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REFLEXIVE PROPERTY ON IDEMPOTENTS

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Abstract. The reflexive property for ideals was introduced by Mason and has important roles in noncommutative ring theory. In this note we study the structure of idempotents satisfying the reflexive property and introduce reflexive-idempotents-property (simply, RIP) as a generalization. It is proved that the RIP can go up to polynomial rings, power series rings, and Dorroh extensions. The structure of non-Abelian RIP rings of minimal order (with or without identity) is completely investigated.

1. Introduction

Throughout this paper all rings are associative with identity unless otherwise stated. Given a ring R the polynomial ring (resp., the power series ring) with an indeterminate x over R is denoted by $R[x]$ (resp., $R[[x]]$). For any polynomial $f(x)$ in $R[x]$, let $C_{f(x)}$ denote the set of all coefficients of $f(x)$. Denote the n by n full matrix ring over R by $\mathrm{Mat}_n(R)$ and the n by n upper triangular matrix ring over R by $U_n(R)$. Use E_{ij} for the matrix with (i,j)-entry 1 and elsewhere 0. Let $Id(R)$ be the set of all idempotent elements of R. Denote $\{a \in U_n(R) \mid \text{the diagonal entries of } a \text{ are all equal}\}$ by $D_n(R)$. $\mathbb{Z}$ and $\mathbb{Z}_n$ denote the ring of integers and the ring of integers modulo n, respectively. $GF(p^n)$ denotes the Galois field of order p^n for a prime p and $n \geq 1$. $J(R)$ denotes the Jacobson radical of R. $|S|$ denotes the cardinality of given a set S.

Mason [18] introduced the reflexive property for ideals, and then this concept was generalized by Kim and Baik [9, 10] by defining idempotent reflexive right ideals and rings. A right ideal I of a ring R (possibly without identity) is called *reflexive* [18] if $aRb \subseteq I$ implies $bRa \subseteq I$ for $a, b \in R$. R is called *reflexive* if 0 is a reflexive ideal (i.e., $aRb = 0$ implies $bRa = 0$ for $a, b \in R$.) In [12], Kwak and Lee characterized aspects of the reflexive and one-sided idempotent reflexive properties, showing that the concept of idempotent reflexive ring is not left-right symmetric [12, Example 3.3]. For a one-sided ideal I of a ring R, I is

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called *right idempotent reflexive* [12, Definition 3.1] if $aRe \subseteq I$ implies $eRa \subseteq I$ for any $a, e^2 = e \in R$, and a ring R is called *right idempotent reflexive* if 0 is a right reflexive ideal. Left idempotent reflexive ideals and rings are defined similarly. If a ring is both left and right idempotent reflexive, then the ring is called an *idempotent reflexive* ring (refs. [9, 10]). Reflexive rings are obviously one-sided idempotent reflexive, but not conversely by [12, Example 2.3(1)]. It is proved that the reflexive condition is Morita invariant [12, Theorem 2.6]. A (right idempotent) reflexive ring which is not semiprime (resp., reflexive) is also constructed from any semiprime (resp., reflexive) ring [12, Proposition 2.5 and Theorem 3.9].

Recall that a ring is *reduced* if it has no nonzero nilpotent elements. Cohn [4] called a ring R *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Due to Bell [2], a right (or left) ideal I of a ring R is said to have the *insertion of factors property* (simply, IFP) if $ab \in I$ implies $aRb \subseteq I$ for $a, b \in R$. A ring R is called *IFP* if the zero ideal of R has the IFP. A ring R is called *2-primal* [3] if the prime radical of R coincides with the set of all nilpotent elements of R. In [17], a ring R is called *NI* if the upper nilradical of R coincides with the set of all nilpotent elements of R. IFP rings are 2-primal and 2-primal rings are NI, but the converses are not true. Also note that the prime radical of a 2-primal ring and the upper nilradical of an NI ring are reflexive ideals.

Rege and Chhawchharia [20] called a ring R *Armendariz* if $ab = 0$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$ whenever any polynomials $f(x), g(x)$ over R satisfy $f(x)g(x) = 0$. A ring is called *Abelian* if every idempotent in it is central. It is well-known that IFP rings and Armendariz rings are Abelian. Every Abelian ring is idempotent reflexive, and hence IFP rings and Armendariz rings are idempotent reflexive. Note that reversible rings are reflexive by [12, Proposition 2.2], but IFP and the reflexive ring property don't imply each other by [12, Example 2.3].

In this paper, we study on rings with reflexive-idempotents-property (simply, RIP) which is a generalization of one-sided idempotent reflexive rings. It is proved that the one-sided idempotent reflexive property and the reflexive-idempotents-property coincide for a right principally quasi-Baer ring (Proposition 2.11), and that R is an RIP ring if and only if $R[x]$ is an RIP ring if and only if $R[[x]]$ is an RIP ring (Theorem 3.1), and that a ring R is RIP if and only if $D_n(R)$ is RIP (Theorem 3.3). If R is a minimal non-Abelian RIP ring, then R is of order 16 and is isomorphic to $\mathrm{Mat}_2(\mathbb{Z}_2)$ (Theorem 4.2). We additionally give an example that RIP rings need not be directly finite (Example 2.13).

2. Properties of RIP rings

We begin with the following.

Definition 2.1. A ring R is called to have *the reflexive-idempotents-property* (simply, *RIP*) if R satisfies the property that

$$eRf = 0 \text{ implies } fRe = 0 \text{ for any } e, f \in Id(R).$$

A ring shall be called *RIP* if it satisfies the reflexive-idempotents-property.

It can be easily checked that every one-sided idempotent reflexive ring is RIP, entailing that Abelian rings are RIP. Hence, the class of RIP rings contains IFP rings and Armendariz rings. Note that the IFP ring property and the Armendariz ring property don't imply each other in general.

The following example shows that there exist RIP rings which are not one-sided idempotent reflexive.

Example 2.2. Let F be a field of characteristic zero and $A = F\langle a, b, c\rangle$ be the free algebra with three non-commuting indeterminates a, b, c over F.

(1) Due to [12, Example 3.3], let I be the ideal of A generated by

$$aAb,\ a^2 - a$$

and $R = A/I$. Then R is a right idempotent reflexive ring but not left idempotent reflexive by the computation in [12, Example 3.3]. Note that R is an RIP ring since R is right idempotent reflexive.

(2) Due to [12, Example 3.3], let I be the ideal of R generated by

$$aAb,\ b^2 - b$$

and $R = A/I$. Then R is left idempotent reflexive but not right idempotent reflexive by the computation in [12, Example 3.3]. Note that R is an RIP ring since R is left idempotent reflexive.

The classes of RIP rings and NI rings do not contain each other by the following example.

Example 2.3. (1) Let F be a field and $R = \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$. Then R is a 2-primal ring and so an NI ring. For $E_{11}, E_{22} \in Id(R)$, we have $E_{22}RE_{11} = 0$ but $E_{11}RE_{22} \neq 0$. Thus, R is not RIP.

We also see that the class of RIP rings is not closed under subrings: Indeed, $R = \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$ is a subring of $\mathrm{Mat}_2(F)$ which is a reflexive ring (and so an RIP ring) by [12, Theorem 2.6(2)].

(2) Let K be a field and $n \geq 2$. Let $R = K\langle a, b \mid a^n = 0\rangle$ be the free algebra with two non-commuting indeterminates a, b over K with $a^n = 0$. Then R is an Armendariz ring such that the set of all nilpotent elements of R is not an ideal by [1, Example 4.8]. Hence, R is RIP but not NI.

For a nonempty subset X of a ring R, we write $r_R(X) = \{c \in R \mid Xc = 0\}$ which is called the *right annihilator* of X in R. The left annihilator is defined similarly and denoted by $\ell_R(X)$.

Proposition 2.4. *For a ring R, the following are equivalent*:

(1) *R is an RIP ring*;

(2) *For $e \in Id(R)$, $r_R(eR) \cap Id(R) = \ell_R(Re) \cap Id(R)$*;

(3) *For any nonempty subsets E and F of $Id(R)$, $ERF = 0$ implies $FRE = 0$*;

(4) *$IJ = 0$ implies $JI = 0$ for all right ideals I, J of R where I and J are right ideals generated by subsets of $Id(R)$; and*

(5) *$IJ = 0$ implies $JI = 0$ for all ideals I, J of R where I and J are ideals generated by subsets of $Id(R)$.*

Proof. (1)$\Leftrightarrow$(2) is shown from the definition.

(1)$\Rightarrow$(3) Let $ERF = 0$ for $E, F \subseteq Id(R)$. Then for any $e \in E, f \in F$, we have $eRf = 0$. Since R is RIP, $fRe = 0$ and so $FRE = 0$.

(3)$\Rightarrow$(4) Let $I = ER$ and $J = FR$ where $E, F \subseteq Id(R)$. Suppose $IJ = 0$. Then $ERF \subseteq IJ = 0$ and so by the condition (3), $FRE = 0$ and so $JI = 0$.

(4)$\Rightarrow$(5) Let $I = RER$ and $J = RFR$ where $E, F \subseteq Id(R)$. Suppose $IJ = 0$. Then $ERFR \subseteq IJ = 0$. By the condition (4), $FRER = 0$ and so $RFRER = 0$. Hence, $JI = 0$.

(5)$\Rightarrow$(1) Suppose that $eRf = 0$ for $e, f \in Id(R)$. Then $ReRRfR = 0$ and so by the condition (5), we have $RfRReR = 0$, entailing that R is an RIP ring. $\square$

Proposition 2.5. (1) *If R is an RIP ring, then so is eRe for each $e \in Id(R)$.*

(2) *Let R/I be an RIP ring for some ideal I of a ring R. If I is a reduced ring (possibly without identity), then R is RIP.*

(3) *For a central idempotent e of a ring R, eR and $(1-e)R$ are RIP if and only if R is RIP.*

Proof. (1) Suppose that R is an RIP ring. Let $f, f' \in Id(eRe)$ such that $f(eRe)f' = 0$. Then $fe = f = ef$ and $f'e = f' = ef'$. Since R is RIP, $fRf' = 0$ implies $0 = f'Rf = f'(eRe)f$. Thus eRe is RIP.

(2) Let $eRf = 0$ with $e, f \in Id(R)$. Since R/I is RIP, we have $fRe \subseteq I$, and so $(fReR)^2 = 0$ implies $fRe = 0$ since I is reduced. Hence, R is RIP.

(3) Assume that eR and $(1-e)R$ are RIP. Let $fRf' = 0$ for $f, f' \in Id(R)$. Then $efRf' = 0$ and $(1-e)fRf' = 0$, and so $ef'Rf = 0$ and $(1-e)f'Rf = 0$ by assumption. Thus, $f'Rf = ef'Rf + (1-e)f'Rf = 0$, proving that R is RIP. The converse is trivial by the result (1). $\square$

If $\mathrm{Mat}_n(R)$ is an RIP ring, then R is an RIP ring by Proposition 2.5(1), since $R \cong RE_{11} = E_{11}\mathrm{Mat}_n(R)E_{11}$. But we don't know whether the converse is true.

Question. If R is an RIP ring, then is $\mathrm{Mat}_n(R)$ RIP?

The condition "I is a reduced ring" in Proposition 2.5(2) cannot be dropped by the following.

Example 2.6. Consider the ring $R = \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$ where F is a field, in Example 2.3, which is not RIP. The only nonzero proper ideals of R are $I_1 = \left(\begin{smallmatrix} F & F \\ 0 & 0 \end{smallmatrix}\right), I_2 = \left(\begin{smallmatrix} 0 & F \\ 0 & F \end{smallmatrix}\right)$ and $I_3 = \left(\begin{smallmatrix} 0 & F \\ 0 & 0 \end{smallmatrix}\right)$, and they are not obviously reduced. But R/I_1 and R/I_2 are isomorphic to F and $R/I_3 = \{\left(\begin{smallmatrix} a & 0 \\ 0 & c \end{smallmatrix}\right) + I_3 \mid a, c \in F\}$ is a reduced ring. Therefore each R/I_i is RIP for $i = 1, 2, 3$.

Recall that a ring R is called *local* if $R/J(R)$ is a division ring. A ring R is called *semilocal* if $R/J(R)$ is semisimple Artinian, and R is called *semiperfect* if R is semilocal and idempotents can be lifted modulo $J(R)$. Local rings are Abelian and semilocal.

Proposition 2.7. (1) *Let R_λ $(\lambda \in \Lambda)$ be rings. The following are equivalent*:

(i) *R_λ is RIP for each $\lambda \in \Lambda$.*

(ii) *The direct product of R_λ $(\lambda \in \Lambda)$ is RIP.*

(iii) *The direct sum* (*possibly without identity*) *of R_λ $(\lambda \in \Lambda)$ is RIP.*

(2) *A ring R is Abelian and semiperfect if and only if R is a finite direct sum of local RIP rings.*

Proof. (1) (i)⇒(ii) Assume (i). Let R be the direct product of RIP rings R_λ $(\lambda \in \Lambda)$. Suppose that $(e_\lambda)R(f_\lambda) = 0$ for $(e_\lambda), (f_\lambda) \in Id(R)$. Then $e_\lambda R_\lambda f_\lambda = 0$ for each $\lambda \in \Lambda$. Note that $e_\lambda, f_\lambda \in Id(R_\lambda)$ for each $\lambda \in \Lambda$. Since R_λ is RIP, $f_\lambda R_\lambda e_\lambda = 0$ for each $\lambda \in \Lambda$. Thus $(f_\lambda)R(e_\lambda) = 0$ and so R is RIP.

(ii)⇒(i) Assume (ii). Let $e = (e_\lambda) \in Id(R)$ such that $e_\lambda = 1$ and $e_\gamma = 0$ for all $\gamma \neq \lambda$. Then $eRe \cong R_\lambda$, and hence R_λ is RIP by Proposition 2.5(1).

(i)⇔(iii) It is similar to the above.

(2) Suppose that R is Abelian and semiperfect. Since R is semiperfect, R has a finite orthogonal set $\{e_1, e_2, \ldots, e_n\}$ of local idempotents whose sum is 1 by [14, Proposition 3.7.2], say $R = \sum_{i=1}^{n} e_i R$ such that each $e_i R e_i$ is a local ring. Since R is Abelian, each $e_i R$ is ideals of R with $e_i R = e_i R e_i$. But each $e_i R$ is also an RIP ring by Proposition 2.5(1) since R is Abelian.

Conversely assume that R is a finite direct sum of local RIP rings. Then R is Abelian and semiperfect since local rings are both Abelian and semiperfect. □

The class of RIP rings is not closed under homomorphic images.

Example 2.8. Let K be a field and $R = K\langle a, b\rangle$. Then R is reduced and so reflexive. Let I be the ideal of R generated by

$$aRb,\ a^2 - a \text{ and } b^2 - b.$$

Then $aRb \subseteq I$, but $ba \in bRa \nsubseteq I$ for $a, b \in Id(R/I)$. We conclude that R/I is not RIP.

Example 2.9. Let R be any ring and $n \geq 2$. Then the n by n upper triangular matrix ring $U_n(R)$ is not an RIP ring. For $E_{11}, E_{nn} \in Id(U_n(R))$, we have $E_{nn}U_n(R)E_{11} = 0$ but $E_{11}U_n(R)E_{nn} = E_{11}(RE_{1n} + RE_{2n} + \cdots + RE_{nn}) = RE_{1n} \neq 0$.

Next let R be any nonzero ring possibly without identity and suppose that there exist nonzero idempotents $e, f \in R$ with relations $ef = e$. Then the n by n upper triangular matrix ring $U_n(R)$ $(n \geq 2)$ is not an RIP ring. Let $A_{ij} = eE_{ij}$ and $B_{ij} = fE_{ij}$ for $i, j = 1, \ldots, n$. Then $A_{ii}, B_{ii} \in Id(U_n(R))$, and we have $B_{nn}U_n(R)A_{11} = 0$ but $A_{11}U_n(R)B_{nn}$ contains $A_{11}(B_{1n} + B_{2n} + \cdots + B_{nn}) = A_{11}B_{1n} = A_{1n} = efE_{in} = eE_{in} \neq 0$.

For a ring R, $e \in Id(R)$ is called *right* (resp., *left*) *semicentral* if $er = ere$ (resp., $re = ere$) for all $r \in R$. We use $S_r(R)$ (resp., $S_\ell(R)$) and $B(R)$ for the sets of right (resp., left) semicentral idempotents and central idempotents of R. Observe that $S_r(R) \cap S_\ell(R) = B(R)$ and if R is a semiprime ring, then $S_\ell(R) = B(R) = S_r(R)$.

For a prime ideal P of a ring R, $O(P) = \{a \in R \mid aRb = 0$ for some $b \in R \backslash P\}$, and a ring R is called *torsion free* if $O(P) = 0$ for some prime ideal P of R.

Proposition 2.10. (1) *Every one-sided semicentral idempotent element of an RIP ring R is central.*

(2) *Let R be an RIP ring. If R is torsion free, then $B(R) = \{0, 1\}$.*

Proof. (1) $e \in S_\ell(R)$ if and only if $(1-e)Re = 0$ if and only if $eR(1-e) = 0$ if and only if $e \in S_r(R)$.

(2) Assume that $O(P) = 0$ for some prime ideal P of R. Let $e \in B(R) = S_\ell(R)$ by (1). Then $(1-e)Re = 0$ and thus $eR(1-e) = 0$ since R is RIP. If $e \notin P$, then $1 - e \in O(P) = \{0\}$, and so $e = 1$. If $e \in P$, then $1 - e \notin P$, and hence $e \in O(P) = \{0\}$, and so $e = 0$. □

Recall that a ring R is called *right* (resp., *left*) *principally quasi-Baer* (or simply, *right* (resp., *left*) *p.q.-Baer*) if the right (resp., left) annihilator of a principal right (resp., left) ideal of R is generated by an idempotent, and that R is called *p.q.-Baer* if it is both left and right p.q.-Baer.

Proposition 2.11. *Let R be a right p.q.-Baer ring. Then the following are equivalent*:

(1) *R is a semiprime ring*;
(2) *R is a reflexive ring*;
(3) *R is a right idempotent reflexive ring*;
(4) *R is a left idempotent reflexive ring*;
(5) *R is an RIP ring*; *and*
(6) $S_\ell(R) = B(R) = S_r(R)$.

Proof. (1)⇔(2)⇔(3)⇔(4)⇔(6) by [12, Proposition 3.15], (4)⇒(5) by definition and (5)⇒(6) by Proposition 2.10(1). □

Corollary 2.12. *Let R be an RIP ring. Then R is a right p.q.-Baer ring if and only if R is a p.q.-Baer ring.*

Proof. Every right p.q.-Baer and RIP ring R is one-sided idempotent reflexive by Proposition 2.11. Thus R is left p.q.-Baer by [12, Proposition 3.16]. □

A ring R is usually called *directly finite* if $ab = 1$ implies $ba = 1$ for $a, b \in R$. Note that both NI rings and Abelian rings are directly finite by [7, Proposition 2.7(1)] and [15, Lemma 3.4], respectively. As in Example 2.3, $\left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$, where F is a field, is not RIP but directly finite. In the following we see RIP rings which are not directly finite.

Example 2.13. (1) Let R be the ring of column finite infinite matrices over a field. Then R is von Neumann regular and so semiprime (hence RIP), but R is not directly finite as can be seen by the matrices $A = E_{12} + E_{23} + \cdots + E_{n(n+1)} + \cdots$ and $B = E_{21} + E_{32} + \cdots + E_{(n+1)n} + \cdots$ such that $AB = 1$ but $BA = E_{22} + E_{33} + \cdots + E_{(n+1)(n+1)} + \cdots \neq 1$.

(2) Shepherdson constructed in [21] a domain D such that $M_2(D)$ is not directly finite. But $M_2(D)$ is prime (hence RIP).

Note. Let $K = \mathbb{Z}_2$ and $A = K\langle a, b\rangle$ be the free algebra generated by the noncommuting indeterminates a, b over K. Let I be the ideal of A generated by $ab - 1$. Set $R = A/I$ and identify a and b with their images in R for simplicity. Then R has the relations $ab = 1$. It is obviously true that R is not directly finite.

$K[x, y]$ denotes the polynomial ring with indeterminates over K. Every element in R is expressed by

$$k + a^i f_0(a) + b^j f_1(b) + b^m f_2(a, b),$$

where $k \in K$, $i, j, m \geq 1$, $f_0(x), f_1(x) \in K[x]$, $f_2(x, y) \in K[x, y]$, and every nonzero non-constant monomial of $f_2(a, b)$ is of the form $b^s a^t$ ($s \geq 0$ and $a \geq 1$). Let $0 \neq e = k + a^i f_0(a) + b^j f_1(b) + b^m f_2(a, b)$ be such that $e^2 = e$.

`Case 1.` $k = 0$, i.e., $e = a^i f_0(a) + b^j f_1(b) + b^m f_2(a, b)$.

In this case we first have

$$\begin{aligned} a^i f_0(a) &= a^i f_0(a)(a^i f_0(a) + b^j f_1(b) + b^m f_2(a, b)) \\ &= f_0(a)(a^{2i} f_0(a) + a^i b^j f_1(b) + a^i b^m f_2(a, b)). \end{aligned}$$

Here assume $f_0(a) \neq 0$. Then $f_1(b) = 0$ and $e = a^i f_0(a) + b^j f_1(b) + b^m f_2(a, b)$. But $e^2 = e$ yields $f_2(a, b) = 0$, entailing $e = a^i f_0(a)$. We also get $e = a^i f_0(a) = 0$ from $e^2 = e$, a contradiction. So $a^i f_0(a) = 0$, i.e., $e = b^j f_1(b) + b^m f_2(a, b)$. Similar computation gives $b^j f_1(b) = 0$, so $e = b^m f_2(a, b)$. Then we have $b^m f_2(a, b) = e = e^2 = b^m f_2(a, b) b^m f_2(a, b)$. Thus we can conclude that $e = b^m f_2(a, b)$ is of the form

$$e = \sum_{\text{finite}} b^s a^t \quad \text{for } s, t \geq 1.$$

`Case 2.` $k = 1$, i.e., $e = 1 + a^i f_0(a) + b^j f_1(b) + b^m f_2(a, b)$.

In this case we first have $1 + f + f + f^2 = e^2 = e = 1 + f$, where $f = a^i f_0(a) + b^j f_1(b) + b^m f_2(a, b)$. Then $f + f^2 = 0$, so $f = -f^2 = -(-f)^2$ and $(-f)^2 = -f$. By the computation in the case 1, we have $-f$ is of the form $\sum_{\text{finite}} b^s a^t$ for $s, t \geq 1$. Thus e is of the form

$$e = 1 - \sum_{\text{finite}} b^s a^t \quad \text{for } s, t \geq 1,$$

where it is understood that $a^0 = b^0 = 1$.

Therefore the set of all idempotents in R is

$$\{0, 1, \sum_{\text{finite}} b^s a^t, 1 - \sum_{\text{finite}} b^s a^t \text{ for } s, t \geq 1\}.$$

For example, we can see $b^{i-1}a^{i-1} - b^i a^i$ (for $i \geq 1$) to be found by Jacobson in [8]. Here we raise the following two questions:

(1) What are more exact expressions of the idempotents $\sum_{\text{finite}} b^s a^t$?

(2) Is the ring R RIP?

3. Extensions of RIP rings

In this section we observe the reflexive-idempotents-property of various kinds of ring extensions. We first consider the cases of polynomial rings and power series rings. The following includes the result of [12, Theorem 3.13].

Theorem 3.1. (1) *R is an RIP ring if and only if $R[x]$ is an RIP ring.*

(2) *R is an RIP ring if and only if $R[[x]]$ is an RIP ring.*

(3) *R is a right (resp., left) idempotent reflexive ring if and only if $R[x]$ is a right (resp., left) idempotent reflexive ring.*

(4) *R is a right (resp., left) idempotent reflexive ring if and only if $R[[x]]$ is a right (resp., left) idempotent reflexive ring.*

Proof. We apply the method in the proof of [12, Theorem 3.13].

(1) Suppose that R is an RIP ring. Note that $f(x)R[x]g(x) = 0$ if and only if $f(x)Rg(x) = 0$ for $f(x), g(x) \in R[x]$. Let $e(x) = \sum_{i=0}^m e_i x^i, f(x) = \sum_{j=0}^n f_j x^j \in Id(R[x])$ such that $e(x)Rf(x) = 0$. From $e^2(x) = e(x)$, we have

$$e_0^2 = e_0, e_h = \sum_{u+v=h} e_u e_v \text{ for } h = 1, 2, \ldots, m.$$

In this situation, note that $e_h \in (e_0, \ldots, e_{h-1})$ for all h where $(e_0, \ldots, e_{h-1})$ denotes the ideal of R generated by $e_0, \ldots, e_{h-1}$. Thus $e_h \in (e_0)$ inductively for all $h = 0, 1, \ldots, m$, where $(e_0) = Re_0R$. Similarly we have $f_k \in (f_0)$ for all $k = 0, 1, \ldots, n$.

If $e_0 = 0$, then $e(x) = 0$ since every e_h is contained in (e_0), entailing $f(x)Re(x) = 0$. So assume $e_0 \neq 0$. From $e(x)Rf(x) = 0$, we get $e_0Rf_0 = 0$. Since R is RIP and $e_0, f_0 \in Id(R)$, we have $f_0Re_0 = 0$. This yields $(Rf_0R)R(Re_0R) = 0$. But $e_i \in Re_0R$ and $f_j \in Rf_0R$ for all i, j. This implies $f_jRe_i = 0$ for all i, j, entailing $f(x)R[x]e(x) = 0$. We conclude that $R[x]$ is RIP.

Conversely, suppose that $R[x]$ is RIP. Let $eRf = 0$ for $e, f \in Id(R)$. Then $eR[x]f = 0$, and so $fR[x]e = 0$ and $fRe = 0$. Thus R is RIP.

(2) The proof is similar by much to the proof of (1).

(3) The proof for an left idempotent reflexive ring is similar to the proof of (1), and the case of a right idempotent reflexive ring is also obtained symmetrically but we write it for completeness. Suppose that R is a right idempotent reflexive ring. Let $f(x) = \sum_{i=0}^m a_i x^i, e^2(x) = e(x) = \sum_{j=0}^n e_j x^j, \in R[x]$ such

that $f(x)Re(x) = 0$. By the proof of (1), we have $e_h \in (e_0)$ inductively for all $h = 0, 1, \ldots, n$. We assume $e_0 \neq 0$. From $f(x)Re(x) = 0$, we get $a_0Re_0 = 0$ and $a_0Re_0R = 0$. This yields $a_0Re_h = 0$ (equivalently, $a_0Re(x) = 0$) for all $h = 0, 1, \ldots, n$ since $e_h \in (e_0)$ for all h. Consequently $(\sum_{i=1}^{m} a_ix^i)Re(x) = 0$. Then $a_1Re_0 = 0$, and we also have $a_1Re(x) = 0$ similarly. Proceeding in this manner, we inductively obtain $a_iRe(x) = 0$ for all $i = 0, 1, \ldots, m$. This implies that

$$a_iRe_j = 0 \text{ for all } i = 0, 1, \ldots, m \text{ and } j = 0, 1, \ldots, n.$$

Especially, $a_iRe_0 = 0$ for all $i = 0, 1, \ldots, m$. Since R is right idempotent reflexive and $e_0 \in Id(R)$, we have $e_0Ra_i = 0$ (equivalently, $e_0Rf(x) = 0$) for all $i = 0, 1, \ldots, m$. This yields $Re_0Rf(x) = 0$. But $e_j \in Re_0R$ for all $j = 0, 1, \ldots, n$. This yields $e_jRa_i = 0$ for all i, j, entailing $e(x)R[x]f(x) = 0$. We conclude that $R[x]$ is right idempotent reflexive.

The converse can be shown by the similar argument to the converse proof of (1).

(4) The proof is similar by much to the proof of (3). □

Let R be an algebra over a commutative ring S. Following Dorroh [5], the *Dorroh extension of* R *by* S *is the Abelian group* $R \oplus S$ *with multiplication given by* $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$ *for* $r_i \in R$ *and* $s_i \in S$.

Theorem 3.2. *Let* R *be an algebra over a commutative ring* S*. Then* R *is RIP if and only if the Dorroh extension* D *of* R *by* S *is RIP.*

Proof. In the following computations, $s \in S$ is identified with $s1 \in R$. Note that $R = \{r + s \mid (r, s) \in D\}$, where $s = s1$.

Suppose that R is RIP. Let $(e_1, s_1)D(e_2, s_2) = 0$ for $(e_1, s_1), (e_2, s_2) \in Id(D)$. Since $(e_i, s_i) = (e_i^2 + 2s_ie_i, s_i^2)$, we have that $(e_i + s_i1)^2 = e_i^2 + 2s_ie_i + s_i^2 1 = e_i + s_i1$ is an idempotent in R. Since $(e_1, s_1)(r, 0)(e_2, s_2) = (e_1re_2 + s_1re_2 + s_2e_1r + s_1s_2r, 0) = (0, 0)$ and $(e_1 + s_11)r(e_2 + s_21) = e_1re_2 + s_1re_2 + s_2e_1r + s_1s_2r$, we have $(e_1 + s_11)R(e_2 + s_21) = 0$. Since R is RIP, we get $(e_2 + s_21)R(e_1 + s_11) = 0$. Hence $e_2re_1 + s_2re_1 + s_1e_2r + s_1s_2r = 0$ for all $r \in R$. Let $(r, s) \in D$. Then

$$\begin{aligned}(e_2, s_2)(r, s)(e_1, s_1) &= (e_2(r+s1)e_1 + s_2(r+s1)e_1 + s_1e_2(r+s1) + s_1s_2r, s_2ss_1)\\ &= (-s_1s_2s1, s_1ss_2) = (0, 0),\end{aligned}$$

where the last equality follows because $(e_1, s_1)D(e_2, s_2) = 0$ implies that $s_1ss_2 = 0$ for all $s \in S$. Therefore D is RIP.

Suppose now that D is RIP. Let $eRf = 0$ for $e, f \in Id(R)$. Then

$$(e, 0)(r, s)(f, 0) = (e(r + s1)f, 0) = (0, 0)$$

for all $(r, s) \in D$. Since D is RIP, we have that $(f, 0)D(e, 0) = 0$. In particular, $(f, 0)(r, s)(e, 0) = (fre, 0) = (0, 0)$ for all $r \in R$. Therefore $fRe = 0$ and thus R is RIP. □

For a ring R and $n \geq 2$, let

$$V_n(R) = \{m = (m_{ij}) \in D_n(R) \mid m_{st} = m_{(s+1)(t+1)} \text{ for } s = 1, \ldots, n-2 \text{ and } t = 2, \ldots, n-1\}.$$

For any ring R and $n \geq 2$, the n by n upper triangular matrix ring $U_n(R)$ is not RIP by Example 2.9; while $\mathrm{Mat}_n(R)$, $D_n(R)$ and $V_n(R)$ over a reflexive ring R are right idempotent reflexive rings by [12, Theorem 2.6(2) and Theorem 3.9], and so they are RIP rings. Moreover, we have the following.

Theorem 3.3. *Let R be a ring and $n \geq 2$.*

(1) *R is an RIP ring if and only if $D_n(R)$ is an RIP ring.*

(2) *R is an RIP ring if and only if $V_n(R)$ is an RIP ring.*

(3) *R is a right (resp., left) idempotent reflexive ring if and only if $D_n(R)$ a right (resp., left) idempotent reflexive ring.*

(4) *R is a right (resp., left) idempotent reflexive ring if and only if $V_n(R)$ is a right (resp., left) idempotent reflexive ring.*

Proof. We apply the method in the proof of [12, Theorem 3.9].

(1) Note that if $E^2 = E = (e_{ij}) \in D_n(R)$ with $e_{ii} = e$ for all $i = 1, \ldots, n$, then $e_{ij} \in ReR$ by the proof of [12, Theorem 3.9]. Suppose that $ED_n(R)F = 0$ for $E = (e_{ij}), F = (f_{kl}) \in Id(D_n(R))$. Let $D = (d_{uv})$ be any in $D_n(R)$. Set $e_{ii} = e, d_{uu} = d, f_{kk} = f$ for all $i, u, k = 1, \ldots, n$. Then $e, f \in Id(R)$. Note that d runs over R. Then $edf = 0$ for all $d \in R$ and hence $eRf = 0$. Since R is RIP and $e, f \in Id(R)$, we have $fRe = 0$. This yields $(RfR)R(ReR) = 0$. But $e_{ij} \in ReR$ and $f_{kl} \in RfR$ for all i, j, k, l. This implies $f_{kl}Re_{ij} = 0$ for all i, j, k, l, entailing $FD_n(R)E = 0$ and so $D_n(R)$ is RIP.

Conversely, assume that $D_n(R)$ is an RIP ring. Let $eRf = 0$ for $e, f \in Id(R)$. We have $ED_n(R)F = 0$ for $E = e\sum_{i=1}^n E_{ii}, F = f\sum_{i=1}^n E_{ii} \in Id(D_n(R))$, and thus $FD_n(R)E = 0$ by assumption. This entails $fRe = 0$ and therefore R is RIP.

(2) is the same as the proof of (1).

(3) The proof for a right idempotent reflexive ring is similar to the proof of (1), but we write it for completeness. Suppose that $AD_n(R)E = 0$ for $A = (a_{ij}), E^2 = E = (e_{kl}) \in D_n(R)$. Let $D = (d_{uv})$ be any in $D_n(R)$. Set $a_{ii} = a, d_{uu} = d, e_{kk} = e = e^2$ for all $i, u, k = 1, \ldots, n$. First, we get $ade = 0$ for all $d \in R$. So, we have $ade_{kl} = 0$ for all $d \in R$, since $aReR = 0$ and $e_{kl} \in ReR$ as noted in the proof of (1). By these two results, we can also obtain $a_{ij}de = 0$ through an induction on $j - i$. Since R is right idempotent reflexive and $e \in Id(R)$, we also get $eRa_{ij} = 0$ for all i, j, and then $ReRa_{ij} = 0$. But $e_{kl} \in ReR$ for any k, j. Then $e_{kl}Ra_{ij} = 0$ for all i, j, k, l. This yields $ED_n(R)A = 0$. Therefore $D_n(R)$ is a right idempotent reflexive ring for $n \geq 2$.

The converse can be shown by the similar argument to the converse proof of (1), and the proof for an left idempotent reflexive ring is symmetrically obtained to the above.

(4) is the same as the proof of (1). □

Recall that for a ring R and an (R,R)-bimodule M, the *trivial extension* of R by M is the ring $T(R,M) = R \oplus M$ with the usual addition and the following multiplication: $(r_1,m_1)(r_2,m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This is isomorphic to the ring of all matrices $\left(\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right)$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

Corollary 3.4. (1) *A ring R is RIP (right (left) idempotent reflexive) if and only if the trivial extension $T(R,R)$ is RIP (right (left) idempotent reflexive).*

(2) *A ring R is RIP (right (left) idempotent reflexive ring) if and only if $R[x]/(x^n)$ is an RIP (right (left) idempotent reflexive) ring for any positive integer n, where (x^n) is an ideal of $R[x]$ generated by x^n.*

Proof. It follows directly from Theorem 3.3 and the fact $V_n(R) \cong R[x]/(x^n)$ by [16]. □

Recall that an element u of a ring R is *right regular* if $ur = 0$ implies $r = 0$ for $r \in R$. Similarly, *left regular* elements can be defined. An element is *regular* if it is both left and right regular (and hence not a zero divisor).

Proposition 3.5. *Let M be a multiplicatively closed subset of a ring R consisting of central regular elements.*

(1) *If $M^{-1}R$ is RIP, then R is RIP.*

(2) *Suppose that every idempotent in $M^{-1}R$ is of the form $u^{-1}e$ with $e \in Id(R)$ and $u \in M$. If R is RIP, then $M^{-1}R$ is RIP.*

Proof. (1) Suppose that $M^{-1}R$ is RIP. Let $eRf = 0$ for $e, f \in Id(R)$. For any $r \in R$ with $w \in M$, $0 = w^{-1}erf = e(w^{-1}r)f$. So we have $e(M^{-1}R)f = 0$ and so $f(M^{-1}R)e = 0$, since $M^{-1}R$ is RIP. Thus $fRe = 0$, showing that R is RIP.

(2) Suppose that R is RIP. Let $\alpha(M^{-1}R)\beta = 0$ where $\alpha = u^{-1}e$, $\beta = v^{-1}f \in Id(M^{-1}R)$ with $e, f \in Id(R)$ and $u, v \in M$. Since M is contained in the center of R, we have $0 = (u^{-1}e)(w^{-1}r)(v^{-1}f) = (uwv)^{-1}(erf)$ for any $w^{-1}r \in M^{-1}R$, and so $eRf = 0$. Since R is RIP, $fRe = 0$ and hence $(uwv)^{-1}(fre) = 0$ for any $r \in R$. This shows that $\beta(M^{-1}R)\alpha = 0$, concluding that $M^{-1}R$ is RIP. □

In connection with the converse of Proposition 3.5(1), it is not true that there exist $e \in Id(R)$ and $u \in M$ such that $\alpha = u^{-1}e$, whenever $\alpha \in Id(M^{-1}R)$. For example, consider the ring $R = K[x;y]/I$, where K is a field and I is the ideal of the polynomial ring $K[x,y]$ generated by $x^2 - xy$. We denote by $\overline{p(x,y)}$ the image of $p(x,y) \in K[x,y]$ under the natural projection $K[x,y] \to R$. It is easy to see that $\bar{y}$ is a regular element in R. Let $\Delta = \{\overline{y^n} \mid n \geq 0\} \subseteq R$. Then in $M^{-1}R$ the element $\bar{y}^{-1}\bar{x}$ is an idempotent, but x is not an idempotent in R. Note that $\overline{y^n}\bar{x}$ is not an idempotent in R for $n \geq 0$.

The ring of *Laurent polynomials* in x, coefficients in a ring R, consists of all formal sums $\sum_{i=k}^{n} r_i x^i$ with obvious addition and multiplication, where $r_i \in R$ and k, n are (possibly negative) integers with $k \leq n$. We denote this ring by $R[x;x^{-1}]$.

Corollary 3.6. *Let R be a ring and suppose that every idempotent in $R[x;x^{-1}]$ is of the form $f(x)x^m$ for some $f(x) \in Id(R[x])$ and $m \in \mathbb{Z}$. Then the following conditions are equivalent*:

(1) *R is RIP*;

(2) *$R[x]$ is RIP*; *and*

(3) *$R[x;x^{-1}]$ is RIP.*

Proof. (1)⇔(2) comes from Theorem 3.1(1), and (2)⇔(3) follows from Proposition 3.5 letting $M = \{1, x, x^2, \ldots\}$. □

Consider the idempotent $f(x)x^m$ in the assumption of Corollary 3.6 for the case of $m \geq 1$. Let $f(x) = a_0 + \cdots + a_l x^l$. Then $f(x)^2 x^{2m} = f(x)x^m$ yields $f(x) = f(x)^2 x^m$, and thus $a_0 = 0$, i.e., $f(x) = a_1 x + \cdots + a_l x^l$. Next we have $a_1 = 0$ by the same computation, and so we inductively obtain $f(x) = 0$. Thus one may investigate the case of $m \leq -1$ to find the structure of the idempotent $f(x)x^m$ in $R[x;x^{-1}]$.

A ring R is called right (resp., left) *Ore* if given $a, b \in R$ with b regular there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$ (resp., $b_1 a = a_1 b$). It is a well-known fact that R is a right (resp., left) Ore ring if and only if the classical right (resp., left) quotient ring of R exists. Suppose that there exists the classical right quotient ring $Q(R)$ of a ring R. If R is reflexive, then so is the right quotient ring $Q(R)$ by [12, Theorem 2.11], but we do not know whether $Q(R)$ is RIP when R is RIP. Recall that every IFP ring is RIP. However, we have the following related fact.

Remark 3.7. Suppose that a ring R is right Ore with the classical right quotient ring $Q(R)$, and that R is IFP. Then we obtain that $au^{-1}Q(R)bv^{-1} = 0$ for $au^{-1}, bv^{-1} \in Q(R)$ implies $aRb = 0$. Let $au^{-1}Q(R)bv^{-1} = 0$ for $au^{-1}, bv^{-1} \in Q(R)$. Then there exist $c \in R$ and regular $w \in R$ such that $u^{-1}b = cw^{-1}$ by [19, Proposition 2.1.16]. This yields $acw^{-1} = 0$ and $ac = 0$. But since R is IFP, $aRc = 0$ and $aRu^{-1}b = aRcw^{-1} = 0$. Especially we obtain $ab = auu^{-1}b = 0$. This implies $aRb = 0$.

4. RIP rings of minimal order

Xu and Xue [22, Theorem 8] proved that a noncommutative IFP ring with identity of minimal order is a local ring of order 16 and if R is such a ring, then $R \cong R_i$ for some $i \in \{1, 2, 3, 4, 5\}$, where R_i's are the rings in the following example.

Example 4.1. We have five kinds of noncommutative finite Abelian rings with 16 elements by the help of [22, Example 7].

(1) Let $R_1 = GF(2)[x,y]/I$, where $GF(2)[x,y]$ is the polynomial ring over $GF(2)$ with non-commuting indeterminates x, y and I is the ideal of $GF(2)[x,y]$ generated by $x^3, y^3, yx, x^2 - xy, y^2 - xy$.

(2) Let $R_2 = \mathbb{Z}_4\langle x, y\rangle/I$, where $\mathbb{Z}_4\langle x, y\rangle$ is the free algebra with non-commuting indeterminates x, y over $\mathbb{Z}_4$ and I is the ideal of $\mathbb{Z}_4\langle x, y\rangle$ generated by $x^3, y^3, yx, x^2 - xy, x^2 - 2, y^2 - 2, 2x, 2y$.

(3) Let $R_3 = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & a^2 \end{smallmatrix}\right) \mid a, b \in GF(2^2) \right\}$.

(4) Let $R_4 = GF(2)[x, y]/I$, where I is the ideal of $GF(2)[x, y]$ generated by $x^3, y^2, yx, x^2 - xy$. R_4 is isomorphic to $D_3(\mathbb{Z}_2)$ through the corresponding $x \mapsto E_{12} + E_{23}$ and $y \mapsto E_{23}$.

(5) Let $R_5 = \mathbb{Z}_4\langle x, y\rangle/I$, where I is the ideal of $\mathbb{Z}_4\langle x, y\rangle$ generated by $x^3, y^2, yx, x^2 - xy, x^2 - 2, 2x, 2y$.

Eldridge proved that if a finite ring has a cube free factorization, then it is commutative [6, Theorem], and that if a ring A is of order p^3, p a prime, then $A \cong U_2(GF(p))$ [6, Proposition]. Thus every noncommutative ring of minimal order is isomorphic to $U_2(\mathbb{Z}_2)$. However, $U_2(\mathbb{Z}_2)$ is not RIP by Example 2.9, and so an RIP ring of minimal order has order ≥ 16. But an RIP ring of minimal order must have order 16, considering the semiprime ring $\mathrm{Mat}_2(\mathbb{Z}_2)$ and the rings of Example 4.1.

Observe that every R_i in Example 4.1 is Abelian. Hence, if R is a noncommutative Abelian RIP ring of minimal order, then R is of order 16 such that R is isomorphic to R_i for some $i \in \{1, 2, 3, 4, 5\}$ in Example 4.1.

Moreover, $\mathrm{Mat}_2(\mathbb{Z}_2)$ is reflexive by [12, Theorem 2.6] and hence RIP. Therefore,

Theorem 4.2. *If R is a non-Abelian RIP ring of minimal order, then R is of order* 16 *and is isomorphic to* $\mathrm{Mat}_2(\mathbb{Z}_2)$.

Proof. Let R be a non-Abelian RIP ring of minimal order. Then it is true that R cannot be local since local rings are Abelian. By the Wedderburn-Artin theorem, $R/J(R) \cong \sum_{i=1}^{n} \mathrm{Mat}_{k_i}(D_i)$ for some k_i's and fields D_i's. Here assume that $k_i = 1$ for all i. Then we have there cases of $|J(R)| = 2$, $|J(R)| = 4$, and $|J(R)| = 8$.

If $|J(R)| = 8$, then $R/J(R) \cong \mathbb{Z}_2$ and so R is local, a contradiction. Thus $|J(R)| = 2$ or $|J(R)| = 4$.

Let $|J(R)| = 4$. Then $R/J(R) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since $J(R)$ is nilpotent, there exist orthogonal nonzero idempotents e_1, e_2 with $e_1 + e_2 = 1$ (i.e., $e_2 = 1 - e_1$) by [14, Proposition 3.7.2], and moreover we have

$$R = \{x + y \mid x \in Id(R), y \in J(R)\},$$

where $Id(R) = \{0, 1, e_1, e_2\}$. Suppose that $eRf = 0$ for $e, f \in Id(R)$. Then e and f are orthogonal each other, say $e = e_1, f = e_2$. Let $r = x + y \in R$ with $x \in Id(R)$ and $y \in J(R)$. Then $0 = e_1re_2 = e_1(x + y)e_2 = e_1ye_2$ since $e_1xe_2 = 0$. But since R is RIP, $e_2Re_1 = 0$ and so we get $e_2ye_1 = 0$ since $e_2xe_1 = 0$. This entails $r = (e_1 + e_2)r(e_1 + e_2) = e_1re_1 + e_2re_2$. Since R is non-Abelian, there exist $g \in \{e_1, e_2\}$ and $s \in R$ such that $gs - sg \neq 0$. Note $gs - sg \in J(R)$. Since $g = e_1$ or $g = e_2$, we have

$$gs - sg = e_1(gs - sg)e_1 + e_2(gs - sg)e_2 = 0,$$

a contradiction. The case of $e = e_2$ and $f = e_1$ also induces a contradiction through a similar computation. This implies $|J(R)| \neq 4$.

Let $|J(R)| = 2$. Then $R/J(R) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since $J(R)$ is nilpotent, there exist orthogonal nonzero idempotents e_1, e_2, e_3 with $e_1 + e_2 + e_3 = 1$ by [14, Proposition 3.7.2], and moreover we have

$$R = \{x + y \mid x \in Id(R), y \in J(R)\},$$

where $Id(R) = \{0, 1, e_1, e_2, e_3, 1 - e_1, 1 - e_2, 1 - e_3\}$. For every $r = x + y \in R$ with $x \in Id(R)$ and $y \in J(R)$, we have $e_i r e_j = e_i y e_j$ for $i \neq j$ since $e_i x e_j = 0$. If $e_i y e_j \neq 0$, then $J(R) = \{0, e_i y e_j\}$. Thus $e_j J(R) e_i = 0$, and so $e_j R e_i = 0$ since $e_j Id(R) e_i = 0$. But since R is RIP, we get $e_i R e_j = 0$ and this yields $e_i y e_j = 0$, a contradiction. Therefore we can conclude that

$$e_i R e_j = 0 \text{ for all } i, j \text{ with } i \neq j.$$

Now suppose that $eRf = 0$ for $e, f \in Id(R)$. Then e and f are orthogonal each other, say $e = e_1$ and $f = e_2$. But since R is RIP, $e_2 R e_1 = 0$ and so we get $e_2 y e_1 = 0$ since $e_2 x e_1 = 0$. This entails

$$r = (e_1 + e_2 + e_3) r (e_1 + e_2 + e_3) = e_1 r e_1 + e_2 r e_2 + e_3 r e_3.$$

Since R is non-Abelian, e_k is non-central for some $k \in \{1, 2, 3\}$. If e_1 is non-central, then there exists $s \in R$ such that $e_1 s - s e_1 \neq 0$. Note $e_1 s - s e_1 \in J(R)$. Then we have

$$\begin{aligned} e_1 s - s e_1 &= (e_1 + e_2 + e_3)(e_1 s - s e_1)(e_1 + e_2 + e_3) \\ &= e_1(e_1 s - s e_1)e_1 + e_2(e_1 s - s e_1)e_2 + e_3(e_1 s - s e_1)e_3 = 0, \end{aligned}$$

a contradiction. Each case of (e_2 is non-central) and (e_3 is non-central) also induces a contradiction through a similar computation. The computations for other cases of e and f are also similar, inducing contradictions.

Thus we must have $k_i \geq 2$ for some i. But R is of order 16 and hence we have $R/J(R) \cong \mathrm{Mat}_2(\mathbb{Z}_2)$ and $J(R) = 0$. This implies $R \cong \mathrm{Mat}_2(\mathbb{Z}_2)$. □

Observe that $\mathrm{Mat}_n(\mathbb{Z}_2)$ is semiprime and so we get the following by Theorem 4.2.

Corollary 4.3. *Let R be a ring. Then R is a non-Abelian RIP ring of minimal order if and only if R is a non-Abelian semiprime ring of minimal order if and only if R is a non-Abelian reflexive ring of minimal order if and only if R is a non-Abelian right idempotent reflexive ring of minimal order if and only if R is a non-Abelian left idempotent reflexive ring of minimal order .*

Recall that an IFP ring with identity is Abelian, but the following example shows that this is no longer valid for the case of rings without identity.

Example 4.4. The rings $R_1 = \begin{pmatrix} GF(p) & GF(p) \\ 0 & 0 \end{pmatrix}$ and $R_2 = \begin{pmatrix} GF(p) & 0 \\ GF(p) & 0 \end{pmatrix}$ are minimal noncommutative IFP rings without identity by [22, p. 71] which have 4 elements, respectively. Notice that they are not Abelian by a simple computation. We will show that R_i's are RIP rings.

The set of all nonzero idempotents in R_1 is $E = \{\left(\begin{smallmatrix} 1 & d \\ 0 & 0 \end{smallmatrix}\right) \mid d \in GF(p)\}$. Suppose $eR_1a = 0$ for $e \in E$ and $a \in R$. Then $a = 0$ since $eR_1 = R_1$, obtaining $aR_1e = 0$. So R_1 is left idempotent reflexive, and hence R_1 is RIP.

The set of all nonzero idempotents in R_2 is $E = \{\left(\begin{smallmatrix} 1 & 0 \\ d & 0 \end{smallmatrix}\right) \mid d \in GF(p)\}$. By a similar computation to above, we can show that R_2 is right idempotent reflexive and so R_2 is RIP.

Lemma 4.5 ([13, Lemma 2.7])**.** *Let R be a ring and N be a nil ideal of R. If $|N| = 4$, then N is a commutative ring with $N^3 = 0$.*

Theorem 4.6. *Let R be a ring without identity. If R is a non-Abelian RIP ring of minimal order, then R is isomorphic to $\left(\begin{smallmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} \mathbb{Z}_2 & 0 \\ \mathbb{Z}_2 & 0 \end{smallmatrix}\right)$.*

Proof. Let R be a non-Abelian RIP ring of minimal order. Then $|R| = 4$ by the existence of the non-Abelian RIP ring $\left(\begin{smallmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{smallmatrix}\right)$ as in Example 4.4. If R is nilpotent then R is commutative by Lemma 4.5, a contradiction. If $|J(R)| = 0$, then R is also commutative by the proof of [11, Theorem 1.15], a contradiction. Thus we have the result of $|J(R)| = 2$, whence we also follow the proof [11, Theorem 1.15] to conclude that R is isomorphic to $\left(\begin{smallmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} \mathbb{Z}_2 & 0 \\ \mathbb{Z}_2 & 0 \end{smallmatrix}\right)$. □

Hence, we have the following by Theorem 4.6.

Corollary 4.7. *Let R be a ring without identity. Then R is a noncommutative RIP ring of minimal order if and only if R is a noncommutative IFP ring of minimal order.*

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