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SOME PROPERTIES OF GENERALIZED LOCAL HOMOLOGY AND COHOMOLOGY MODULES

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Abstract. We study some properties of representable generalized local homology modules. By duality, we get some properties of good generalized local cohomology modules.

1. Introduction

Let I be an ideal of a local Noetherian commutative ring R and M, N R-modules. In [8], [10] we defined the i-th generalized local homology module $H_i^I(M,N)$ of M, N with respect to I by

$$H_i^I(M,N) = \varprojlim_t \operatorname{Tor}_i^R(M/I^tM, N).$$

This definition is in some sense dual to J. Herzog's definition of generalized local cohomology modules [5] and in fact a generalization of the usual local homology modules

$$H_i^I(M) = \varprojlim_t \operatorname{Tor}_i^R(R/I^tR, M) \text{ ([3], [4]).}$$

In [7] Macdonald defined a non-zero R-module M to be secondary if its multiplication endomorphism by any element x of R is either surjective or nilpotent. It is immediate that the nil-radical of M is a prime ideal $\mathfrak{p}$ and M is call $\mathfrak{p}$-secondary. A secondary representation for an R-module M is an expression for M as a finite sum of secondary modules. If such a representation exists, we will say that M is representable. For the convenient, a zero module is considered as a representable module. If M has a reduced secondary representation $M = M_1 + M_2 + \cdots + M_n$ and N_i is $\mathfrak{p}_i$-secondary, we write $\operatorname{Att}(M) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$.

In [3, 4.7] we showed that for any artinian R-module M,

$$\inf\{i : H_i^I(M) \text{ is not artinian}\} = \inf\{i : I \not\subseteq \sqrt{\operatorname{Ann}(H_i^I(M))}\}.$$

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Then Rezaei [12] extended the above result and proved that

$$\inf\{i : H_i^I(M) \text{ is not representable}\} = \inf\{i : I \not\subseteq \sqrt{\mathrm{Ann}(H_i^I(M))}\}.$$

In this paper, we study representable generalized local homology modules $H_i^I(M,N)$ and get some general results. The first main result is Theorem 2.4 in which we show that if M is a non-zero representable I-separated R-module, then

$$I \subseteq \bigcap_{\mathfrak{p}\in \mathrm{Att}(M)} \mathfrak{p} = \sqrt{\mathrm{Ann}_R(M))}.$$

Next, Theorem 2.7 gives us the following equalities when M is finitely generated and N is artinian:

$$\begin{aligned}\inf\{i : I \not\subseteq \sqrt{\mathrm{Ann}(H_i^I(M,N))}\} &= \inf\{i : H_i^I(M,N) \text{ is not I-stable}\}\\ &= \inf\{i : H_i^I(M,N) \text{ is not artinian}\}\\ &= \inf\{i : H_i^I(M,N) \text{ is not representable}\}.\end{aligned}$$

In Theorem 2.8 we see that if

$$t = \sup\{i : H_i^I(M,N) \text{ is not representable}\} \geq pd(M),$$

then $H_t^I(M,N) \neq 0$.

By duality, we get some properties of good generalized local cohomology modules $H_I^i(M,N)$ (Theorems 3.2 and 3.3). Theorem 3.3 especially gives us a nice consequence: If M and N are finitely generated R-modules such that $\dim N = d$ and $\mathrm{ext}^+(M, H_I^d(N)) = pd(M) = r < \infty$, then $H_I^{r+d}(M,N)$ is not good (Corollary 3.4).

2. Representable generalized local homology modules

We first recall some basic properties of generalized local homology modules $H_i^I(M,N)$ that we shall use.

Lemma 2.1 ([8, 2.7]). *Let M be a finitely generated R-module and N an artinian R-module. If N is complete with respect to I-adic topology (i.e., $\Lambda_I(N) \cong N$), then there is an isomorphism for all $i \geq 0$,*

$$\mathrm{Tor}_i^R(M,N) \cong H_i^I(M,N).$$

Theorem 2.2 ([8, 2.12]). *Let M be a finitely generated R-module and N an artinian R-module. Let s be a positive integer. Then the following statements are equivalent*:

(i) *$H_i^I(M,N)$ is artinian for all $i < s$;*
(ii) *$I \subseteq \sqrt{\mathrm{Ann}_R(H_i^I(M,N))}$ for all $i < s$.*

Let $pd(M)$ be the projective dimension of M and $\mathrm{Ndim}\, N$ the Noetherian dimension of N, we have a vanishing theorem for generalized local homology modules.

Theorem 2.3 ([9, 3.11])**.** *Let M be a finitely generated R-module with $pd(M) < \infty$ and N an artinian R-module. Then*

$$H_i^I(M,N) = 0$$

for all $i > pd(M) + \operatorname{Ndim} N$.

Let I be an ideal of R, an R-module M is called I-separated if $\underset{t>0}{\cap} I^t M = 0$. Note that the generalized local homology modules $H_i^I(M,N)$ are I-separated ([8, 2.3 (i)]). We have the following properties of non-zero representable I-separated R-modules.

Theorem 2.4. *Let M be a non-zero I-separated R-module. If M is representable, then*

$$I \subseteq \bigcap_{\mathfrak{p}\in \operatorname{Att}(M)} \mathfrak{p} = \sqrt{\operatorname{Ann}_R(M)}.$$

Proof. It follows from [2, 7.2.11] that

$$\bigcap_{\mathfrak{p}\in \operatorname{Att}(M)} \mathfrak{p} = \sqrt{\operatorname{Ann}_R(M)}.$$

We now assume that

$$M = T_1 + T_2 + \cdots + T_m$$

is a minimal secondary representation of M and T_j is $\mathfrak{p}_j$-secondary for $j = 1, 2, \ldots, m$. Then $\operatorname{Att}(M) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_m\}$.

If $I \not\subseteq \overset{m}{\underset{j=1}{\cap}} \mathfrak{p}_j$, then $I \not\subseteq \mathfrak{p}_s$ for some $s \in \{1, 2, \ldots, m\}$. Thus there exists $x \in I - \mathfrak{p}_s$. It follows

$$T_s = xT_s = \underset{t>0}{\cap} x^t T_s \subseteq \underset{t>0}{\cap} x^t M = 0$$

which is a contraction. The proof is complete. □

By [8, 2.3 (i)] the generalized local homology module $H_i^I(M,N)$ is I-separated, so we have the following immediate consequence.

Corollary 2.5. *Let M and N be R-modules and i an integer. If $H_i^I(M,N)$ is a representable R-module, then*

$$I \subseteq \sqrt{\operatorname{Ann}_R(H_i^I(M,N))}.$$

An R-module N is called I-stable if for each element $x \in I$, there is a positive integer n such that $x^t N = x^n N$ for all $t \geq n$. There are many I-stable modules. For example, for an R-module M the quotient module $N = M/IM$ is I-stable. Artinian modules especially are I-stable (see [10]). To prove Theorem 2.7 we need the following lemma.

Lemma 2.6. *Let M be an I-separated R-module. Then M is I-stable if and only if $I \subseteq \sqrt{\operatorname{Ann}_R(M)}$.*

Proof. "If" is clear.

"Only if". Note that I is finitely generated, as R is Noetherian. Since M is I-stable, there is a positive integer n such that

$$I^n M = I^{n+1} M = \bigcap_{t>0} I^t M = 0.$$

Thus $I \subseteq \sqrt{\mathrm{Ann}_R(M)}$. □

Theorem 2.7. *Let M be a finitely generated R-module and N an artinian R-module. Let s be a positive integer. Then the following statements are equivalent*:

(i) $H_i^I(M,N)$ *is I-stable for all $i < s$;*
(ii) $H_i^I(M,N)$ *is artinian for all $i < s$;*
(iii) $H_i^I(M,N)$ *is representable for all $i < s$.*

Proof. (i) ⇒ (ii) by Theorem 2.2 and Lemma 2.6.

(ii) ⇒ (iii) is clear, as every artinian module is representable.

(iii) ⇒ (i) by Corollary 2.5 and Lemma 2.6. □

In the following theorem we find some equivalent conditions when generalized local homology modules $H_i^I(M,N)$ are representable for all $i > pd(M) + s$.

Theorem 2.8. *Let M be a finitely generated R-module and N an artinian R-module. Let s be a non-negative integer. Then the following statements are equivalent*:

(i) $H_i^I(M,N)$ *is I-stable for all $i > pd(M) + s$;*
(ii) $H_i^I(M,N)$ *is artinian for all $i > pd(M) + s$;*
(iii) $\mathrm{Ass}_R(H_i^I(M,N)) \subseteq \{\mathfrak{m}\}$ *for all $i > pd(M) + s$;*
(iv) $H_i^I(M,N) = 0$ *for all $i > pd(M) + s$;*
(v) $H_i^I(M,N)$ *is representable for all $i > pd(M) + s$.*

Proof. (i) ⇒ (ii). We proceed by induction on $d = \mathrm{Ndim}\, M$.

If $d = 0$, $H_i^I(M,N) = 0$ for all $i > pd(M)$ by Theorem 2.3 and we have the result.

Let $d > 0$. There is a positive integer n such that $I^t N = I^n N$ for all $t \geq n$. Set $K = I^n N$, the short exact sequence of artinian R-modules

$$0 \longrightarrow K \longrightarrow N \longrightarrow N/K \longrightarrow 0$$

induces an exact sequence of generalized local homology modules

$$\cdots \to H_{i+1}^I(M, N/K) \to H_i^I(M,K) \to H_i^I(M,N) \to H_i^I(M,N/K) \to \cdots .$$

It is clear that N/K is complete in the I-adic topology. By Lemma 2.1, there is an isomorphism $H_i^I(M,N/K) \cong \mathrm{Tor}_i^R(M,N/K) = 0$ for all $i > pd(M)$. Then $H_i^I(M,K) \cong H_i^I(M,N)$ for all $i > pd(M)$. Thus, the proof will be complete if we show that $H_i^I(M,K)$ is artinian for all $i > pd(M) + s$. As K is an artinian R-moduls and $IK = K$, there is an element $x \in I$ such that $xK = K$. Moreover, there is a positive integer r such that $x^r H_i^I(M,K) = 0$ for all $i > pd(M) + s$. Now the short exact sequence

$$0 \longrightarrow 0 :_K x^r \longrightarrow K \xrightarrow{.x^r} K \longrightarrow 0$$

induces a short exact sequence of generalized local homology modules

$$0 \longrightarrow H_{i+1}^I(M,K) \longrightarrow H_i^I(M, 0 :_K x^r) \longrightarrow H_i^I(M,K) \longrightarrow 0$$

for all $i > pd(M) + s$. It follows $I \subseteq \sqrt{\mathrm{Ann}_R(H_i^I(M, 0 :_K x^r))}$ for all $i > pd(M) + s$. That means $H_i^I(M, 0 :_K x^r)$ is I-stable for all $i > pd(M) + s$. By the inductive hypothesis, $H_i^I(M, 0 :_K x^r)$ is artinian and then $H_i^I(M,K)$ is also artinian for all $i > pd(M) + s$.

(ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (iv). We use induction on $d = \mathrm{Ndim}\, M$.

If $d = 0$, $H_i^I(M,N) = 0$ for all $i > pd(M)$ and we have the result.

Let $d > 0$. As in the proof of (i) $\Rightarrow$ (ii), there is a positive integer n such that $I^t N = I^n N$ for all $t \geq n$. Set $K = I^n N$, the short exact sequence of artinian R-modules

$$0 \longrightarrow K \longrightarrow N \longrightarrow N/K \longrightarrow 0$$

induces an exact sequence of generalized local homology modules

$$\cdots \to H_{i+1}^I(M, N/K) \to H_i^I(M,K) \to H_i^I(M,N) \to H_i^I(M, N/K) \to \cdots .$$

By the arguments above, $H_i^I(M,K) \cong H_i^I(M,N)$ for all $i > pd(M)$. Thus, the proof will be complete if we show that $H_i^I(M,K) = 0$ for all $i > pd(M) + s$. As $IK = K$, there is an element $x \in I$ such that $xK = K$. Now the short exact sequence

$$0 \longrightarrow 0 :_K x \longrightarrow K \xrightarrow{.x} K \longrightarrow 0$$

induces an exact sequence of generalized local homology modules

$$\cdots \to H_{i+1}^I(M,K) \to H_i^I(M, 0 :_K x) \to H_i^I(M,K) \xrightarrow{.x} H_i^I(M,K) \to \cdots .$$

As $\mathrm{Ass}_R(H_i^I(M,K)) = \mathrm{Ass}_R(H_i^I(M,N)) \subseteq \{\mathfrak{m}\}$ for all $i > pd(M) + s$, the exact sequence yields $\mathrm{Ass}_R(H_i^I(M, 0 :_K x)) \subseteq \{\mathfrak{m}\}$ for all $i > pd(M) + s$. By [4, 4.7], $\mathrm{Ndim}(0 :_K x) \leq \mathrm{Ndim}(0 :_N x) \leq d - 1$. Then the inductive hypothesis gives $H_i^I(M, 0 :_K x) = 0$ for all $i > pd(M) + s$ and we have an exact sequence

$$0 \longrightarrow H_i^I(M,K) \xrightarrow{.x} H_i^I(M,K)$$

for all $i > pd(M) + s$. If $H_j^I(M,K) \neq 0$ for some $j > pd(M) + s$, then $\mathrm{Ass}_R(H_j^I(M,K)) = \{\mathfrak{m}\}$. Thus, there exists a non-zero element $a \in H_j^I(M,K)$ such that $\mathfrak{m} = \mathrm{Ann}_R(a)$, that means $\mathfrak{m}\, a = 0$. Hence $xa = 0$, so $a = 0$ which is a contraction. Therefore, $H_i^I(M,N) = H_i^I(M,K) = 0$ for all $i > pd(M) + s$.

(iv) $\Rightarrow$ (v) is clear.

(v) $\Rightarrow$ (i) by Corollary 2.5 and Lemma 2.6. □

3. Generalized local cohomology modules

Note that the i-th generalized local cohomology module of M, N with respect to I is defined by Herzog as follows

$$H_I^i(M,N) = \varinjlim_t \operatorname{Ext}_R^i(M/I^tM, N).$$

Let $D(N) = \operatorname{Hom}_R(N, E(R/\mathfrak{m}))$ be the Matlis dual of N, we have the dual formula.

Lemma 3.1 ([8, 2.3 (ii)]). *Let M be a finitely generated module over the local ring $(R,\mathfrak{m})$ and N an R-module. Then for all $i \geq 0$,*

$$H_i^I(M, D(N)) \cong D(H_I^i(M,N)).$$

An R-module M is called *good* if its zero submodule has a primary decomposition in M. It is clear that finitely generated R-modules are good modules.

Theorem 3.2. *Let M and N be finitely generated R-modules. Let s be a positive integer. Then the following statements are equivalent*:

(i) $H_I^i(M,N)$ *is I-stable for all $i < s$;*
(ii) $H_I^i(M,N)$ *is finitely generated for all $i < s$;*
(iii) $H_I^i(M,N)$ *is good for all $i < s$.*

Proof. (i) $\Rightarrow$ (ii). It follows from Lemma 3.1 that $H_i^I(M,D(N)) \cong D(H_I^i(M,N))$. Then $H_i^I(M,D(N))$ is I-stable for all $i < s$. By Theorem 2.7,

$$I \subseteq \sqrt{\operatorname{Ann}_R(H_i^I(M,D(N))}$$

for all $i < s$. Then, $I \subseteq \sqrt{\operatorname{Ann}_R(D(H_I^i(M,N)))} = \sqrt{\operatorname{Ann}_R(H_I^i(M,N))}$ for all $i < s$. Thus $H_I^i(M,N)$ is finitely generated for all $i < s$ by [6, 2.9].

(ii) $\Rightarrow$ (iii) is clear, since every finitely generated R-module is good.

(iii) $\Rightarrow$ (i). We have $H_i^I(M,D(N)) \cong D(H_I^i(M,N))$. As $H_I^i(M,N)$ is good for all $i < s$, $D(H_I^i(M,N))$ is representable for all $i < s$ by [1, 3.2]. Thus $H_i^I(M,D(N))$ is representable for all $i < s$. It follows from Theorem 2.7 that $I \subseteq \sqrt{\operatorname{Ann}_R(D(H_I^i(M,N)))} = \sqrt{\operatorname{Ann}_R(H_I^i(M,N))}$ for all $i < s$. Therefore $H_I^i(M,N)$ is I-stable for all $i < s$. □

In the following theorem we find some equivalent conditions when generalized local cohomology modules $H_i^I(M,N)$ are good for all $i > pd(M) + s$.

Theorem 3.3. *Let M and N be finitely generated R-modules. Let s be a non-negative integer. Then the following statements are equivalent*:

(i) $H_I^i(M,N) = 0$ *for all $i > pd(M) + s$;*
(ii) $H_I^i(M,N)$ *is finitely generated for all $i > pd(M) + s$;*
(iii) $\operatorname{Coass}_R(H_I^i(M,N)) \subseteq \{\mathfrak{m}\}$ *for all $i > pd(M) + s$;*
(iv) $I \subseteq \sqrt{\operatorname{Ann}_R(H_I^i(M,N))}$ *for all $i > pd(M) + s$;*
(v) $H_I^i(M,N)$ *is I-stable for all $i > pd(M) + s$;*
(vi) $H_I^i(M,N)$ *is good for all $i > pd(M) + s$.*

Proof. (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii) follows from [13, 2,10].

(iii) $\Rightarrow$ (iv). We have $H_i^I(M, D(N)) \cong D(H_I^i(M, N))$. By [13, 1.7]

$$\underset{R}{\mathrm{Ass}}(D(H_I^i(M, N)) = \underset{R}{\mathrm{Coass}}(H_I^i(M, N)) \subseteq \{\mathfrak{m}\}.$$

Then $\mathrm{Ass}_R(H_i^I(M, D(N))) \subseteq \{\mathfrak{m}\}$. It follows from 3.2 that

$$\begin{aligned} I &\subseteq \sqrt{\mathrm{Ann}_R(H_i^I(M, D(N)))} \\ &= \sqrt{\mathrm{Ann}_R(D(H_I^i(M, N)))} \\ &= \sqrt{\mathrm{Ann}_R(H_I^i(M, N))} \end{aligned}$$

for all $i > pd(M) + s$.

(iv) $\Rightarrow$ (i). From the hypothesis and the isomorphism $H_i^I(M, D(N)) \cong D(H_I^i(M, N))$ we get $I \subseteq \sqrt{\mathrm{Ann}_R(H_i^I(M, D(N)))}$ for all $i > pd(M) + s$. By 2.8, $H_i^I(M, D(N))) = 0$ for all $i > pd(M) + s$. It follows $D(H_I^i(M, N)) = 0$ and then $H_I^i(M, N) = 0$ for all $i > pd(M) + s$.

(iv) $\Leftrightarrow$ (v). The arguments are similar to that in the proof of Theorem 3.2.

(i) $\Rightarrow$ (vi) is clear.

(vi) $\Rightarrow$ (i). We have $H_i^I(M, D(N)) \cong D(H_I^i(M, N))$. Note that Matlis dual of a good R-module R-module is representable. That means $H_i^I(M, D(N))$ is representable for all $i > pd(M) + s$. Therefore $H_i^I(M, D(N)) = 0$ by Theorem 2.8. That means $D(H_I^i(M, N)) = 0$ and then $H_I^i(M, N) = 0$ for all $i > pd(M) + s$. □

For two R-modules M and N, we put

$$\mathrm{ext}^+(M, N) = \sup\{i \mid \mathrm{Ext}_R^i(M, N) \neq 0\}.$$

Theorem 3.3 gives us the following consequence.

Corollary 3.4. *Let M and N be finitely generated R-modules such that* $\dim N = d$ *and* $\mathrm{ext}^+(M, H_I^d(N)) = pd(M) = r < \infty$. *Then $H_I^{r+d}(M, N)$ is not good.*

Proof. It follows from [11, 2.2] that $H_I^{r+d}(M, N) \neq 0$ and $H_I^i(M, N) = 0$ for all $i > r + d$. By Theorem 3.3 we have the result. □

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