In recent years, there have been many results on generalized Petersen graph and related to domination parameters [1, 2, 3]. The signed total domination number of a graph $G$ was introduced by Zelinka [11]. In [6], Henning gave a lower bounds on a signed total domination number of general graphs, an upper bound on the signed upper total domination number of a regular graph, an upper and lower bounds for trees and characterized the extremal trees achieving these bounds. In [8], Shan and Cheng showed a lower bound on the signed total domination number for a k-partite graph and gave a lower bounds on the signed total domination for triangle-free graphs and characterized the extremal graphs achieving these bounds. In [10], Xing et al. gave some lower bounds on signed total domination number and showed the relationship between the signed total domination and total domination in graphs. In [7], Li, Xing and Sohn computed the exact value of the signed domination number of generalized Petersen graph $P(n,2)$. In this paper, we will study the signed total domination number of generalized Petersen graph $P(n,2)$. The following is our main theorem.

**Main Theorem.** *For any positive integer $n \geq 6$, $\gamma_t^s(P(n,2)) = 2\lfloor \frac{n}{3} \rfloor + 2t$, where $t \equiv n \pmod 3$ and $0 \leq t \leq 2$.*

## 2. Proof of the main theorem

Let $f$ be a STDF on $G$. Let $M_f = \{v \in V \mid f(v) = -1\}$ and $P_f = \{v \in V \mid f(v) = +1\}$. For our convenience, we denote $M_f$ and $P_f$ by $M$ and $P$, respectively.

**Lemma 1.** *If $n = 3k (k \geq 1)$, then $\gamma_t^s(P(n,2)) = 2k$.*

*Proof.* Let $f$ be a $\gamma_t^s$-function of $P(n,2)$. Then $\sum_{v \in V} f(N(v)) \geq 2n$. Since $P(n,2)$ is a 3-regular graph, we have $\sum_{v \in V} f(N(v)) = 3w(f)$. Therefore $\gamma_t^s(P(n,2)) = w(f) \geq \frac{2n}{3} = 2k$.

On the other hand, consider the function $g : V(P(n,2)) \to \{-1,+1\}$ such that

$$g(x) = \begin{cases} -1, & x \in \{w_{3i}, u_{3i} \mid 0 \leq i \leq k-1\}; \\ +1, & \text{otherwise}. \end{cases}$$

Then $g$ is a STDF of $P(n,2)$ and $w(g) = 2k$. Thus $\gamma_t^s(P(n,2)) \leq 2k$, which implies that $\gamma_t^s(P(n,2)) = 2k$. □

**Lemma 2.** *Let $f$ be a STDF on $P(n,2)$ with $n \geq 5$. Then $|N(v) \cap M| \leq 1$ for each $v \in V$.*

*Proof.* Assume that there exists a vertex $u \in V$ with $|N(u) \cap M| \geq 2$. Since $|N(u)| = 3$, we have that $f(N(u)) \leq -1$, a contradiction. □

**Corollary 1.** *Let $f$ be a STDF on $P(n,2)$ with $n \geq 5$. Then $|N(v) \cap M| \leq 1$ for any $v \in M$.*

Corollary 1 implies that $\langle M \rangle$ consists of isolated vertices or paths of order 2.

**Lemma 3.** *Let $f$ be a STDF on $P(n,2)$ with $n \geq 5$. If there exists a vertex $u \in M$ with $N(u) \cap M = \emptyset$, then $f(w) = +1$ for every $w \in \{v \in V | \, d(v,u) = 1 \text{ or } 2\}$.*

*Proof.* Let $w \in \{v \in V \mid d(v,u) = 1 \text{ or } 2\}$. If $d(w,u) = 1$, then obviously $f(w) = +1$. If $d(w,u) = 2$ and $f(w) = -1$, then let $x$ be the vertex adjacent to $u$ and $w$. Since $|N(x)| = 3$, we have that $f(N(x)) \leq f(w) + f(u) + 1 = -1$, a contradiction. □

**Lemma 4.** *Assume that $f$ be a STDF on $P(n,2)$ for $n \geq 5$. Let $u \in M$. If $N(u) \cap M = \emptyset$, then $f(N(u)) = 3$ and there exists a vertex $q \in V$ $(q \neq u)$ such that $f(N(q)) = 3$.*

*Proof.* Since $|N(u)| = 3$, by Lemma 3, we have that $f(N(u)) = 3$. In the following, we will show that there exists another vertex $q \in V$, $q \neq u$, such that $f(N(q)) = 3$.

**Case 1** $u \in U$.

By Lemma 3, $f(w) = +1$ for every $w \in \{v \in V \mid d(v,u) = 1 \text{ or } 2\}$, which is shown in Fig. 1(a). If $f(v) = +1$, then $f(N(x)) = 3$. If $f(v) = -1$, then by Lemma 2, $|N(y) \cap M| \leq 1$. Thus $f(z) = +1$, which implies that $f(N(s)) = 3$.

**Case 2** $u \in W$.

By Lemma 3, $f(w) = +1$ for every $w \in \{v \in V \mid d(v,u) = 1 \text{ or } 2\}$, which is shown in Fig. 1(b). If $f(v) = +1$, then $f(N(x)) = 3$. If $f(v) = -1$, then by Lemma 2, $|N(y) \cap M| \leq 1$. Therefore $f(z) = +1$, which implies that $f(N(s)) = 3$.

From above, the assertion follows. □

**Lemma 5.** *Let $f$ be a STDF on $P(n,2)$. If $n = 3k + t(k \geq 2, 1 \leq t \leq 2)$, then $|M_f| \leq 2k$.*

*Proof.* Let $f$ be a STDF on $P(n,2)$ and $n = 3k + t(k \geq 2, 1 \leq t \leq 2)$. We assume that $|M_f| \geq 2k + l$, where $l \in \mathbb{Z}^+$. Consider the generalized Petersen graph $P(3n,2)$ constructed as follows: it consists of three copies of $P(n,2)$, which are denoted by $P^j(n,2)(0 \leq j \leq 2)$ and we delete the edges in $\{u^j_{n-1}u^j_0, w^j_{n-2}w^j_0, w^j_{n-1}w^j_1 \mid 0 \leq j \leq 2\}$ and add the edges in $\{u^j_{n-1}u^{j+1}_0, w^j_{n-2}w^{j+1}_0, w^j_{n-1}w^{j+1}_1 \mid 0 \leq j \leq 2$, superscripts module 3$\}$. Consider the mapping $g : V(P(3n,2)) \to \{-1,+1\}$ such that $g(w^j_i) = f(w_i)$ and $g(u^j_i) = f(u_i)$ for $0 \leq j \leq 2$ and $0 \leq i \leq n-1$. Then $g$ is a STDF on $P(3n,2)$ and

$$w(g) = 3w(f) = 3(2n - 2|M_f|) \leq 6k + 6t - 6l.$$

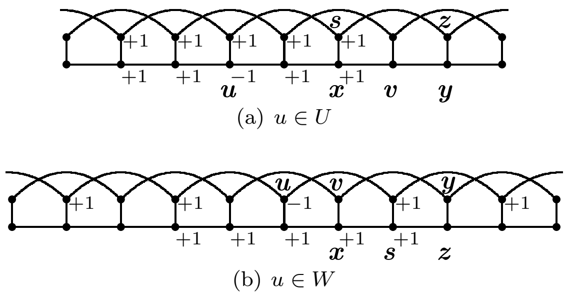


FIGURE 1. The graph for the proof of Lemma 4.

On the other hand, by Lemma 1, we have that $\gamma_t^s(P(3n,2)) = 2n = 6k+2t$. If $t=1$ or $t=2$ and $l \geq 2$, then $w(g) < \gamma_t^s(P(3n,2))$, which contradicts the definition of the signed total domination number. Thus $t=2$ and $|M_f| = 2k+1$. By Corollary 1, there exists at least one isolated vertex in $\langle M \rangle$. Let $u \in M$ be a isolated vertex in $\langle M \rangle$. Then by Lemma 4, $f(N(u)) = 3$ and there exists a vertex $q$ ($q \neq u$) such that $f(N(q)) = 3$. Since $P(n,2)$ is a 3-regular graph, we have

$$\begin{aligned} 3w(f) &= \sum_{v \in V} f(N(v)) \\ &= f(N(u)) + f(N(q)) + \sum_{v \in V - \{u,q\}} f(N(v)) \\ &\geq 2n+4. \end{aligned}$$

Thus, $w(f) \geq \frac{2n+4}{3} = \frac{2(3k+2)+4}{3} = 2k + \frac{8}{3}$. On the other hand,

$$\begin{aligned} w(f) &= |P_f| - |M_f| \\ &= 2n - 2|M_f| \\ &= 2k+2 \\ &< 2k + \frac{8}{3}, \end{aligned}$$

a contradiction. Therefore, the result holds. □

**Lemma 6.** *If* $n = 3k+t (k \geq 2, 1 \leq t \leq 2)$, *then* $\gamma_t^s(P(n,2)) = 2k+2t$.

*Proof.* Let $f$ be a $\gamma_t^s$-function of $P(n,2)$. By Lemma 5, $\gamma_t^s(P(n,2)) = w(f) = 2n - 2|M_f| \geq 2(3k+t) - 2 \cdot 2k = 2k+2t$.

On the other hand, if $t=1$, consider the function $g : V \to \{-1,+1\}$ such that

$$g(x) = \begin{cases} -1, & x \in \{w_{3i}, u_{3i} \mid 0 \leq i \leq k-2\} \cup \{u_{n-4}, u_{n-3}\}; \\ +1, & \text{otherwise.} \end{cases}$$

Then $g$ is a STDF of $P(n,2)$ and $w(g) = 2n-2|M_g| = 2(3k+1)-2\cdot 2k = 2k+2$. Thus $\gamma_t^s(P(n,2)) \leq 2k+2$, which implies that $\gamma_t^s(P(n,2)) = 2k+2$.

If $t = 2$, consider the function $h : V \to \{-1,+1\}$ such that

$$h(x) = \begin{cases} -1, & x \in \{w_{3i}, u_{3i} \mid 0 \leq i \leq k-1\}; \\ +1, & \text{otherwise.} \end{cases}$$

Then $h$ is a STDF of $P(n,2)$ and $w(h) = 2n-2|M_h| = 2(3k+2)-2\cdot 2k = 2k+4$. Thus $\gamma_t^s(P(n,2)) \leq 2k+4$, which implies that $\gamma_t^s(P(n,2)) = 2k+4$. This completes the proof of Lemma 7. □

By Lemma 1 and Lemma 6, we have the following theorem.

**Theorem 1.** *For any integer $n \geq 6$, $\gamma_t^s(P(n,2)) = 2\lfloor \frac{n}{3} \rfloor + 2t$, where $t \equiv n (mod\ 3)$ and $0 \leq t \leq 2$.*

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