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THE COMPETITION INDEX OF A NEARLY REDUCIBLE BOOLEAN MATRIX

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Abstract. Cho and Kim [4] have introduced the concept of the competition index of a digraph. Similarly, the competition index of an $n \times n$ Boolean matrix A is the smallest positive integer q such that $A^{q+i}(A^T)^{q+i} = A^{q+r+i}(A^T)^{q+r+i}$ for some positive integer r and every nonnegative integer i, where A^T denotes the transpose of A. In this paper, we study the upper bound of the competition index of a Boolean matrix. Using the concept of Boolean rank, we determine the upper bound of the competition index of a nearly reducible Boolean matrix.

1. Preliminaries and notations

In this paper, we follow the terminology and notation used in [3, 7]. A *Boolean matrix* is a matrix over the binary Boolean algebra $\{0, 1\}$. For $m \times n$ Boolean matrices $A = (a_{ij})$ and $B = (b_{ij})$, we say that B is dominated by A (denoted by $B \le A$) if $b_{ij} \le a_{ij}$ for all i and j. We denote the $m \times n$ all-ones Boolean matrix by $J_{m,n}$ (and by J_n if $m = n$), the $m \times n$ all-zeros Boolean matrix by $O_{m,n}$ (and by O_n if $m = n$), and the $n \times n$ identity Boolean matrix by I_n. The subscripts m and n will be omitted whenever their values are clear from the context.

Let $D = (V, E)$ denote a *digraph* (directed graph) with vertex set $V = V(D)$ and arc set $E = E(D)$. Loops are permitted but multiple arcs are not. An $x \to y$ *walk* in a digraph D is a sequence of vertices $x, v_1, \ldots, v_t, y \in V(D)$ and a sequence of arcs $(x, v_1), (v_1, v_2), \ldots, (v_t, y) \in E(D)$, where the vertices and arcs are not necessarily distinct. A *closed walk* is an $x \to y$ walk where $x = y$. A *cycle* is a closed $x \to y$ walk in which all vertices except x and y are distinct. The *length of a walk* W is the number of arcs in W. The notation $x \xrightarrow{k} y$ is used to indicate that there is a $x \to y$ walk of length k. An l-cycle is a cycle of

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length l. If the digraph D has at least one cycle, the length of a shortest cycle in D is called the *girth* of D, denoted by $s(D)$.

For an $n \times n$ Boolean matrix $A = (a_{ij})$, its digraph, denoted by $D(A)$, is the digraph with vertex set $V(D(A)) = \{v_1, v_2, \ldots, v_n\}$, and (v_i, v_j) is an arc of $D(A)$ if and only if $a_{ij} = 1$. Using Boolean arithmetic ($1+1 = 1$, $0+0 = 0$, $1+0 = 1$), AB and $A+B$ are Boolean matrices if A and B are Boolean matrices. Note that for a positive integer k, the (Boolean) k-th power $A^k = [b_{ij}]$ of A is a Boolean matrix such that $b_{ij} = 1$ if and only if there is a directed walk of length k from v_i to v_j in $D(A)$.

A digraph D is called *strongly connected* if for each pair of vertices x and y in $V(D)$, there is a walk from x to y. For a strongly connected digraph D, the *index of imprimitivity* of D is the greatest common divisor of the lengths of the cycles in D, and it is denoted by $p(D)$. If D is a trivial digraph of order 1, $p(D)$ is undefined. A strongly connected digraph D is *primitive* if $p(D) = 1$. If D is primitive, there exists some positive integer l such that there is a walk of length exactly l from each vertex x to each vertex y. The smallest such l is called the *exponent* of D, denoted by exp(D). Exponents have been studied by several researchers [3, 7, 8, 9, 10].

We say that a Boolean matrix A is *permutationally similar* to a Boolean matrix B if there exists a permutation Boolean matrix P satisfying $B = PAP^T$, where P^T denotes the transpose of P. The Boolean matrix A is called *reducible* if A is permutationally similar to a Boolean matrix of the form

$$\begin{bmatrix} A_1 & O \\ A_{21} & A_2 \end{bmatrix},$$

where A_1 and A_2 are square Boolean matrices of order at least one. If A is not reducible, it is called *irreducible.* A is irreducible if and only if $D(A)$ is strongly connected (see [3]). The Boolean matrix A is called *primitive* if $D(A)$ is primitive.

Let D be a digraph (with or without loops) with the vertex set $\{v_1, v_2, \ldots, v_n\}$. Given a positive integer m, we say that a vertex v_k of D is an *m-step common prey* of v_i and v_j if $v_i \overset{m}{\to} v_k$ and $v_j \overset{m}{\to} v_k$. Then, the *m-step competition graph of* D, denoted by $C^m(D)$, has the same vertex set as D, and there is an edge between vertices v_i and v_j ($v_i \neq v_j$) if and only if v_i and v_j have an m-step common prey in D. The m-step digraph of D, denoted by D^m, has the same vertex set as D and an arc (v_i, v_j) if and only if $v_i \overset{m}{\to} v_j$. Then, we have $C^m(D) = C(D^m)$ for each positive integer m (see [5]).

Consider the sequence

$$D, D^2, D^3, \ldots, D^m, \ldots.$$

Then, there exists the smallest positive integer q such that $D^q = D^{q+r}$ for some positive integer r. Such an integer q is called the *index* of D, and it is denoted by index(D). There also exists the smallest positive integer p such that $D^q = D^{q+p}$; such an integer is called the *period* of D, and it is denoted by period(D).

Now, consider the competition graph sequence

$$C(D), C(D^2), C(D^3), \ldots, C(D^m), \ldots.$$

There exists the smallest positive integer q such that $C(D^{q+i}) = C(D^{q+r+i})$ for some positive integer r and every nonnegative integer i. Such an integer q is called the *competition index* of D, and it is denoted by cindex(D). Let $q = \text{cindex}(D)$. Then, there exists the smallest positive integer p such that $C(D^{q+i}) = C(D^{q+p+i})$ for every nonnegative integer i. Such an integer p is called the *competition period* of D, and it is denoted by cperiod(D).

An analogous definition for the competition index and competition period can be given for a Boolean matrix. The *competition index* of a Boolean matrix A, denoted by cindex(A), is the smallest positive integer q such that $A^{q+i}(A^T)^{q+i} = A^{q+r+i}(A^T)^{q+r+i}$ for some positive integer r and every nonnegative integer i. The *competition period* of a Boolean matrix A, denoted by cperiod(A), is the smallest positive integer p such that $A^{q+i}(A^T)^{q+i} = A^{q+p+i}(A^T)^{q+p+i}$ for $q = \text{cindex}(A)$ and every nonnegative integer i. If A is the adjacency matrix of a digraph D, then we have $\text{cindex}(A) = \text{cindex}(D)$ and $\text{cperiod}(A) = \text{cperiod}(D)$. As a result, throughout the paper, as long as no confusion occurs, we use the digraph D and the adjacency matrix $A(D)$ interchangeably.

Akelbek and Kirkland [2] introduced the *scrambling index* of a primitive digraph. The scrambling index is the smallest positive integer k such that for every pair of vertices u and v, there exists a vertex w such that $u \xrightarrow{k} w$ and $v \xrightarrow{k} w$ in D. Akelbek and Kirkland's definition of the scrambling index is the same as our definition of the competition index in the case of a primitive digraph (see [6]). In [2], they presented the following result regarding the scrambling index.

Proposition 1.1 (Akelbek and Kirkland [2]). *Let D be a primitive digraph of order n and girth s. Then,*

$$\text{cindex}(D) \le \begin{cases} n - s + \left(\frac{s-1}{2}\right) n, & \textit{when } s \textit{ is odd,} \\ \\ n - s + \left(\frac{n-1}{2}\right) s, & \textit{when } s \textit{ is even.} \end{cases}$$

For a positive integer $n \ge 3$, we define

$$W_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

and $\omega_n = \left\lceil \frac{(n-1)^2+1}{2} \right\rceil$, $\omega_1 = 1$, $\omega_2 = 2$.

Cho and Kim [4] presented the following result regarding the upper bound of the competition index of a strongly connected digraph.

Proposition 1.2 (Cho and Kim [4])**.** *Let A be an irreducible $n \times n$ Boolean matrix, where $n \geq 3$. Then, we have*

$$\text{cindex}(A) \leq \omega_n.$$

The equality holds if and only if A is permutationally similar to W_n.

2. A bound on the competition index of an irreducible Boolean matrix using Boolean rank

For a pair of vertices u and v, let $\text{cindex}(D : u, v)$ denote the smallest positive integer m such that u and v have an l-step common prey whenever $l \geq m$. If there is no such positive integer m for u and v, we let $\text{cindex}(D : u, v) = 1$. Further, we let $\text{cindex}(D : u, u) = 1$. If D is strongly connected,

$$\text{cindex}(D) = \max\{\text{cindex}(D : u, v) \mid u, v \in V(D)\}.$$

Theorem 2.1. *Let A be an $n \times n$ irreducible Boolean matrix, with $p(A) = p$. If we denote $r = \lfloor n/p \rfloor$ and $s = n - pr$, we have*

$$\text{cindex}(A) \leq \begin{cases} p \cdot \omega_r + s, & \textit{when } r > 1, \\ s, & \textit{when } r = 1 \textit{ and } s > 0, \\ 1, & \textit{when } r = 1 \textit{ and } s = 0. \end{cases}$$

Proof. Let $D = D(A)$ and $V_0, V_1, \ldots, V_{p-1}$ be p nonempty sets, with $V_p = V_0$, where each arc of D issues from V_i and enters V_{i+1} for some i with $0 \leq i \leq p-1$. Let E_i be the subgraph of D^p induced by V_i, where $0 \leq i \leq p-1$. Then, E_i is primitive.

If $r = 1$ and $s = 0$, we have $\text{cindex}(A) = 1$. Further, if $r = 1$ and $s > 0$, we have $\text{cindex}(D : u, v) \leq s$. Suppose that $r > 1$. We claim that $\text{cindex}(D : u, v) \leq p \cdot \omega_r + s$ for any two vertices u and v. If $u \in V_i$ and $v \in V_j$ where $i \neq j$, u and v do not have an l-step common prey for any positive integer l. Thus, $\text{cindex}(D : u, v) = 1$. We may suppose that $u, v \in V_j$ for some $0 \leq j \leq p-1$. Then, there exists V_q such that $|V_q| \leq r$, and there exist walks

$$u \xrightarrow{f} u' \in V_q \text{ and } v \xrightarrow{f} v' \in V_q,$$

where $0 \leq f \leq s$. Since $\text{cindex}(D^p : u', v') \leq \omega_{|V_q|} \leq \omega_r$, we have

$$\begin{aligned} \text{cindex}(D : u, v) &\leq f + \text{cindex}(D : u', v') \\ &\leq s + p \cdot \text{cindex}(D^p : u', v') \\ &\leq s + p \cdot \omega_r. \end{aligned}$$

Thus, we have $\text{cindex}(D) = \max\{\text{cindex}(D : u, v) \mid u, v \in V(D)\} \leq p \cdot \omega_r + s$.

This establishes the result. $\square$

For an $m \times n$ Boolean matrix A, we define its *Boolean rank* $b(A)$ to be the smallest positive integer b such that for some $m \times b$ Boolean matrix X and $b \times n$ Boolean matrix Y, $A = XY$. The Boolean rank of the zero matrix is defined to be zero. $A = XY$ is called a *Boolean rank factorization* of A.

Proposition 2.2 (Akelbek, Fital, and Shen [1]). *Suppose that X and Y are $n \times m$ and $m \times n$ Boolean matrices, respectively, and that neither has a zero line (i.e., row or column).*
(i) *XY is primitive if and only if YX is primitive.*
(ii) *If XY and YX are primitive,*

$$|\mathrm{cindex}(XY) - \mathrm{cindex}(YX)| \leq 1.$$

Lemma 2.3. *Suppose A is an $n \times m$ Boolean matrix and $A = XY$ is a Boolean rank factorization of A, where $b(A) = b$. If A has no zero lines, neither X nor Y has a zero line.*

Proof. Since A has no zero lines, X has no zero rows and Y has no zero columns. Suppose that X has a zero column, and without loss of generality, let it be the ith column. Let X' be the matrix obtained from X by deleting its ith column, and let Y' be the matrix obtained from Y by deleting its ith row. Then, X' is an $n \times (b-1)$ matrix, Y' is a $(b-1) \times m$ matrix, and $X'Y' = A$. Therefore, the Boolean rank of A is at most $b - 1$. This is a contradiction. Hence, X has no zero columns. Similarly, Y has no zero rows. This establishes the result. $\square$

Lemma 2.4. *Let A be an $n \times n$ Boolean irreducible matrix, with $p(A) = p$, and let $A = XY$ be a Boolean rank factorization of A, with $b(A) = b$. Then,*
(i) *YX is irreducible, with $p(YX) \geq p$.*
(ii) $\mathrm{cindex}(XY) \leq \mathrm{cindex}(YX) + 1$.

Proof. If A is primitive, we have the result by Proposition 2.2. Suppose that $p(A) \geq 2$. Then, we may suppose that

$$A = \begin{bmatrix} O & A_0 & O & \cdots & O \\ O & O & A_1 & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & A_{p-2} \\ A_{p-1} & O & O & \cdots & O \end{bmatrix},$$

in which the zero matrices on the diagonal are square matrices of orders $n_0, n_1, \ldots, n_{p-1}$, respectively (see [3]). Further, there exists a permutation matrix P such that

$$XP = \begin{bmatrix} X_0 & O & O & \cdots & O \\ O & X_1 & O & \cdots & O \\ O & O & X_2 & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & X_{p-1} \end{bmatrix}, \quad P^T Y = \begin{bmatrix} O & Y_0 & O & \cdots & O \\ O & O & Y_1 & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & Y_{p-2} \\ Y_{p-1} & O & O & \cdots & O \end{bmatrix},$$

where $A_i = X_iY_i$ is a Boolean rank factorization of A_i, with $b(A_i) = b_i$. Moreover, YX is permutationally similar to

$$(1)\qquad \begin{bmatrix} O & Y_0X_1 & O & \cdots & O \\ O & O & Y_1X_2 & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & Y_{p-2}X_{p-1} \\ Y_{p-1}X_0 & O & O & \cdots & O \end{bmatrix},$$

in which the zero matrices on the diagonal are square matrices of orders $b_0, b_1, \ldots, b_{p-1}$, respectively. Let

$$\begin{aligned} \bar{A}_0 &= A_0A_1\cdots A_{p-2}A_{p-1} = X_0Y_0X_1Y_1\cdots X_{p-2}Y_{p-2}X_{p-1}Y_{p-1}, \\ \bar{A}_1 &= A_1A_2\cdots A_{p-1}A_0 = X_1Y_1X_2Y_2\cdots X_{p-1}Y_{p-1}X_0Y_0, \\ &\cdots \\ \bar{A}_{p-1} &= A_{p-1}A_0\cdots A_{p-3}A_{p-2} = X_{p-1}Y_{p-1}X_0Y_0\cdots X_{p-3}Y_{p-3}X_{p-2}Y_{p-2}. \end{aligned}$$

For each i, there exists a positive integer l such that $\bar{A}_i^l = J_{n_i}$ since $\bar{A}_i$ is primitive. For each i, neither X_i nor Y_i has a zero line by Lemma 2.3. Then, we have

$$\begin{aligned} & (Y_0X_1Y_1X_2\cdots Y_{p-2}X_{p-1}Y_{p-1}X_0)^{l+1} \\ = \ & Y_0\bar{A}_1^l(X_1Y_1X_2\cdots Y_{p-2}X_{p-1}Y_{p-1}X_0) \\ = \ & Y_0J_{n_1}(X_1Y_1X_2\cdots Y_{p-2}X_{p-1}Y_{p-1}X_0) = J_{b_0}, \end{aligned}$$

$$\begin{aligned} & (Y_1X_2Y_2X_3\cdots Y_{p-1}X_0Y_0X_1)^{l+1} \\ = \ & Y_1\bar{A}_2^l(X_2Y_2X_3\cdots Y_{p-1}X_0Y_0X_1) \\ = \ & Y_1J_{n_2}(X_2Y_2X_3\cdots Y_{p-1}X_0Y_0X_1) = J_{b_1}, \end{aligned}$$

$$\cdots$$

$$\begin{aligned} & (Y_{p-1}X_0Y_0X_1\cdots Y_{p-3}X_{p-2}Y_{p-2}X_{p-1})^{l+1} \\ = \ & Y_{p-1}\bar{A}_0^l(X_0Y_0X_1\cdots Y_{p-3}X_{p-2}Y_{p-2}X_{p-1}) \\ = \ & Y_{p-1}J_{n_0}(X_0Y_0X_1\cdots Y_{p-3}X_{p-2}Y_{p-2}X_{p-1}) = J_{b_{p-1}}. \end{aligned}$$

Therefore, YX is irreducible and $p(YX) \geq p = p(XY)$ by (1).

Suppose that $\operatorname{cindex}(XY) = k$. By the definition of the competition index of an irreducible Boolean matrix,

$$(2)\qquad A^k(A^T)^k = (XY)^k((XY)^T)^k = \begin{bmatrix} J_{n_0} & O & O & \cdots & O \\ O & J_{n_1} & O & \cdots & O \\ O & O & J_{n_2} & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & J_{n_{p-1}} \end{bmatrix}.$$

For each i, neither X_i nor Y_i has a zero line by Lemma 2.3. Then, we have

$$X_i X_i^T \geq I_{n_i} \text{ and } Y_i Y_i^T \geq I_{b_i}.$$

If we suppose that all subscripts are taken by modulo p, we have

$$\begin{aligned}(A_i A_{i+1} \cdots A_{i+k})(A_i A_{i+1} \cdots A_{i+k})^T &= J_{n_i}, \\ (X_i Y_i X_{i+1} Y_{i+1} \cdots X_{i+k} Y_{i+k})(X_i Y_i X_{i+1} Y_{i+1} \cdots X_{i+k} Y_{i+k})^T &= J_{n_i}\end{aligned}$$

by (2). Therefore, we have

$$\begin{aligned}&(Y_{i-1} X_i Y_i X_{i+1} Y_{i+1} \cdots Y_{i+k} X_{i+k+1})(Y_{i-1} X_i Y_i X_{i+1} Y_{i+1} \cdots Y_{i+k} X_{i+k+1})^T \\ &= (Y_{i-1} X_i Y_i X_{i+1} Y_{i+1} \cdots Y_{i+k})(X_{i+k+1} X_{i+k+1}^T)(Y_{i-1} X_i Y_i X_{i+1} Y_{i+1} \cdots Y_{i+k})^T \\ &\geq (Y_{i-1} X_i Y_i X_{i+1} Y_{i+1} \cdots Y_{i+k}) I_{n_{i+k+1}} (Y_{i-1} X_i Y_i X_{i+1} Y_{i+1} \cdots Y_{i+k})^T \\ &= Y_{i-1} (X_i Y_i X_{i+1} Y_{i+1} \cdots Y_{i+k}) I_{n_{i+k+1}} (X_i Y_i X_{i+1} Y_{i+1} \cdots Y_{i+k})^T Y_{i-1}^T \\ &= Y_{i-1} J_{n_i} Y_{i-1}^T = J_{b_{i-1}}.\end{aligned}$$

Then, we have

$$(YX)^{k+1}((YX)^T)^{k+1} = \begin{bmatrix} J_{b_0} & O & O & \cdots & O \\ O & J_{b_1} & O & \cdots & O \\ O & O & J_{b_2} & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & J_{b_{p-1}} \end{bmatrix}.$$

Thus, we have $\operatorname{cindex}(YX) \leq k+1 = \operatorname{cindex}(XY)+1$. This establishes the result. $\square$

In Lemma 2.4, the condition that A is irreducible is required. See Example 2.5.

Example 2.5. Consider the Boolean reducible matrix A such that

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, we have $b(A) = 4$ and a Boolean rank factorization $A = XY$ for X and Y such that

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; Y = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Then, we have

$$YX = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have $\text{cindex}(A) = 3$ and $\text{cindex}(YX) = 1$. Therefore, we have

$$\text{cindex}(XY) > \text{cindex}(YX) + 1.$$

Proposition 2.6 (Akelbek, Fital, and Shen [1]). *Suppose that A is an $n \times n$ ($n \geq 2$) primitive Boolean matrix with Boolean rank $b(A) = b$. Then,*

$$\text{cindex}(A) \leq \omega_b + 1.$$

If $3 \leq b \leq n-1$, the equality holds if and only if A is permutationally similar to one of the forms M_1, M_3, and M_5 in Table 1.

In Table 1, the rows and columns of M_1, M_3, and M_5 are partitioned conformally, so that each diagonal block is square, and the top left-hand side submatrix common to each has b blocks in its partition.

Theorem 2.7. *Suppose that A is an $n \times n$ irreducible Boolean matrix with Boolean rank $b(A) = b$, where $3 \leq b \leq n-1$. Then, we have*

$$\text{cindex}(A) \leq \omega_b + 1.$$

The equality holds if and only if A is permutationally similar to one of the forms M_1, M_3, and M_5 in Table 1.

Proof. If $p(A) = 1$, we have the result by Proposition 2.6. Suppose that $p(A) \geq 2$. We claim that $\text{cindex}(A) < \omega_b + 1$. Let $A = XY$ be a Boolean rank factorization of A. Then, YX is a $b \times b$ irreducible matrix, with $p = p(YX) \geq p(XY) \geq 2$ by Lemma 2.4. By Lemma 2.4 and Proposition 1.2, we have

$$\text{cindex}(A) \leq \text{cindex(YX)} + 1 \leq \begin{cases} p \cdot \omega_r + p, & \text{when } r \geq 2, \\ p, & \text{when } r < 2, \end{cases}$$

where $r = \left\lfloor \frac{b}{p} \right\rfloor$. If $r < 2$, we obtain the result. Suppose that $2 \leq p \leq \left\lfloor \frac{b}{2} \right\rfloor$. Then, we have

$$\text{cindex}(A) \leq \text{cindex(YX)} + 1 \leq p \cdot \omega_r + p \leq \frac{b^2}{2p} + \frac{5}{2}p - b.$$

Let $g(p) = \frac{b^2}{2p} + \frac{5}{2}p - b(2 \leq p \leq \left\lfloor \frac{b}{2} \right\rfloor)$. Then, $g(p)$ attains the maximum value when $p = 2$. $g(2) = \frac{b^2-4b+20}{4} < \left\lceil \frac{b^2-2b+2}{2} \right\rceil + 1 = \omega_b + 1$ since $b \geq 2p \geq 4$. Then,

$$\text{cindex}(A) < \omega_b + 1.$$

This establishes the result. $\square$

3. A bound on the competition index of a nearly reducible matrix

The irreducible Boolean matrix A is called *nearly reducible* if each matrix obtained from A by the replacement of a 1 with a 0 is a reducible Boolean matrix. Thus, the digraph D is *minimally strong* if and only if its adjacency matrix A is nearly reducible.

The *term rank* of a Boolean matrix A, denoted by $t(A)$, is defined to be the largest number of 1s in A, with at most one 1 in each column and at most one 1 in each row. Then, we have $b(A) \leq t(A)$.

Proposition 3.1 (Cho and Kim [4])**.** *Let D be a strongly connected digraph of order $n(\geq 3)$. If $p(D) > \frac{n}{2}$, we have*

$$\operatorname{cindex}(D) \leq \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Theorem 3.2. *Let A be a nearly reducible $n \times n$ Boolean matrix, where $n \geq 8$. Then, we have*

$$\operatorname{cindex}(A) \leq \left\lceil \frac{(n-2)^2+1}{2} \right\rceil + 1.$$

The equality holds if and only if A is permutationally similar to

$$\left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & \cdots & 0 & 0 & 0 & 0 \end{array}\right].$$

Proof. Denote $p = p(A)$.
Case 1. $p > \frac{n}{2}$.
By Proposition 3.1, we have $\operatorname{cindex}(D) \leq \lfloor \frac{n-1}{2} \rfloor < \left\lceil \frac{(n-2)^2+1}{2} \right\rceil + 1$.
Case 2. $2 \leq p \leq \frac{n}{2}$.
By Theorem 2.1, we have

$$\operatorname{cindex}(A) \leq p \cdot \omega_{n/p} + p - 1 \leq \frac{n^2}{2p} + \frac{5}{2}p - n - 1.$$

Let $g(p) = \frac{n^2}{2p} + \frac{5}{2}p - n - 1 (2 \leq p \leq \frac{n}{2})$. Then, $g(p)$ attains the maximum value when $p = 2$. $g(2) = \frac{n^2-4n+16}{4} < \left\lceil \frac{(n-2)^2+1}{2} \right\rceil + 1$, where $n \geq 8$.
Case 3. $p = 1$.
If $b(A) \leq n-2$, we have $\operatorname{cindex}(A) \leq \omega_{n-2} + 1 < \left\lceil \frac{(n-2)^2+1}{2} \right\rceil + 1$ by Theorem 2.7.

If $b(A) = n - 1$, we have

$$\text{cindex}(A) \leq \omega_{n-1} + 1 = \left\lceil \frac{(n-2)^2 + 1}{2} \right\rceil + 1,$$

and the equality holds if and only if A is permutationally similar to

$$\left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & \cdots & 0 & 0 & 0 & 0 \end{array}\right],$$

by Theorem 2.7.

Suppose that $b(A) = n$. Then we have $t(A) = n$ since $t(A) \geq b(A)$. Thus we have an $n \times n$ permutation submatrix in A. If there is no n-cycle in $D = D(A)$, $s(D) = s \leq \left\lfloor \frac{n}{2} \right\rfloor$. By Proposition 1.1 we have cindex$(A) < \left\lceil \frac{(n-2)^2+1}{2} \right\rceil + 1$ since $n \geq 8$. If there is an n-cycle in D, D is isomorphic to an n-cycle since A is a nearly reducible Boolean matrix. However, $p(C_n) = n$ is not primitive. This establishes the result. □

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