

KNOTS WITH ARBITRARILY HIGH DISTANCE BRIDGE DECOMPOSITIONS

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ABSTRACT. We show that for any given closed orientable 3-manifold M with a Heegaard surface of genus g , any positive integers b and n , there exists a knot K in M which admits a (g, b) -bridge splitting of distance greater than n with respect to the Heegaard surface except for $(g, b) = (0, 1), (0, 2)$.

1. Introduction

A *Heegaard splitting* of a closed, orientable 3-manifold M is a decomposition of M into two handlebodies, which was originally introduced and studied by Heegaard [11]. About fifty years later, Moise proved in [22] that every closed orientable 3-manifold can admit a Heegaard splitting. It is then shown that certain properties of Heegaard splittings reflect topological characteristics of 3-manifolds: For example, any Heegaard splitting of a reducible 3-manifold is reducible [7], any irreducible Heegaard splitting of a non-Haken 3-manifold is strongly irreducible [4].

Motivated by such works, Hempel [12] introduced an integer-valued invariant of a Heegaard splitting, called the *distance*, or commonly called the *Hempel distance*. This is a natural refinement of strong irreducibility, and measures certain complexity of Heegaard splittings. A lot of studies have been done about the distance of Heegaard splittings. See [20, Section 6] for an aspect of such works.

Among them, on the existence of high distance splittings, there are several known results. We here summarize a part of them in the following.

First, Hempel [12] exhibited that there exist Heegaard splittings of closed orientable 3-manifolds with distance at least n for arbitrarily large n (adapting

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an idea of Kobayashi [17]). Later, Evans [5] explicitly defined an infinite sequence of closed 3-manifolds $\{M_n\}$ and proved that the distance of a Heegaard splitting of M_n is at least n by using purely combinatorial techniques.

About the non-closed compact 3-manifold case, in particular, the knot exterior case, Minsky, Moriah and Schleimer [21] constructed knots in the 3-sphere S^3 with the exteriors having Heegaard splittings of arbitrarily high distance, in any genus. Extending their results, Campisi and Rathbun [3] showed that for any closed 3-manifold M with a minimal genus Heegaard splitting of genus $g(M)$ and any integers $g \geq g(M) + 1$ and $n > 0$, there is a knot K in M with its exterior admitting a genus g Heegaard splitting of distance greater than n .

A natural generalization of the Heegaard splitting for a link in a 3-manifold is given by the bridge splitting. A (g, b) -bridge splitting of a link L in a closed 3-manifold M is a decomposition of (M, L) into two pairs of a genus g handlebody and b trivial arcs. For bridge splittings, the notion of distance is naturally defined as a generalization of the case of Heegaard splittings for closed manifolds. See the next section for details.

The second author [23] showed that for any closed 3-manifold admitting a Heegaard splitting of genus one, there is a knot in the manifold with a $(1, 1)$ -bridge splitting of arbitrarily high distance. Recently, Blair, Tomova and Yoshizawa [2] showed that for given integers b, c, g , and n , there exists a manifold M containing a c -component link L so that (M, L) admits a (g, b) -bridge splitting of distance at least n .

Extending their results, in this paper, we show the following.

Theorem 1. *For any given closed orientable 3-manifold M with a Heegaard surface of genus g , any positive integers b and n , there exists a knot K in M which admits a (g, b) -bridge splitting of distance greater than n with respect to the Heegaard surface except for $(g, b) = (0, 1), (0, 2)$.*

Remark 1. If we regard a $(g, 0)$ -bridge splitting for a knot as a Heegaard splitting of the exterior of the knot, then the case where $b = 0$ corresponds to the result given in [3].

Remark 2. Note that the case $(g, b) = (0, 1)$ have to be excluded, for the distance of such a splitting cannot be defined. Also, for the case $(g, b) = (0, 2)$, if we change some definitions to define the distance for $(0, 2)$ -bridge splittings suitably, then the same result can be obtained. See the remarks in Section 2.

Remark 3. In general, we can also make a link of at most b components in stead of the knot enjoying the desired properties in the theorem. See remarks in Section 3.

As an application, in Section 4, we will study the *meridional destabilizing numbers* of knots in S^3 introduced by the second author in [24]. Recall that, for a knot in S^3 , an invariant is defined by the minimal genus of Heegaard surfaces of its exterior minus one, called the *tunnel number* of a knot. This gives an interesting subject to study in Knot Theory, and there are many works to

study it. By considering bridge splittings of knots simultaneously, for example, tunnel number one knots are classified into 3 classes: the non-trivial two-bridge knots, i.e., knots admitting $(0, 2)$ -bridge splittings, the $(1, 1)$ -knots, i.e., knots admitting $(1, 1)$ -bridge splittings, and the other knots, i.e., knots admitting $(2, 0)$ -bridge splittings. A natural generalization of this filtration is given by considering the meridional destabilizing number, and as a corollary to Theorem 1, we have the following.

Corollary 2. *For any integers $t \geq 1$ and $m \geq 0$ with $m \leq t + 1$, there exists a knot $K \subset S^3$ of tunnel number t and of meridional destabilizing number m .*

Precise definition of the meridional destabilizing number of knots and a proof of this corollary will be given in the last section.

2. Distance of Heegaard splitting and (g, b) -bridge splitting

In this section, we recall definitions of a Heegaard splitting of a 3-manifold, a (g, b) -bridge splitting of a knot, and their (Hempel) distance.

Let M be a closed orientable 3-manifold. A closed orientable surface S of genus g embedded in M is called a (genus g) *Heegaard surface* of M if S divides M into two handlebodies V_1 and V_2 . The triplet $(V_1, V_2; S)$ is called a (genus g) *Heegaard splitting* of M .

Let K be a knot in a closed orientable 3-manifold M . We say that K admits a (g, b) -*bridge splitting* ($b > 0$) if there is a genus g Heegaard splitting $(V_1, V_2; S)$ of M such that $V_i \cap K$ consists of b arcs which are mutually boundary parallel for each $i = 1, 2$. Set $\mathcal{V}_i = (V_i, V_i \cap K)$ and $\mathcal{S} = (S, S \cap K)$. We also call the triplet $(\mathcal{V}_1, \mathcal{V}_2; \mathcal{S})$ a (g, b) -*bridge splitting* of (M, K) , and \mathcal{S} is called a (g, b) -*bridge surface*, or a *bridge surface* for short. We say that K admits a $(g, 0)$ -*bridge splitting* if there is a genus g Heegaard splitting $(V_1, V_2; S)$ of M such that $K \subset V_i$ ($i = 1$ or 2), say $i = 2$, and that $V_2 - \text{int}N(K)$ is a compression body. Note that a $(g, 0)$ -bridge splitting of (M, K) is also called a *Heegaard splitting* of (M, K) , and a $(g, 0)$ -bridge surface of (M, K) is called a *Heegaard surface* of (M, K) .

To define the (Hempel) distance of the splittings above, we first prepare the terminology about the curve complex, originally introduced in [8, 9]. Let F be a compact orientable surface of genus g possibly with non-empty boundary. Then the *curve complex* $\mathcal{C}(F)$ of F is defined as the simplicial complex whose k -simplexes are the isotopy classes of $k + 1$ collections of mutually non-isotopic essential loops (i.e., non-contractible and not boundary-parallel) which can be realized disjointly. It is then proved in [19] that the curve complex is connected if F is not sporadic, i.e., F has at least 5 (resp. 2) boundary components if $g = 0$ (resp. $g = 1$). For vertices $[x]$ and $[y]$ of $\mathcal{C}(F)$, the distance $d([x], [y])$ between $[x]$ and $[y]$ is defined as the minimal number of 1-simplexes in a simplicial path joining $[x]$ to $[y]$.

For a bridge splitting $(\mathcal{V}_1, \mathcal{V}_2; \mathcal{S})$ of (M, K) , set $W_i = V_i - \text{int}N(V_i \cap K)$ ($i = 1, 2$) and $S_0 = S - \text{int}N(S \cap K)$. For $i = 1, 2$, the set of vertices in $\mathcal{C}(S_0)$

each of which corresponds to a curve bounding a disk in W_i is called the *disk set* $\mathcal{D}(W_i)$. Then the *Hempel distance* $d(\mathcal{S})$ is defined as $d(\mathcal{D}(W_1), \mathcal{D}(W_2)) = \min\{d([x], [y]) \mid [x] \in \mathcal{D}(W_1), [y] \in \mathcal{D}(W_2)\}$.

Remark 4. For the case of $(g, b) = (0, 2)$, equivalently, for the case of two-bridge knots in S^3 , the corresponding curve complex in the above definition is not connected, and so the distance of such a splitting cannot be well-defined. However, as remarked in Section 1, the same result as Theorem 1 can be obtained for this case if we change some definition suitably. To do this, we define the curve complex of the 4-punctured sphere as the simplicial complex whose k -simplexes are the isotopy classes of $k+1$ collections of mutually non-isotopic essential loops (i.e., non-contractible and not boundary-parallel) which can be realized with just two points as intersections. Then the curve complex can be naturally identified with the well-known Farey diagram. The distance for a given splitting, equivalently a given two-bridge knot, can be related to the minimal length of the continued fractional expansions for the rational number corresponding to the two-bridge knot. See [10] for example.

3. Proof of Theorem 1

In this section, we give a proof of Theorem 1. We can make use of the strategy for the proof of the main theorem in [21]. Also see [3, Section 4] as a summary. Actually their theorems concern only closed 3-manifolds, and so, we modify the arguments to fit for the case of manifolds with boundary.

3.1. Settings

We start with preparing our setting.

Let M be a closed orientable 3-manifold and F a Heegaard surface of M of genus g . That is, F decomposes M into two handlebodies V_1 and V_2 . Choose arbitrary integers $n, b \geq 1$. We only consider the case $b \geq 3$ if $g = 0$ by the assumption.

As stated in Section 1, the second author [23] showed that for any closed 3-manifold admitting a Heegaard splitting of genus one, there is a knot in the manifold with a $(1, 1)$ -bridge splitting of arbitrarily high distance. Thus, in the following, we consider the case $b \geq 2$ if $g = 1$.

Let us first take a $2b$ -tuple of points $\{x_1, x_2, \dots, x_{2b-1}, x_{2b}\}$ on F . Take embedded b arcs $\alpha_1, \dots, \alpha_b$ on F disjoint from each other such that α_j connects x_{2j-1} and x_{2j} for $j = 1, \dots, b$. Pushing the interior of each arc α_j into V_i , we have a pair of b boundary-parallel arcs in both handlebodies, say, $\alpha_{i,1}, \alpha_{i,2}, \dots, \alpha_{i,b} \subset V_i$ for $i = 1, 2$. Then they compose a b -component trivial link in M .

Take a small regular neighborhood $N(F)$ of F in M , which is topologically $F \times [0, 1]$, and regard it as lying in V_2 with $F = F \times \{0\}$. Now we define $W_1 = V_1 - \text{int}N(\alpha_{1,1} \cup \dots \cup \alpha_{1,b})$ and $W_2 = V_2 - \text{int}N(\alpha_{2,1} \cup \dots \cup \alpha_{2,b})$.

3.2. Key ingredients

We here prepare two key ingredients used in the proof of Theorem 1. One is a pseudo-Anosov homeomorphism ϕ on $F_0 := F - \text{int}N(x_1 \cup \cdots \cup x_{2b})$, and the other is a simple closed curve c on F_0 , both satisfying some particular conditions.

3.2.1. Pseudo-Anosov map. A *pseudo-Anosov* homeomorphism $\phi : F_0 \rightarrow F_0$ is defined as a homeomorphism on F_0 with $\phi|_{\partial} = id$ such that there exist a constant $k_\phi > 1$ and a pair of transverse measured foliations λ^\pm satisfying that $\phi(\lambda^+) = \frac{1}{k_\phi}\lambda^+$, $\phi(\lambda^-) = k_\phi\lambda^-$, and that each boundary component of F_0 is a cycle of leaves of λ^+ and of λ^- and contains singularities of these two foliations. See [25] for original definition, and see also [6] for a survey.

Now let us take a pseudo-Anosov homeomorphism $\phi : F_0 \rightarrow F_0$ such that the capped off homeomorphism $\widehat{\phi} : F \rightarrow F$ naturally induced from ϕ is isotopic to the identity map. Moreover $\widehat{\phi} : F \rightarrow F$ can be taken to satisfy that the point x_{j+1} is mapped to x_j for each j with $1 \leq j \leq 2b - 1$ and x_1 is mapped to x_{2b} . The existence of such a homeomorphism is essentially guaranteed by the result given in [14]. See also related works [13, 15, 16].

Precisely, for the case of $g \geq 2$, we have such a homeomorphism by extending [14, Example 2, p. 531], which gives an example for the genus 2 with 3 marked points case.

When $g = 0$ or 1, we make a pseudo-Anosov homeomorphism $\phi : F_0 \rightarrow F_0$ as follows. Let $F_0^1 = F - \text{int}N(x_3 \cup \cdots \cup x_{2b})$, which contains F_0 as a subsurface. Since we are assuming $b \geq 3$ if $g = 0$ and $b \geq 2$ if $g = 1$, the Euler characteristic of F_0^1 is negative. Then, as in the above, by generalizing [14, Examples 1 and 2, p. 531], we can find a pseudo-Anosov homeomorphism $\phi_1 : F_0 \rightarrow F_0$ such that ϕ_1 naturally induces a homeomorphism $\widehat{\phi}_1 : F_0^1 \rightarrow F_0^1$ which is isotopic to the identity map and interchanges the two points x_1 and x_2 . Note that the capped off homeomorphism of ϕ_1 on F is also isotopic to the identity map on F . We prepare $\phi_2, \dots, \phi_{b-1}$ in the same way, and compose them to obtain $\phi = \phi_{b-1} \circ \cdots \circ \phi_1$. Then the pseudo-Anosov homeomorphism $\phi : F_0 \rightarrow F_0$ satisfies all the conditions we desired above.

Remark 5. If we want to produce just a b -component link, not a knot, then it suffices to use the result obtained by Kra [18]. The result given in [14] is an extension of that in [18]. Also if we want to make a link of components less than b and at least 2, then it suffices to use a natural extension of [14, Example 1].

3.2.2. Dehn twist. Next let us take a Dehn twist along a curve c on F_0 such that the curve c is constructed as follows.

Prepare diskal neighborhoods D_j of the previously chosen arcs α_j ($j = 1, \dots, b$) on $F = F \times \{0\}$, and pushing the interior of them into V_i to obtain properly embedded disks $D_{i,1}, \dots, D_{i,b}$ in V_i for $i = 1, 2$. We assume that each

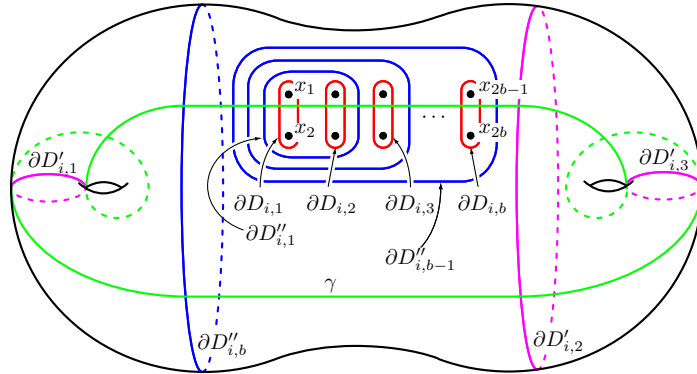


FIGURE 1. $\partial D''_{i,1}, \dots, \partial D''_{i,b-1}, \partial D''_{i,b}$ and γ for closed genus two case.

$D_{i,j}$ cuts a ball including $\alpha_{i,j}$ from V_i . Let P_j be the pair of pants which is bounded by ∂D_j together with the two components of ∂F_0 corresponding to x_{2j-1} and x_{2j} .

We here use the result given as [3, Lemma 5.2]: Given any genus g Heegaard splitting $(V_1, V_2; F)$ of a closed 3-manifold, there exist essential disks $D'_{i,1}, \dots, D'_{i,3g-3}$ for V_i for $i = 1, 2$ such that their boundary curves give a pair of pants decompositions on F (i.e., their boundary curves decompose F into pairs of pants maximally) and that there exists a curve c_0 in F which traverses all the seams of all pairs of pants obtained from the both pants decompositions. We further assume that $D_{i,1}, \dots, D_{i,b}$ is contained in the same region obtained by cutting V_i along $D'_{i,1}, \dots, D'_{i,3g-3}$ for $i = 1, 2$.

Here let us recall the definitions used in the above. For a pair of pants P , a seam of P is an essential arc in P connecting two distinct boundary components of P . Let \mathcal{P} be a set of pairs of pants embedded into a surface with mutually disjoint interiors and γ a simple closed curve on the surface. We say that γ traverses a seam in $P \in \mathcal{P}$ if γ intersects P minimally, up to isotopy, and a component of $\gamma \cap P$ is a seam. We say that γ traverses all the seams of \mathcal{P} if γ traverses every seam of every pair of pants in \mathcal{P} . Please remark that these definitions are slightly different with those in [21] and [3].

We further take disks $D''_{i,1}, \dots, D''_{i,b-1}, D''_{i,b}$ in V_i for $i = 1, 2$ such that, together with $D_{i,1}, \dots, D_{i,b}$ and $D'_{i,1}, \dots, D'_{i,3g-3}$, the collection of simple closed curves $\{\partial D_{i,j}\}_{1 \leq j \leq b} \cup \{\partial D'_{i,j'}\}_{1 \leq j' \leq 3g-3} \cup \{\partial D''_{i,j''}\}_{1 \leq j'' \leq b}$ divides F_0 into a set of pairs of pants, say \mathcal{P}_i . Note that P_1, \dots, P_b are included in both \mathcal{P}_i 's. Then we have a simple closed curve γ on F_0 traverses all the seams of pairs of pants in $\mathcal{P}'_i = \mathcal{P}_i - \{P_1, \dots, P_b\}$. See Figure 1 for closed genus two case for example.

From the simple closed curve γ regarded as on F , by smoothing γ and the arc α_1 at the intersection point, we have a simple arc on F with endpoints x_1

and x_2 which traverses all the seams of pairs of pants in \mathcal{P}'_i for $i = 1, 2$. We take a diskal neighborhood of the arc on F , and let c be its boundary curve. Then c is also a simple closed curve on F_0 traverses all the seams of pairs of pants in \mathcal{P}'_i for $i = 1, 2$.

Now recall that τ is a Dehn twist along a curve c on F_0 . This gives a non-trivial map on F_0 , but its natural extension $\hat{\tau}$ on F is isotopic to the identity map on F , for it bounds a disk on F , which contains exactly two marked points x_1 and x_2 .

3.3. Claims

Here we prove two key claims.

The first claim is a modification of [21, Lemma 2.3]. To state the claim, we need to prepare some notation. Let $\overline{\mathcal{D}(W_1)}$ be the closure of the disk set $\mathcal{D}(W_1)$ of W_1 in the space of projective measured laminations $\mathcal{PML}(F_0)$ on F_0 . Let $\overline{\mathcal{D}(W_2)}$ be the closure of the disk set $\mathcal{D}(W_2)$ of W_2 , which is also regarded as in $\mathcal{PML}(F_0)$. We here omit the definitions of the terms used here. Please refer [6] for example.

Claim 1. *With sufficiently large $N > 0$, the map $\phi_N = \tau^N \circ \phi \circ \tau^{-N}$ is a pseudo-Anosov homeomorphism of F_0 with stable and unstable laminations $\lambda_N^\pm \in \mathcal{PML}(F_0)$ such that $\lambda_N^+ \notin \overline{\mathcal{D}(W_1)}$ and $\lambda_N^- \notin \overline{\mathcal{D}(W_2)}$.*

Proof. Since ϕ is pseudo-Anosov, its conjugate ϕ_N is also a pseudo-Anosov map of F_0 . Let λ^\pm be the stable/unstable laminations of ϕ . Then the stable/unstable laminations λ_N^\pm of ϕ_N are just $\tau^N(\lambda^\pm)$ as claimed in [21]. Both laminations intersect the curve c , since λ^\pm is filling. Thus, as $N \rightarrow \infty$, the both laminations λ_N^\pm converge to $[c]$ in $\mathcal{PML}(F_0)$, which is guaranteed by calculations given in [6, Appendix C].

On the other hand, we see $[c] \notin \overline{\mathcal{D}(W_1)}$ as follows. Set

$$\mathcal{C}_1 = \{\partial D'_{1,j'}\}_{1 \leq j' \leq 3g-3} \cup \{\partial D''_{1,j''}\}_{1 \leq j'' \leq b}.$$

Then c is not isotopic to any element of \mathcal{C}_1 , otherwise c does not traverse all the seams of pairs of pants in \mathcal{P}'_1 , contradicting the assumption. Suppose that there exists a sequence of meridians $\boldsymbol{\mu} = \{\mu_1, \mu_2, \dots\}$ on F_0 which gives a sequence $\{[\mu_1], [\mu_2], \dots\} \subset \overline{\mathcal{D}(W_1)}$ converging to $[c] \in \mathcal{PML}(F_0)$. By passing to a subsequence, we may assume that none of the curves in $\boldsymbol{\mu}$ is isotopic to a curve in \mathcal{C}_1 . Then we see that every curve in $\boldsymbol{\mu}$ has a *wave* (i.e., an essential arc in a pair of pants P with endpoints on the same boundary component of P) in some pair of pants in \mathcal{P}'_1 , since any meridian is either parallel to an element of \mathcal{C}_1 or has a wave in some pair of pants in \mathcal{P}'_1 by outermost disk arguments as shown in [21, Lemma 2.2]. Passing to subsequences again, we may assume that all the curves in $\boldsymbol{\mu}$ have the same wave w in the same pair of pants $Y \in \mathcal{P}'_1$. Since c traversed all the seams of pairs of pants in \mathcal{P}'_1 , the curve μ in $\boldsymbol{\mu}$ with a large enough index would also traverse all the seams of pairs of pants in \mathcal{P}'_1 . Since one of the seams of Y intersects w essentially, it would follow that

the curve μ also has a self-intersection, a contradiction. Consequently, for a sufficiently large N , we see $\lambda_N^+ \notin \overline{\mathcal{D}(W_1)}$.

By considering W_2 and by regarding $\mathcal{P}'_2 \subset \partial W_2$, we can prove $\lambda_N^- \notin \overline{\mathcal{D}(W_2)}$ in the same way. □

The next claim is a bridge decomposition version of [21, Lemma 2.1], which is essentially proved as [1, Theorem 2.3], and is originally due to [12]. We here follow the outline of their proof of [1, Theorem 2.3].

Claim 2. *Let $(\phi_N)^m(\mathcal{D}(W_2))$ be the set of vertices in $\mathcal{C}(F_0)$ each of which corresponds to the image of a curve bounding a disk in W_2 by $(\phi_N)^m$ for $m \in \mathbb{N}$. Then $d(\mathcal{D}(W_1), (\phi_N)^m(\mathcal{D}(W_2))) \rightarrow \infty$ as $m \rightarrow \infty$.*

Proof. Suppose for a contradiction that $d(\mathcal{D}(W_1), (\phi_N)^m(\mathcal{D}(W_2)))$ is bounded by some integer Z for all $m \in \mathbb{N}$. Then, for every m , there is a set of essential curves $\{\alpha_k^m\}_{k=0}^Z$ such that $[\alpha_0^m] \in \mathcal{D}(W_1)$, $[\alpha_Z^m] \in (\phi_N)^m(\mathcal{D}(W_2))$, and $\alpha_k^m \cap \alpha_{k+1}^m = \emptyset$. It follows that there is a curve $\beta_m \in \mathcal{D}(W_2)$ such that $(\phi_N)^m(\beta_m) = \alpha_Z^m$. For these β_m 's, the corresponding points in $\mathcal{PM}\mathcal{L}(F_0)$ avoid an open neighborhood of the unstable lamination λ_N^- , since λ_N^- is not contained in $\overline{\mathcal{D}(W_2)}$ by the above claim. Thus, the points $[(\phi_N)^m(\beta_m)] = [\alpha_Z^m]$ converge to λ_N^+ as a sequence in $\mathcal{PM}\mathcal{L}(F_0)$. Inductively pass to subsequences exactly Z times to ensure that the k -th sequence $\{\alpha_k^m\}$ also gives a convergent sequence in $\mathcal{PM}\mathcal{L}(F_0)$ for $k = Z - 1, Z - 2, \dots, 0$. Denote the limit of the sequence $\{[\alpha_k^m]\}$ by λ_k . Note that $\lambda_Z = \lambda_N^+$, in particular, $\lambda_Z \neq \lambda_0$ also by the above claim since $\lambda_0 \in \overline{\mathcal{D}(W_1)}$ but $\lambda_N^+ \notin \overline{\mathcal{D}(W_1)}$. Then, as $\alpha_k^m \cap \alpha_{k+1}^m = \emptyset$, we have $\iota(\lambda_k, \lambda_{k+1}) = 0$, abusing notation slightly, where $\iota(\cdot, \cdot)$ denotes the geometric intersection number. Since ι is a continuous function $\mathcal{ML}(F_0) \times \mathcal{ML}(F_0) \rightarrow \mathbb{R}$, it follows that λ_Z and λ_{Z-1} must be the same point of $\mathcal{PM}\mathcal{L}(F_0)$ for λ_N^+ is minimal and uniquely ergodic. Inductively, $\lambda_k = \lambda_{k-1}$ in $\mathcal{PM}\mathcal{L}(F_0)$ which implies that $\lambda_Z = \lambda_0$. This is a contradiction. □

3.4. Replacement

To complete the proof of Theorem 1, we replace V_2 in M via an extension of the map $(\widehat{\phi_N})^m$ on ∂V_2 as follows.

Recall that, by constructions, the extensions $\widehat{\phi}$ and $\widehat{\tau}$ on F are isotopic to the identity on F . Thus $\phi_N = \tau^N \circ \phi \circ \tau^{-N}$ has a natural extension $\widehat{\phi_N}$ on F which is isotopic to the identity on F . Set $m = 2bm_0 + 1$ for some integer m_0 , and consider the map $(\phi_N)^m$, whose natural extension $(\widehat{\phi_N})^m$ is isotopic to the identity on F also.

Let $\Phi_m : F \times [0, 1] \rightarrow F \times [0, 1]$ be an isotopy from $(\widehat{\phi_N})^m$ to the identity with $\Phi_m(x, 0) = ((\widehat{\phi_N})^m(x), 0)$ and $\Phi_m(x, 1) = (x, 1)$. Note that $\Phi_m(x_j, 0) = ((\widehat{\phi_N})^m(x_j), 0) = (x_{j-1}, 0)$ for $1 \leq j \leq 2b-1$ and $\Phi_m(x_1, 0) = ((\widehat{\phi_N})^m(x_1), 0) = (x_{2b}, 0)$, since $\widehat{\phi}$ maps x_{j+1} to x_j and x_1 to x_{2b} , $\widehat{\tau}$ keeps x_j 's invariant, and

$m = 2bm_0 + 1$ with $m_0 \in \mathbb{Z}$. Regarding $F \times [0, 1] \subset V_2$, this map $\Phi_m : F \times [0, 1] \rightarrow F \times [0, 1]$ can be naturally extended to the map $\widehat{\Phi}_m : V_2 \rightarrow V_2$, since $\Phi_m|_{F \times \{1\}}$ is the identity map.

Now we replace V_2 by $\widehat{\Phi}_m(V_2)$. Precisely, when we regard M as $V_1 \cup_h V_2$ with a gluing map $h : \partial V_1 \rightarrow \partial V_2$, we set M_m as $V_1 \cup_h \widehat{\Phi}_m(V_2)$.

Note that this manifold M_m can be regarded as $V_1 \cup_{(\widehat{\Phi}_m)^{-1} \circ h} V_2$ with the gluing map $(\widehat{\Phi}_m)^{-1} \circ h : \partial V_1 \rightarrow \partial V_2$. Since $(\widehat{\Phi}_m)^{-1}$ is isotopic to the identity map on ∂V_2 , the gluing map $(\widehat{\Phi}_m)^{-1} \circ h$ is isotopic to h . This implies that M_m is homeomorphic to M . In particular, $(\widehat{\Phi}_m)^{-1} \circ h(\partial V_1) = \partial V_2$ in M_m is identified with F in M .

In M_m , we have a knot K_m , as a union of $\alpha_{1,1} \cup \dots \cup \alpha_{1,b}$ in V_1 and $\widehat{\Phi}_m(\alpha_{2,1} \cup \dots \cup \alpha_{2,b})$ in $\widehat{\Phi}_m(V_2)$. This K_m has a b -bridge presentation with respect to the Heegaard surface obtained from ∂V_i 's. Under the identification observed as above, the knot K_m can be regarded as in M , and it has a b -bridge presentation with respect to F .

Note that the distance of the bridge splitting of K_m with respect to F is equal to $d(\mathcal{D}(W_1), \mathcal{D}(\widehat{\Phi}_m(W_2)))$ in $\mathcal{C}(F_0)$. On the other hand, by construction, we see that $\widehat{\Phi}_m|_{F_0} = (\phi_N)^m$. It follows that $\mathcal{D}(\widehat{\Phi}_m(W_2)) = (\phi_N)^m(\mathcal{D}(W_2))$ in $\mathcal{C}(F_0)$. Thus the distance of the bridge splitting of K_m is equal to

$$d(\mathcal{D}(W_1), (\phi_N)^m(\mathcal{D}(W_2)))$$

in $\mathcal{C}(F_0)$.

Consequently, together with Claim 2, we see that the distance of the bridge splitting of K_m goes to infinity as $m \rightarrow \infty$. This completes the proof of Theorem 1.

4. Application

Let K be a knot in a closed 3-manifold M . The *meridional destabilizing number* of K , denoted by $md(K)$, is defined by the maximal number of m such that (M, K) admits a $(t(K) - m + 1, m)$ -bridge splitting. Here $t(K)$ is the tunnel number of K , i.e., the minimal number t such that (M, K) admits a Heegaard surface of genus $t + 1$. See [24] for the original definition. Note that $md(K) \leq t(K) + 1$. As an application of Theorem 1, we prove Corollary 2 described in Section 1 by using the following Tomova's theorem.

Lemma 1 (c.f. [26, Theorem 10.3]). *Let K be a non-trivial knot in a closed orientable irreducible 3-manifold M and $\mathcal{S} = (S, S \cap K)$ a bridge surface of (M, K) such that $|S \cap K| > 4$ if S is a 2-sphere. If $\mathcal{S}' = (S', S' \cap K)$ is also a bridge surface or a Heegaard surface of (M, K) , then*

- (1) $d(\mathcal{S}) \leq 2 - \chi(S' - K)$ or
- (2) \mathcal{S}' is obtained from \mathcal{S} by stabilization, meridional stabilization and perturbation.

Proof of Corollary 2. Let t, m be a pair of integers satisfying $t \geq 1$ and $m \geq 0$ with $m \leq t+1$. We will show that there exists a knot $K \subset S^3$ of tunnel number t and of meridional destabilizing number m .

First consider the case of $m = 0$. It follows from [21, Theorem 4.2] that for any positive integer $t \geq 1$, there is a knot $K \subset S^3$ with $t(K) = t$ so that K admits no $(t, 1)$ -bridge surface. This implies that there exists a knot $K \subset S^3$ with $t(K) = t$ and $md(K) = 0$.

Next we consider $m > 0$. It follows from Theorem 1 that there exists a knot $K \subset S^3$ so that K admits a $(t - m + 1, m)$ -bridge surface \mathcal{S} of distance greater than $2t + 2$. Then the knot K is shown to satisfy $t(K) = t$ and $md(K) = m$ as follows.

We have a $(t + 1, 0)$ -bridge surface from \mathcal{S} by m -times meridional stabilizations, and hence we see $t(K) \leq t$. Suppose that $t(K) < t$, i.e., K admits a $(t, 0)$ -bridge surface \mathcal{S}' . Then, since $d(\mathcal{S}) > 2t = 2 - \chi(\mathcal{S}' - K)$, it follows from Lemma 1 that \mathcal{S}' is obtained from \mathcal{S} by stabilization, meridional stabilization and perturbation. We can observe that a $(t - m + 2, m)$ -bridge surface is obtained from \mathcal{S} by one-time stabilization, a $(t - m + 2, m - 1)$ -bridge surface by one-time meridional stabilization, and a $(t - m + 1, m + 1)$ -bridge surface by one-time perturbation. These imply that we cannot have a $(t, 0)$ -bridge surface, which contradicts that \mathcal{S}' is obtained from \mathcal{S} by such operations. Hence we see $t(K) = t$. Similarly, we see that K admits no $(t - m, m + 1)$ -bridge surface and hence $md(K) \leq m$. This implies $md(K) = m$. \square

Remark 6. We can improve Corollary 2 as follows: For any integers $t \geq 1$, $m \geq 0$ with $m \leq t + 1$ and $b \geq 0$, there exists a knot $K \subset S^3$ with $t(K) = t$ and $md(K) = m$ which admits no $(t - m, b)$ -bridge splitting. This is pointed out by Yo'av Rieck.

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