

## UNION-SOFT SETS WITH APPLICATIONS IN *BCK/BCI*-ALGEBRAS

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ABSTRACT. The aim of this article is to lay a foundation for providing a soft algebraic tool in considering many problems that contain uncertainties. In order to provide these soft algebraic structures, the notion of union-soft sets is introduced, and its application to *BCK/BCI*-algebras is considered. The notions of union-soft algebras, union-soft (commutative) ideals and closed union-soft ideals are introduced, and related properties and relations are investigated. Conditions for a union-soft ideal to be closed are provided. Conditions for a union-soft ideal to be a union-soft commutative ideal are also provided. Characterizations of (closed) union-soft ideals and union-soft commutative ideals are established. Extension property for a union-soft commutative ideal is established.

### 1. Introduction

Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [24]. In response to this situation Zadeh [25] introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [26]. To solve complicated problem in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [22]. Maji et al. [19] and Molodtsov

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Received July 17, 2012.

2010 *Mathematics Subject Classification.* 06F35, 03G25, 06D72.

*Key words and phrases.* exclusive set, union-soft algebra, (closed) union-soft ideal, union-soft commutative ideal.

[22] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [22] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [19] described the application of soft set theory to a decision making problem. Maji et al. [18] also studied several operations on the theory of soft sets. Chen et al. [7] presented a new definition of soft set parametrization reduction, and compared this definition to the related concept of attributes reduction in rough set theory. The algebraic structure of set theories dealing with uncertainties has been studied by some authors. Çağman et al. [6] introduced fuzzy parameterized (FP) soft sets and their related properties. They proposed a decision making method based on FP-soft set theory, and provided an example which shows that the method can be successfully applied to the problems that contain uncertainties. Feng [8] considered the application of soft rough approximations in multicriteria group decision making problems. Feng et al. [10] established an interesting connection between two mathematical approaches to vagueness: rough sets and soft sets. Aktaş and Çağman [2] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets, providing examples to clarify their differences. They also discussed the notion of soft groups. After that, many algebraic properties of soft sets are studied (see [1, 3, 9, 12, 13, 14, 15, 16, 17, 23, 27]). In [5], Çağman and Enginoğlu defined soft matrices which are representations of soft sets. This representation has several advantages. It is easy to store and manipulate matrices and hence the soft sets represented by them in a computer. They also constructed a soft decision making which is more practical and can be successfully applied to many problems that contain uncertainties without using the rough sets and fuzzy soft sets.

The goal of this article is to lay a foundation for providing a soft algebraic tool in considering many problems that contain uncertainties. In order to provide these soft algebraic structures, we first introduce the notion of union-soft sets, and consider its application to *BCK/BCI*-algebras. We introduce the notions of union-soft algebras, union-soft (commutative) ideals and closed union-soft ideals. We display several examples, and investigate related properties and relations. We provide conditions for a union-soft ideal to be closed, and give conditions for a union-soft ideal to be a union-soft commutative ideal. We discuss characterizations of (closed) union-soft ideals and union-soft commutative ideals. We establish extension property for a union-soft commutative ideal.

### 2. Preliminaries

A *BCK/BCI*-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra  $(X; *, 0)$  of type  $(2, 0)$  is called a *BCI-algebra* if it satisfies the following conditions:

- (I)  $(\forall x, y, z \in X) (((x * y) * (x * z)) * (z * y) = 0)$ ,
- (II)  $(\forall x, y \in X) ((x * (x * y)) * y = 0)$ ,
- (III)  $(\forall x \in X) (x * x = 0)$ ,
- (IV)  $(\forall x, y \in X) (x * y = 0, y * x = 0 \Rightarrow x = y)$ .

If a *BCI*-algebra  $X$  satisfies the following identity:

$$(V) (\forall x \in X) (0 * x = 0),$$

then  $X$  is called a *BCK-algebra*. Any *BCK/BCI*-algebra  $X$  satisfies the following axioms:

- (a1)  $(\forall x \in X) (x * 0 = x)$ ,
- (a2)  $(\forall x, y, z \in X) (x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x)$ ,
- (a3)  $(\forall x, y, z \in X) ((x * y) * z = (x * z) * y)$ ,
- (a4)  $(\forall x, y, z \in X) ((x * z) * (y * z) \leq x * y)$ ,

where  $x \leq y$  if and only if  $x * y = 0$ . In a *BCI*-algebra  $X$ , the following hold:

- (b1)  $(\forall x, y \in X) (x * (x * (x * y)) = x * y)$ ,
- (b2)  $(\forall x, y \in X) (0 * (x * y) = (0 * x) * (0 * y))$ .

A *BCK*-algebra  $X$  is said to be *commutative* if  $x \wedge y = y \wedge x$  for all  $x, y \in X$  where  $x \wedge y = y * (y * x)$ . A nonempty subset  $S$  of a *BCK/BCI*-algebra  $X$  is called a *subalgebra* of  $X$  if  $x * y \in S$  for all  $x, y \in S$ . A subset  $A$  of a *BCK/BCI*-algebra  $X$  is called an *ideal* of  $X$  if it satisfies:

- (2.1)  $0 \in A$ ,
- (2.2)  $(\forall x \in X) (\forall y \in A) (x * y \in A \Rightarrow x \in A)$ .

A subset  $A$  of a *BCK*-algebra  $X$  is called a *commutative ideal* if it satisfies (2.1) and

$$(2.3) (\forall x, y \in X) (\forall z \in A) ((x * y) * z \in A \Rightarrow x * (y * (y * x)) \in A).$$

Observe that every commutative ideal is an ideal, but the converse is not true (see [20]).

**Theorem 2.1.** *An ideal  $A$  of a *BCK*-algebra  $X$  is commutative if and only if the following implication holds:*

$$(2.4) (\forall x, y \in X) (x * y \in A \Rightarrow x * (y * (y * x)) \in A).$$

We refer the reader to the books [11, 20] for further information regarding *BCK/BCI*-algebras.

A soft set theory is introduced by Molodtsov [22], and Çağman et al. [4] provided new definitions and various results on soft set theory.

In what follows, let  $U$  be an initial universe set and  $E$  be a set of parameters. We say that the pair  $(U, E)$  is a *soft universe*. Let  $\mathcal{P}(U)$  denotes the power set of  $U$  and  $A, B, C, \dots \subseteq E$ .

**Definition 2.2** ([4, 22]). A soft set  $\mathcal{F}_A$  over  $U$  is defined to be the set of ordered pairs

$$\mathcal{F}_A := \{(x, f_A(x)) : x \in E, f_A(x) \in \mathcal{P}(U)\},$$

where  $f_A : E \rightarrow \mathcal{P}(U)$  such that  $f_A(x) = \emptyset$  if  $x \notin A$ .

The function  $f_A$  is called the approximate function of the soft set  $\mathcal{F}_A$ . The subscript  $A$  in the notation  $f_A$  indicates that  $f_A$  is the approximate function of  $\mathcal{F}_A$ .

In what follows, denote by  $S(U)$  the set of all soft sets over  $U$  by Çağman et al. [4].

**Definition 2.3** ([18]). Let  $\mathcal{F}_A, \mathcal{G}_B \in S(U)$ . We say that  $\mathcal{F}_A$  is a *soft subset* of  $\mathcal{G}_B$ , denoted by  $\mathcal{F}_A \subseteq \mathcal{G}_B$ , if

- (i)  $A \subseteq B$ ,
- (ii)  $(\forall e \in A) (f_A(e) \text{ and } g_B(e) \text{ are identical approximations})$ .

### 3. Union-soft sets

In this section, let  $U$  denote an initial universe set and assume that  $E$ , a set of parameters, has a binary operation  $\rightsquigarrow$ .

**Definition 3.1.** For any non-empty subset  $A$  of  $E$ , let  $\mathcal{F}_A \in S(U)$ . Then  $\mathcal{F}_A$  is called a *union-soft set* over  $U$  if it satisfies:

$$(3.1) \quad (\forall x, y \in A) (x \rightsquigarrow y \in A \Rightarrow f_A(x \rightsquigarrow y) \subseteq f_A(x) \cup f_A(y)).$$

**Example 3.2.** (1) Suppose that there are five houses in the initial universe set  $U$  given by

$$U = \{h_1, h_2, h_3, h_4, h_5\}.$$

Let a set of parameters  $E = \{e_1, e_2, e_3, e_4\}$  be a set of status of houses which stand for the parameters “beautiful”, “cheap”, “in good location” and “in green surroundings”, respectively, with the following binary operation:

$\rightsquigarrow$	$e_1$	$e_2$	$e_3$	$e_4$
$e_1$	$e_1$	$e_1$	$e_1$	$e_1$
$e_2$	$e_2$	$e_1$	$e_1$	$e_2$
$e_3$	$e_3$	$e_3$	$e_1$	$e_3$
$e_4$	$e_4$	$e_4$	$e_4$	$e_1$

(1) Consider a soft set  $\mathcal{F}_E$  over  $U$  as follows:

$$\mathcal{F}_E = \{(e_1, \{h_3, h_4\}), (e_2, \{h_2, h_3, h_4\}), (e_3, \{h_2, h_3, h_4, h_5\}), (e_4, \{h_1, h_3, h_4\})\}.$$

Then  $\mathcal{F}_E$  is a union-soft set over  $U$ .

(2) Let  $B = \{e_1, e_2, e_4\} \subsetneq E$ . Then the soft set  $\mathcal{G}_B$  over  $U$  which is given by

$$\mathcal{G}_B = \{(e_1, \{h_1, h_5\}), (e_2, \{h_1, h_3, h_5\}), (e_3, \emptyset), (e_4, \{h_1, h_2, h_3, h_5\})\}$$

is a union-soft set over  $U$ .

**Theorem 3.3.** Let  $\mathcal{F}_A, \mathcal{G}_B \in S(U)$  be such that  $\mathcal{F}_A$  is a soft subset of  $\mathcal{G}_B$ . If  $\mathcal{G}_B$  is a union-soft set over  $U$ , then so is  $\mathcal{F}_A$ .

*Proof.* Let  $x, y \in A$  with  $x \rightsquigarrow y \in A$ . Then  $x \rightsquigarrow y \in B$  since  $A \subseteq B$ . Hence

$$f_A(x \rightsquigarrow y) = g_B(x \rightsquigarrow y) \subseteq g_B(x) \cup g_B(y) = f_A(x) \cup f_A(y).$$

Therefore  $\mathcal{F}_A$  is a union-soft set over  $U$ . □

The converse of Theorem 3.3 may not be true as seen in the following example.

**Example 3.4.** Let  $U = \{h_1, h_2, h_3, h_4, h_5\}$  be the initial universe set which consists of five houses. Let a set of parameters  $E = \{e_1, e_2, e_3, e_4\}$  be a set of status of houses which stand for the parameters “beautiful”, “cheap”, “in good location” and “in green surroundings”, respectively, with the following binary operation:

$\rightsquigarrow$	$e_1$	$e_2$	$e_3$	$e_4$
$e_1$	$e_1$	$e_2$	$e_3$	$e_4$
$e_2$	$e_2$	$e_1$	$e_3$	$e_2$
$e_3$	$e_3$	$e_4$	$e_1$	$e_2$
$e_4$	$e_4$	$e_3$	$e_2$	$e_1$

Let  $A = \{e_1, e_2\} \subseteq E$  and consider a soft set  $\mathcal{F}_E$  over  $U$  as follows:

$$\mathcal{F}_A = \{(e_1, \{h_1, h_3\}), (e_2, \{h_1, h_3, h_4\}), (e_3, \emptyset), (e_4, \emptyset)\}.$$

Then  $\mathcal{F}_A$  is a union-soft set over  $U$ . Consider a soft set  $\mathcal{G}_E$  over  $U$  as follows:

$$\mathcal{G}_E = \{(e_1, \{h_1, h_3\}), (e_2, \{h_1, h_3, h_4\}), (e_3, \{h_2, h_4\}), (e_4, \{h_4, h_5\})\}.$$

Then  $\mathcal{F}_A$  is a soft subset of  $\mathcal{G}_E$ . But  $\mathcal{G}_E$  is not a union-soft set over  $U$  since

$$g_E(e_3 \rightsquigarrow e_4) = g_E(e_2) = \{h_1, h_3, h_4\} \not\subseteq \{h_2, h_4, h_5\} = g_E(e_3) \cup g_E(e_4).$$

Let  $\mathcal{F}_A \in S(U)$  and let  $\tau \subseteq U$ . Then the  $\tau$ -exclusive set of  $\mathcal{F}_A$  is defined to be the set

$$e(\mathcal{F}_A; \tau) := \{x \in A \mid f_A(x) \subseteq \tau\}.$$

Obviously, we have the following properties:

- (1)  $e(\mathcal{F}_A; U) = A$ .
- (2)  $f_A(x) = \cap \{\tau \subseteq U \mid x \in e(\mathcal{F}_A; \tau)\}$ .
- (3)  $(\forall \tau_1, \tau_2 \subseteq U) (\tau_1 \subseteq \tau_2 \Rightarrow e(\mathcal{F}_A; \tau_1) \subseteq e(\mathcal{F}_A; \tau_2))$ .

**4. Union-soft algebras in BCK/BCI-algebras**

**Definition 4.1.** Let  $(U, E) = (U, X)$  where  $X$  is a BCK/BCI-algebra. Given a subalgebra  $A$  of  $E$ , let  $\mathcal{F}_A \in S(U)$ . Then  $\mathcal{F}_A$  is called a *union-soft algebra* over  $U$  if the approximate function  $f_A$  of  $\mathcal{F}_A$  satisfies:

$$(4.1) \quad (\forall x, y \in A) (f_A(x * y) \subseteq f_A(x) \cup f_A(y)).$$

**Example 4.2.** Let  $(U, E) = (U, X)$  where  $X = \{0, a, b\}$  is a BCK-algebra with the following Cayley table:

*	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Let  $\tau_1, \tau_2$  and  $\tau_3$  be subsets of  $U$  such that  $\tau_1 \subsetneq \tau_2 \subsetneq \tau_3$ . Define a soft set  $\mathcal{F}_E$  over  $U$  as follows:

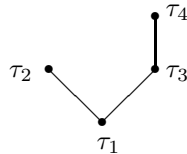
$$\mathcal{F}_E = \{(0, \tau_1), (a, \tau_2), (b, \tau_3)\}.$$

Routine calculations show that  $\mathcal{F}_E$  is a union-soft algebra over  $U$ .

**Example 4.3.** Let  $(U, E) = (U, X)$  where  $X = \{0, a, b, c\}$  is a BCK-algebra with the following Cayley table:

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

Let  $\{\tau_1, \tau_2, \tau_3, \tau_4\}$  be a class of subsets of  $U$  which is a poset with the following Hasse diagram:



Define a soft set  $\mathcal{F}_E$  over  $U$  as follows:

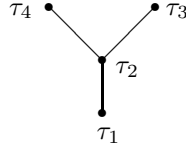
$$\mathcal{F}_E = \{(0, \tau_1), (a, \tau_3), (b, \tau_4), (c, \tau_2)\}.$$

Routine calculations show that  $\mathcal{F}_E$  is a union-soft algebra over  $U$ .

**Example 4.4.** Let  $(U, E) = (U, X)$  where  $X = \{0, a, b, c, d\}$  is a BCK-algebra with the following Cayley table:

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	a	a
b	b	b	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

Let  $\{\tau_1, \tau_2, \tau_3, \tau_4\}$  be a class of subsets of  $U$  which is a poset with the following Hasse diagram:



For a subalgebra  $A = \{0, a, b, c\}$  of  $E$ , define a soft set  $\mathcal{F}_A$  over  $U$  as follows:

$$\mathcal{F}_A = \{(0, \tau_1), (a, \tau_2), (b, \tau_3), (c, \tau_4), (d, \emptyset)\}.$$

Routine calculations show that  $\mathcal{F}_A$  is a union-soft algebra over  $U$ .

**Theorem 4.5.** *Let  $(U, E) = (U, X)$  where  $X$  is a BCK/BCI-algebra. Given a subalgebra  $A$  of  $E$ , let  $\mathcal{F}_A \in S(U)$ . Then  $\mathcal{F}_A$  is a union-soft algebra over  $U$  if and only if the nonempty  $\tau$ -exclusive set of  $\mathcal{F}_A$  is a subalgebra of  $A$  for all  $\tau \subseteq U$ .*

*Proof.* Assume that  $\mathcal{F}_A$  is a union-soft algebra over  $U$ . Let  $\tau \subseteq U$  and  $x, y \in e(\mathcal{F}_A; \tau)$ . Then  $f_A(x) \subseteq \tau$  and  $f_A(y) \subseteq \tau$ . It follows from (4.1) that  $f_A(x * y) \subseteq f_A(x) \cup f_A(y) \subseteq \tau$ . Hence  $x * y \in e(\mathcal{F}_A; \tau)$ . Therefore  $e(\mathcal{F}_A; \tau)$  is a subalgebra of  $A$ .

Conversely, suppose that the nonempty  $\tau$ -exclusive set of  $\mathcal{F}_A$  is a subalgebra of  $A$  for all  $\tau \subseteq U$ . Let  $x, y \in A$  be such that  $f_A(x) = \tau_1$  and  $f_A(y) = \tau_2$ . Taking  $\tau = \tau_1 \cup \tau_2$  implies that  $x, y \in e(\mathcal{F}_A; \tau)$ , and so  $x * y \in e(\mathcal{F}_A; \tau)$ . Hence

$$f_A(x * y) \subseteq \tau = \tau_1 \cup \tau_2 = f_A(x) \cup f_A(y).$$

Therefore  $\mathcal{F}_A$  is a union-soft algebra over  $U$ . □

**Proposition 4.6.** *Let  $(U, E) = (U, X)$  where  $X$  is a BCK/BCI-algebra. Given a subset  $A$  of  $E$ , let  $\mathcal{F}_A \in S(U)$ . If  $\mathcal{F}_A$  is a union-soft algebra over  $U$ , then the approximate function  $f_A$  of  $\mathcal{F}_A$  satisfies*

$$(4.2) \quad (\forall x \in A) (f_A(0) \subseteq f_A(x)).$$

*Proof.* If  $0 \notin A$ , then  $f_A(0) = \emptyset \subseteq f_A(x)$  for all  $x \in A$ . If  $0 \in A$ , then

$$f_A(0) = f_A(x * x) \subseteq f_A(x) \cup f_A(x) = f_A(x)$$

for all  $x \in A$ . □

**Proposition 4.7.** *Let  $(U, E) = (U, X)$  where  $X$  is a BCI-algebra. Given a subalgebra  $A$  of  $E$ , let  $\mathcal{F}_A \in S(U)$ . If  $\mathcal{F}_A$  is a union-soft algebra over  $U$ , then the approximate function  $f_A$  of  $\mathcal{F}_A$  satisfies*

$$(4.3) \quad (\forall x, y \in A) (f_A(x * (0 * y)) \subseteq f_A(x) \cup f_A(y)).$$

*Proof.* Using Proposition 4.6, we have

$$\begin{aligned} f_A(x * (0 * y)) &\subseteq f_A(x) \cup f_A(0 * y) \\ &\subseteq f_A(x) \cup f_A(0) \cup f_A(y) = f_A(x) \cup f_A(y) \end{aligned}$$

for all  $x, y \in A$ . □

**Proposition 4.8.** *Let  $(U, E) = (U, X)$  where  $X$  is a BCK/BCI-algebra. Given a subalgebra  $A$  of  $E$ , let  $\mathcal{F}_A \in S(U)$ . If  $\mathcal{F}_A$  is a union-soft algebra over  $U$ , then the approximate function  $f_A$  of  $\mathcal{F}_A$  satisfies*

$$(\forall x, y \in A) (f_A(x * y) \subseteq f_A(y) \Leftrightarrow f_A(x) = f_A(0)).$$

*Proof.* Assume that  $f_A(x * y) \subseteq f_A(y)$  for all  $x, y \in A$ . Taking  $y = 0$  induces  $f_A(x) = f_A(x * 0) \subseteq f(0)$ . It follows from (4.2) that  $f_A(x) = f_A(0)$  for all  $x \in A$ .

Conversely, suppose that  $f_A(x) = f_A(0)$  for all  $x \in A$ . Then

$$f_A(x * y) \subseteq f_A(x) \cup f_A(y) = f_A(0) \cup f_A(y) = f_A(y)$$

for all  $x, y \in A$ . □

**Theorem 4.9.** *Let  $(U, E) = (U, X)$  where  $X$  is a BCK/BCI-algebra. Given a subalgebra  $A$  of  $E$ , let  $\mathcal{F}_A \in S(U)$ . Let  $\mathcal{F}_A^* \in S(U)$  with the approximate function  $f_A^*$  defined by*

$$f_A^* : E \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} f_A(x) & \text{if } x \in e(\mathcal{F}_A; \tau), \\ U & \text{otherwise.} \end{cases}$$

*If  $\mathcal{F}_A$  is a union-soft algebra over  $U$ , then so is  $\mathcal{F}_A^*$ .*

*Proof.* If  $\mathcal{F}_A$  is a union-soft algebra over  $U$ , then  $e(\mathcal{F}_A; \tau)$  is a subalgebra of  $A$  for all  $\tau \subseteq U$ . Let  $x, y \in A$ . If  $x, y \in e(\mathcal{F}_A; \tau)$ , then  $x * y \in e(\mathcal{F}_A; \tau)$  and so

$$f_A^*(x * y) = f_A(x * y) \subseteq f_A(x) \cup f_A(y) = f_A^*(x) \cup f_A^*(y).$$

If  $x \notin e(\mathcal{F}_A; \tau)$  or  $y \notin e(\mathcal{F}_A; \tau)$ , then  $f_A^*(x) = U$  or  $f_A^*(y) = U$ . Thus

$$f_A^*(x * y) \subseteq U = f_A^*(x) \cup f_A^*(y).$$

Therefore  $\mathcal{F}_A^*$  is a union-soft algebra over  $U$ . □

### 5. Union-soft ideals in BCK/BCI-algebras

**Definition 5.1.** Let  $(U, E) = (U, X)$  where  $X$  is a BCK/BCI-algebra. Given a subalgebra  $A$  of  $E$ , let  $\mathcal{F}_A \in S(U)$ . Then  $\mathcal{F}_A$  is called a *union-soft ideal* over  $U$  if the approximate function  $f_A$  of  $\mathcal{F}_A$  satisfies (4.2) and

$$(5.1) \quad (\forall x, y \in A) (f_A(x) \subseteq f_A(x * y) \cup f_A(y)).$$

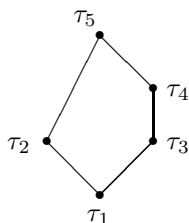
**Example 5.2.** The soft set  $\mathcal{F}_A$  over  $U$  in Example 4.4 is a union-soft ideal over  $U$ .



**Example 5.3.** Let  $(U, E) = (U, X)$  where  $X = \{0, 1, 2, a, b\}$  is a BCI-algebra with the following Cayley table:

*	0	1	2	a	b
0	0	0	0	a	a
1	1	0	1	b	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

Let  $\{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$  be a class of subsets of  $U$  which is a poset with the following Hasse diagram:



Define a soft set  $\mathcal{F}_E$  over  $U$  as follows:

$$\mathcal{F}_E = \{(0, \tau_1), (1, \tau_2), (2, \tau_3), (a, \tau_4), (b, \tau_5)\}.$$

Then  $\mathcal{F}_E$  is a union-soft ideal over  $U$ .

**Example 5.4.** Let  $(U, E) = (U, X)$  where  $X = \{2^n \mid n \in \mathbb{Z}\}$  is a BCI-algebra with a binary operation “ $\div$ ” (usual division). Let  $\mathcal{F}_E \in S(U)$  be defined by

$$f_E : E \rightarrow \mathcal{P}(U), x \mapsto \begin{cases} \tau_1 & \text{if } n \geq 0, \\ \tau_2 & \text{if } n < 0, \end{cases}$$

where  $\tau_1$  and  $\tau_2$  are subsets of  $U$  with  $\tau_1 \subsetneq \tau_2$ . Then  $\mathcal{F}_E$  is a union-soft ideal over  $U$ ,

**Lemma 5.5.** Let  $(U, E) = (U, X)$  where  $X$  is a BCK/BCI-algebra. Given a subalgebra  $A$  of  $E$ , let  $\mathcal{F}_A \in S(U)$ . If  $\mathcal{F}_A$  is a union-soft ideal over  $U$ , then

$$(5.2) \quad (\forall x, y \in A) (x \leq y \Rightarrow f_A(x) \subseteq f_A(y)).$$

*Proof.* Let  $x, y \in A$  be such that  $x \leq y$ . Then  $x * y = 0$ , and so

$$f_A(x) \subseteq f_A(x * y) \cup f_A(y) = f_A(0) \cup f_A(y) = f_A(y)$$

by (5.1) and (4.2). □

**Proposition 5.6.** Let  $(U, E) = (U, X)$  where  $X$  is a BCK/BCI-algebra. Given a subalgebra  $A$  of  $E$ , let  $\mathcal{F}_A \in S(U)$ . If  $\mathcal{F}_A$  is a union-soft ideal over  $U$ , then the approximate function  $f_A$  of  $\mathcal{F}_A$  satisfies:

- (1)  $(\forall x, y, z \in A) (f_A(x * y) \subseteq f_A(x * z) \cup f_A(z * y)).$
- (2)  $(\forall x, y \in A) (f_A(x * y) = f_A(0) \Rightarrow f_A(x) \subseteq f_A(y)).$

*Proof.* Since  $(x * y) * (x * z) \leq z * y$ , it follows from Lemma 5.5 that

$$f_A((x * y) * (x * z)) \subseteq f_A(z * y).$$

Hence

$$\begin{aligned} f_A(x * y) &\subseteq f_A((x * y) * (x * z)) \cup f_A(x * z) \\ &\subseteq f_A(x * z) \cup f_A(z * y). \end{aligned}$$

(2) If  $f_A(x * y) = f_A(0)$ , then

$$f_A(x) \subseteq f_A(x * y) \cup f_A(y) = f_A(0) \cup f_A(y) = f_A(y)$$

for all  $x, y \in A$ . □

**Proposition 5.7.** *Let  $(U, E) = (U, X)$  where  $X$  is a BCK/BCI-algebra. For any union-soft ideal  $\mathcal{F}_A$  where  $A$  is a subalgebra of  $E$ , the following conditions are equivalent:*

- (1)  $(\forall x, y \in A) (f_A(x * y) \subseteq f_A((x * y) * y))$ .
- (2)  $(\forall x, y, z \in A) (f_A((x * z) * (y * z)) \subseteq f_A((x * y) * z))$ .

*Proof.* Assume that (1) is valid and let  $x, y, z \in A$ . Since

$$((x * (y * z)) * z) * z = ((x * z) * (y * z)) * z \leq (x * y) * z,$$

it follows from (a3), (1) and (5.2) that

$$\begin{aligned} f_A((x * z) * (y * z)) &= f_A((x * (y * z)) * z) \\ &\subseteq f_A(((x * (y * z)) * z) * z) \subseteq f_A((x * y) * z). \end{aligned}$$

Conversely, suppose that (2) holds. If we take  $y = z$  in (2), then

$$f_A((x * z) * z) \supseteq f_A((x * z) * (z * z)) = f_A((x * z) * 0) = f_A(x * z)$$

by (III) and (a1). This proves (1). □

**Theorem 5.8.** *Let  $(U, E) = (U, X)$  where  $X$  is a BCK-algebra. Then every union-soft ideal is a union-soft algebra.*

*Proof.* Let  $\mathcal{F}_A$  be a union-soft ideal over  $U$  where  $A$  is a subalgebra of  $E$ . Note that  $x * y \leq x$  for all  $x, y \in E$ . Using Lemma 5.5, we have

$$f_A(x * y) \subseteq f_A(x) \subseteq f_A(x * y) \cup f_A(y) \subseteq f_A(x) \cup f_A(y).$$

Hence  $\mathcal{F}_A$  is a union-soft algebra over  $U$ . □

If  $X$  is a BCI-algebra, then Theorem 5.8 is not true as seen in the following example.

**Example 5.9.** Let  $(Y, *, 0)$  be a BCI-algebra and let  $(\mathbb{Z}, -, 0)$  be the adjoint BCI-algebra of the additive group  $(\mathbb{Z}, +, 0)$  of integers. Let  $(U, E) = (U, X)$  where  $X = Y \times \mathbb{Z}$  is a BCI-algebra with a binary operation  $\otimes$  defined as follows:

$$(\forall (x, m), (y, n) \in X) ((x, m) \otimes (y, n) = (x * y, m - n)).$$

For a subset  $A = Y \times (\mathbb{N} \cup \{0\})$  of  $X$ , let  $\mathcal{F}_A$  be a soft set over  $U$  in which the approximate function  $f_A$  is given by

$$f_A : E \rightarrow \mathcal{P}(U), (x, m) \mapsto \begin{cases} \tau & \text{if } x \in A, \\ U & \text{otherwise,} \end{cases}$$

where  $\tau \subsetneq U$ . Then  $\mathcal{F}_A$  is a union-soft ideal over  $U$ . Note that  $(0, 2) \in A$  and  $(0, 3) \in A$ , but  $(0, 2) \otimes (0, 3) = (0, -1) \notin A$ . Thus

$$f_A((0, 2) \otimes (0, 3)) = U \not\subseteq \tau = f_A(0, 2) \cup f_A(0, 3).$$

Therefore  $\mathcal{F}_A$  is not a union-soft algebra over  $U$ .

**Proposition 5.10.** *Let  $(U, E) = (U, X)$  where  $X$  is a BCK/BCI-algebra. Given a subalgebra  $A$  of  $E$ , let  $\mathcal{F}_A \in S(U)$ . If  $\mathcal{F}_A$  is a union-soft ideal over  $U$ , then the approximate function  $f_A$  satisfies the following condition:*

$$(5.3) \quad (\forall x, y, z \in A) (x * y \leq z \Rightarrow f_A(x) \subseteq f_A(y) \cup f_A(z)).$$

*Proof.* Let  $x, y, z \in A$  be such that  $x * y \leq z$ . Then  $(x * y) * z = 0$ , and so

$$f_A(x * y) \subseteq f_A((x * y) * z) \cup f_A(z) = f_A(0) \cup f_A(z) = f_A(z)$$

by (5.1) and (4.2). It follows that

$$f_A(x) \subseteq f_A(x * y) \cup f_A(y) \subseteq f_A(y) \cup f_A(z).$$

This completes the proof.  $\square$

**Proposition 5.11.** *Let  $(U, E) = (U, X)$  where  $X$  is a BCK/BCI-algebra. Given a subalgebra  $A$  of  $E$ , let  $\mathcal{F}_A \in S(U)$ . If the approximate function  $f_A$  of  $\mathcal{F}_A$  satisfies (4.2) and (5.3), then  $\mathcal{F}_A$  is a union-soft ideal over  $U$ .*

*Proof.* Note that  $x * (x * y) \leq y$  by (II), and thus  $f_A(x) \subseteq f_A(x * y) \cup f_A(y)$  by (5.3). Therefore  $\mathcal{F}_A$  is a union-soft ideal over  $U$ .  $\square$

The following could be easily proved by induction.

**Corollary 5.12.** *Let  $(U, E) = (U, X)$  where  $X$  is a BCK/BCI-algebra. Given a subalgebra  $A$  of  $E$ , let  $\mathcal{F}_A \in S(U)$  satisfy the condition (4.2). Then  $\mathcal{F}_A$  is a union-soft ideal over  $U$  if and only if the approximate function  $f_A$  of  $\mathcal{F}_A$  satisfies: for all  $x, a_1, a_2, \dots, a_n \in A$ ,*

$$x * \prod_{i=1}^n a_i = 0 \Rightarrow f_A(x) \subseteq \bigcup_{i=1,2,\dots,n} f_A(a_i),$$

where  $x * \prod_{i=1}^n a_i = (\dots(x * a_1) * \dots) * a_n$ .

**Proposition 5.13.** *Let  $(U, E) = (U, X)$  where  $X$  is a BCK-algebra such that*

$$(5.4) \quad (x * a) * b = 0,$$

$$(5.5) \quad a * \prod_{i=1}^n a_i = 0,$$

$$(5.6) \quad b * \prod_{j=1}^m b_j = 0$$

for all  $x, a, b, a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m \in X$ . If  $\mathcal{F}_A$  is a union-soft ideal over  $U$ , then the approximate function  $f_A$  of  $\mathcal{F}_A$  satisfies: for all  $x, a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m \in A$ ,

$$f_A(x) \subseteq \bigcup_{\substack{i=1,2,\dots,n \\ j=1,2,\dots,m}} (f_A(a_i) \cup f_A(b_j)).$$

*Proof.* The condition (5.4) implies from (a3) that  $(x * b) * a = 0$ , i.e.,  $x * b \leq a$ . It follows from (a2), (a3) and (5.5) that

$$\left( x * \prod_{i=1}^n a_i \right) * b \leq a * \prod_{i=1}^n a_i = 0$$

so that  $\left( x * \prod_{i=1}^n a_i \right) * b = 0$ , i.e.,  $x * \prod_{i=1}^n a_i \leq b$ . Using (a2), (a3) and (5.6), we have

$$\left( x * \prod_{i=1}^n a_i \right) * \prod_{j=1}^m b_j \leq b * \prod_{j=1}^m b_j = 0,$$

and so  $\left( x * \prod_{i=1}^n a_i \right) * \prod_{j=1}^m b_j = 0$ . Thus, by Corollary 5.12, we have

$$f_A(x) \subseteq \bigcup_{\substack{i=1,2,\dots,n \\ j=1,2,\dots,m}} (f_A(a_i) \cup f_A(b_j))$$

for all  $x, a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m \in A$ .  $\square$

**Theorem 5.14.** Let  $(U, E) = (U, X)$  where  $X$  is a BCK/BCI-algebra. Given a subalgebra  $A$  of  $E$ , let  $\mathcal{F}_A \in S(U)$ . If  $\mathcal{F}_A$  is a union-soft ideal over  $U$ , then the nonempty  $\tau$ -exclusive set of  $\mathcal{F}_A$  is an ideal of  $A$  for all  $\tau \subseteq U$ .

*Proof.* Assume that  $\mathcal{F}_A$  is a union-soft ideal over  $U$ . Let  $\tau \subseteq U$  be such that  $e(\mathcal{F}_A; \tau) \neq \emptyset$ . Then  $f_A(x) \subseteq \tau$  for some  $x \in A$ . It follows from (4.2) that  $f_A(0) \subseteq f_A(x) \subseteq \tau$ . Hence  $0 \in e(\mathcal{F}_A; \tau)$ . Let  $x, y \in A$  be such that  $x * y \in e(\mathcal{F}_A; \tau)$  and  $y \in e(\mathcal{F}_A; \tau)$ . Then  $f_A(x * y) \subseteq \tau$  and  $f_A(y) \subseteq \tau$ . It follows from (5.1) that

$$f_A(x) \subseteq f_A(x * y) \cup f_A(y) \subseteq \tau.$$

Thus  $x \in e(\mathcal{F}_A; \tau)$ . Therefore  $e(\mathcal{F}_A; \tau)$  is an ideal of  $A$ .  $\square$

**Theorem 5.15.** Let  $(U, E) = (U, X)$  where  $X$  is a BCK/BCI-algebra. Given a subalgebra  $A$  of  $E$ , let  $\mathcal{F}_A \in S(U)$ . If the nonempty  $\tau$ -exclusive set of  $\mathcal{F}_A$  is an ideal of  $A$  for all  $\tau \subseteq U$ , then  $\mathcal{F}_A$  is a union-soft ideal over  $U$ .

*Proof.* Assume that the nonempty  $\tau$ -exclusive set of  $\mathcal{F}_A$  is an ideal of  $A$  for all  $\tau \subseteq U$ . Then  $0 \in e(\mathcal{F}_A; \tau)$ . If there exists  $a \in A$  such that  $f_A(0) \not\subseteq f_A(a)$ , then  $f_A(0) \not\subseteq \tau$  for  $\tau = f_A(a) \setminus f_A(0)$ . Hence  $0 \notin e(\mathcal{F}_A)$ , a contradiction. Therefore  $f_A(0) \subseteq f_A(x)$  for all  $x \in A$ . Let  $x, y \in A$  be such that  $f_A(x * y) = \tau_1$  and  $f_A(y) = \tau_2$ . Let us take  $\tau = \tau_1 \cup \tau_2$ . Then  $x * y \in e(\mathcal{F}_A; \tau)$  and  $y \in e(\mathcal{F}_A; \tau)$ . Since  $e(\mathcal{F}_A; \tau)$  is an ideal of  $A$ , it follows from (2.2) that  $x \in e(\mathcal{F}_A; \tau)$ . Hence  $f_A(x) \subseteq \tau = \tau_1 \cup \tau_2 = f_A(x * y) \cup f_A(y)$ . Consequently,  $\mathcal{F}_A$  is a union-soft ideal over  $U$ .  $\square$

**Definition 5.16.** Let  $(U, E) = (U, X)$  where  $X$  is a BCI-algebra. Given a subalgebra  $A$  of  $E$ , let  $\mathcal{F}_A \in S(U)$ . A union-soft ideal  $\mathcal{F}_A$  over  $U$  is said to be *closed* if the approximate function  $f_A$  of  $\mathcal{F}_A$  satisfies:

$$(5.7) \quad (\forall x \in A) (f_A(0 * x) \subseteq f_A(x)).$$

**Example 5.17.** The union-soft ideal  $\mathcal{F}_E$  in Example 5.3 is closed.

Note that if  $E = X$  is a BCK-algebra, then every union-soft ideal over  $U$  is closed. The following example shows that if  $E = X$  is a BCI-algebra, then there exists a union-soft ideal over  $U$  which is not closed.

**Example 5.18.** In Example 5.4, the union-soft ideal  $\mathcal{F}_E$  over  $U$  is not closed since

$$f_E(1 \div 2^2) = f_E(2^{-2}) = \tau_2 \not\subseteq \tau_1 = f_E(1) \cup f_E(2^2).$$

**Theorem 5.19.** Let  $(U, E) = (U, X)$  where  $X$  is a BCI-algebra. Then a union-soft ideal over  $U$  is closed if and only if it is a union-soft algebra over  $U$ .

*Proof.* Let  $\mathcal{F}_A$  be a union-soft ideal over  $U$ . If  $\mathcal{F}_A$  is closed, then  $f_A(0 * x) \subseteq f_A(x)$  for all  $x \in A$ . It follows from (5.1) that

$$f_A(x * y) \subseteq f_A((x * y) * x) \cup f_A(x) = f_A(0 * y) \cup f_A(x) \subseteq f_A(x) \cup f_A(y)$$

for all  $x, y \in A$ . Hence  $\mathcal{F}_A$  is a union-soft algebra over  $U$ .

Conversely, if  $\mathcal{F}_A$  is a union-soft algebra over  $U$ , then

$$f_A(0 * x) \subseteq f_A(0) \cup f_A(x) = f_A(x)$$

for all  $x \in A$ . Therefore  $\mathcal{F}_A$  is closed.  $\square$

Let  $X$  be a BCI-algebra and  $B(X) := \{x \in X \mid 0 \leq x\}$ . For any  $x \in X$  and  $n \in \mathbb{N}$ , we define  $x^n$  by

$$x^1 = x, \quad x^{n+1} = x * (0 * x^n).$$

If there is an  $n \in \mathbb{N}$  such that  $x^n \in B(X)$ , then we say that  $x$  is of *finite period* (see [21]), and we denote its period  $|x|$  by

$$|x| = \min\{n \in \mathbb{N} \mid x^n \in B(X)\}.$$

Otherwise,  $x$  is of infinite period and denoted by  $|x| = \infty$ .

**Theorem 5.20.** Let  $(U, E) = (U, X)$  where  $X$  is a BCI-algebra in which every element is of finite period. Then every union-soft ideal over  $U$  is closed.

*Proof.* Let  $\mathcal{F}_E$  be a union-soft ideal over  $U$ . For any  $x \in E$ , assume that  $|x| = n$ . Then  $x^n \in B(X)$ . Note that

$$\begin{aligned} (0 * x^{n-1}) * x &= (0 * (0 * (0 * x^{n-1}))) * x \\ &= (0 * x) * (0 * (0 * x^{n-1})) \\ &= 0 * (x * (0 * x^{n-1})) = 0 * x^n = 0, \end{aligned}$$

and so  $f_E((0 * x^{n-1}) * x) = f_E(0) \subseteq f_E(x)$  by (4.2). It follows from (5.1) that

$$(5.8) \quad f_E(0 * x^{n-1}) \subseteq f_E((0 * x^{n-1}) * x) \cup f_E(x) \subseteq f_E(x).$$

Also, note that

$$\begin{aligned} (0 * x^{n-2}) * x &= (0 * (0 * (0 * x^{n-2}))) * x \\ &= (0 * x) * (0 * (0 * x^{n-2})) \\ &= 0 * (x * (0 * x^{n-2})) = 0 * x^{n-1}, \end{aligned}$$

which implies from (5.8) that

$$f_E((0 * x^{n-2}) * x) = f_E(0 * x^{n-1}) \subseteq f_E(x).$$

Using (5.1), we have

$$f_E(0 * x^{n-2}) \subseteq f_E((0 * x^{n-2}) * x) \cup f_E(x) \subseteq f_E(x).$$

Continuing this process, we have  $f_E(0 * x) \subseteq f_E(x)$  for all  $x \in E$ . Therefore  $\mathcal{F}_E$  is closed.  $\square$

### 6. Union-soft commutative ideals in BCK-algebras

**Definition 6.1.** Let  $(U, E) = (U, X)$  where  $X$  is a BCK-algebra. Given a subalgebra  $A$  of  $E$ , let  $\mathcal{F}_A \in S(U)$ . Then  $\mathcal{F}_A$  is called a *union-soft commutative ideal* over  $U$  if it satisfies (4.2) and

$$(6.1) \quad (\forall x, y, z \in A) (f_A(x * (y * (y * x)))) \subseteq f_A((x * y) * z) \cup f_A(z).$$

**Example 6.2.** Let  $(U, E) = (U, X)$  where  $X = \{0, a, b, c\}$  is a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Define a soft set  $\mathcal{F}_E$  over  $U$  by

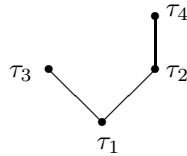
$$\mathcal{F}_E = \{(0, \tau_1), (a, \tau_3), (b, \tau_3), (c, \tau_2)\},$$

where  $\tau_1 \subsetneq \tau_2 \subsetneq \tau_3 \subseteq U$ . Routine calculations give that  $\mathcal{F}_E$  is a union-soft commutative ideal over  $U$ .

**Example 6.3.** Let  $(U, E) = (U, X)$  where  $X = \{0, 1, 2, 3, 4\}$  is a BCK-algebra with the following Cayley table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	1	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

Let  $\{\tau_1, \tau_2, \tau_3, \tau_4\}$  be a class of subsets of  $U$  which is a poset with the following Hasse diagram:



Define a soft set  $\mathcal{F}_E$  over  $U$  as follows:

$$\mathcal{F}_E = \{(0, \tau_1), (1, \tau_3), (2, \tau_3), (3, \tau_2), (4, \tau_4)\}.$$

Routine calculations show that  $\mathcal{F}_E$  is a union-soft commutative ideal over  $U$ .

**Theorem 6.4.** Let  $(U, E) = (U, X)$  where  $X$  is a BCK-algebra. Then any union-soft commutative ideal over  $U$  is a union-soft ideal over  $U$ .

*Proof.* Let  $\mathcal{F}_A$  be a union-soft commutative ideal over  $U$  where  $A$  is a subalgebra of  $E$ . If we take  $y = 0$  in (6.1) and use (a1) and (V), then

$$f_A(x) = f_A(x * (0 * (0 * x))) \subseteq f_A((x * 0) * z) \cup f_A(z) = f_A(x * z) \cup f_A(z)$$

for all  $x, z \in A$ . Hence  $\mathcal{F}_A$  is a union-soft ideal over  $U$ . □

The following example shows that the converse of Theorem 6.4 is not true.

**Example 6.5.** Let  $(U, E) = (U, X)$  where  $X = \{0, a, b, c, d\}$  is a BCK-algebra with the following Cayley table:

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	d	c	0

Define a soft set  $\mathcal{F}_E \in S(U)$  by

$$\mathcal{F}_E = \{(0, \tau_1), (a, \tau_2), (b, \tau_3), (c, \tau_3), (d, \tau_3)\},$$

where  $\tau_1 \subsetneq \tau_2 \subsetneq \tau_3 \subseteq U$ . Routine calculations give that  $\mathcal{F}_E$  is a union-soft ideal over  $U$ . But it is not a union-soft commutative ideal over  $U$  since

$$f_E(b * (c * (c * b))) = \tau_3 \not\subseteq \tau_1 = f_E((b * c) * 0) \cup f_E(0).$$

We provide conditions for a union-soft ideal over  $U$  to be a union-soft commutative ideal over  $U$ .

**Theorem 6.6.** *Let  $(U, E) = (U, X)$  where  $X$  is a BCK-algebra. Given a subalgebra  $A$  of  $E$ , let  $\mathcal{F}_A \in S(U)$ . Then  $\mathcal{F}_A$  is a union-soft commutative ideal over  $U$  if and only if  $\mathcal{F}_A$  is a union-soft ideal over  $U$  satisfying the following condition:*

$$(6.2) \quad (\forall x, y \in A) (f_A(x * (y * (y * x))) \subseteq f_A(x * y)).$$

*Proof.* Assume that  $\mathcal{F}_A$  is a union-soft commutative ideal over  $U$ . Then  $\mathcal{F}_A$  is a union-soft ideal over  $U$  by Theorem 6.4. Taking  $z = 0$  in (6.1) and using (4.2) and (a1), we have (6.2).

Conversely, let  $\mathcal{F}_A$  be a union-soft ideal over  $U$  satisfying the condition (6.2). Then

$$(6.3) \quad (\forall x, y, z \in A) (f_A(x * y) \subseteq f_A((x * y) * z) \cup f_A(z))$$

by (5.1). Combining (6.2) and (6.3) induces (6.1). Hence  $\mathcal{F}_A$  is a union-soft commutative ideal over  $U$ .  $\square$

**Corollary 6.7.** *Let  $(U, E) = (U, X)$  where  $X$  is a BCK-algebra and  $\mathcal{F}_E \in S(U)$ . Then  $\mathcal{F}_E$  is a union-soft commutative ideal over  $U$  if and only if  $\mathcal{F}_E$  is a union-soft ideal over  $U$  satisfying the following condition:*

$$(\forall x, y \in E) (f_E(x * (y * (y * x))) \subseteq f_E(x * y)).$$

**Theorem 6.8.** *Let  $(U, E) = (U, X)$  where  $X$  is a commutative BCK-algebra. Then every union-soft ideal over  $U$  is a union-soft commutative ideal over  $U$ .*

*Proof.* Let  $\mathcal{F}_A$  be a union-soft ideal over  $U$ , where  $A$  is a subalgebra of  $E$ . Note that

$$\begin{aligned} & ((x * (y * (y * x))) * ((x * y) * z)) * z \\ &= ((x * (y * (y * x))) * z) * ((x * y) * z) \\ &\leq (x * (y * (y * x))) * (x * y) \\ &= (x * (x * y)) * (y * (y * x)) = 0, \end{aligned}$$

that is,  $((x * (y * (y * x))) * ((x * y) * z)) \leq z$  for all  $x, y, z \in A$ . It follows from Proposition 5.10 that

$$f_A(x * (y * (y * x))) \subseteq f_A(((x * y) * z)) \cup f_A(z).$$

Therefore  $\mathcal{F}_A$  is a union-soft commutative ideal over  $U$ .  $\square$

**Theorem 6.9.** *Let  $(U, E) = (U, X)$  where  $X$  is a BCK-algebra. Given a subalgebra  $A$  of  $E$ , let  $\mathcal{F}_A \in S(U)$  in which the following conditions are valid:*

- (i)  $(\forall x, y \in A) (x * (x * y) \leq y * (y * x))$ ,
- (ii)  $\mathcal{F}_A$  is a union-soft ideal over  $U$ .

*Then  $\mathcal{F}_A$  is a union-soft commutative ideal over  $U$ .*



*Proof.* For any  $x, y \in A$ , we have

$$(x * (y * (y * x))) * (x * y) = (x * (x * y)) * (y * (y * x)) = 0$$

by (a3) and (i). Hence  $x * (y * (y * x)) \leq x * y$  for all  $x, y \in A$ , which implies from Lemma 5.5 that  $f_A(x * (y * (y * x))) \subseteq f_A(x * y)$ . Using Theorem 6.6, we conclude that  $\mathcal{F}_A$  is a union-soft commutative ideal over  $U$ .  $\square$

Using the notion of  $\tau$ -exclusive sets, we consider a characterization of a union-soft commutative ideal.

**Theorem 6.10.** *Let  $(U, E) = (U, X)$  where  $X$  is a BCK-algebra, Given a subalgebra  $A$  of  $E$ , let  $\mathcal{F}_A \in S(U)$ . Then the following are equivalent.*

- (1)  $\mathcal{F}_A$  is a union-soft commutative ideal over  $U$ .
- (2) The nonempty  $\tau$ -exclusive set of  $\mathcal{F}_A$  is a commutative ideal of  $A$  for any  $\tau \subseteq U$ .

*Proof.* Assume that  $\mathcal{F}_A$  is a union-soft commutative ideal over  $U$ . Let  $\tau$  be a subset of  $U$  such that  $e(\mathcal{F}_A; \tau) \neq \emptyset$ . Then  $f_A(x) \subseteq \tau$  for some  $x \in A$ , which implies from (4.2) that  $f_A(0) \subseteq f_A(x) \subseteq \tau$ . Hence  $0 \in e(\mathcal{F}_A; \tau)$ . Let  $x, y, z \in A$  be such that  $(x * y) * z \in e(\mathcal{F}_A; \tau)$  and  $z \in e(\mathcal{F}_A; \tau)$ . Then  $f_A((x * y) * z) \subseteq \tau$  and  $f_A(z) \subseteq \tau$ . It follows from (6.1) that

$$\tau \supseteq f_A((x * y) * z) \cup f_A(z) \supseteq f_A(x * (y * (y * x)))$$

so that  $x * (y * (y * x)) \in e(\mathcal{F}_A; \tau)$ . Therefore  $e(\mathcal{F}_A; \tau)$  is a commutative ideal of  $A$  for any  $\tau \subseteq U$ .

Conversely, suppose that the nonempty  $\tau$ -exclusive set of  $\mathcal{F}_A$  is a commutative ideal of  $A$  for any  $\tau \subseteq U$ . Then  $e(\mathcal{F}_A; \tau)$  is a subalgebra of  $A$ . Let  $x, y \in A$  be such that  $f_A(x) = \tau_1$  and  $f_A(y) = \tau_2$ . Taking  $\tau = \tau_1 \cup \tau_2$  implies that  $x, y \in e(\mathcal{F}_A; \tau)$ , and so  $x * y \in e(\mathcal{F}_A; \tau)$ . Thus

$$f_A(x * y) \subseteq \tau = \tau_1 \cup \tau_2 = f_A(x) \cup f_A(y),$$

which implies that  $f_A(0) = f_A(x * x) \subseteq f_A(x)$  for all  $x \in A$ . Let  $x, y, z \in A$  be such that  $f_A((x * y) * z) = \tau_1$  and  $f_A(z) = \tau_2$ . Let us take  $\tau = \tau_1 \cup \tau_2$ . Then  $(x * y) * z \in e(\mathcal{F}_A; \tau)$  and  $z \in e(\mathcal{F}_A; \tau)$ . Since  $e(\mathcal{F}_A; \tau)$  is a commutative ideal, it follows from (2.3) that  $x * (y * (y * x)) \in e(\mathcal{F}_A; \tau)$ . Hence

$$f_A(x * (y * (y * x))) \subseteq \tau = \tau_1 \cup \tau_2 = f_A((x * y) * z) \cup f_A(z).$$

Therefore  $\mathcal{F}_A$  is a union-soft commutative ideal over  $U$ .  $\square$

The commutative ideals  $e(\mathcal{F}_A; \tau)$  in Theorem 6.10 are called the *exclusive commutative ideals* of  $\mathcal{F}_A$ .

**Theorem 6.11.** *Let  $(U, E) = (U, X)$  where  $X$  is a BCK-algebra and let  $\mathcal{F}_A \in S(U)$ . For a subset  $\tau$  of  $U$ , define a soft set  $\mathcal{F}_A^*$  over  $U$  by*

$$f_A^* : E \rightarrow \mathcal{P}(U), x \mapsto \begin{cases} f_A(x) & \text{if } x \in e(\mathcal{F}_A; \tau), \\ U & \text{otherwise.} \end{cases}$$

*If  $\mathcal{F}_A$  is a union-soft commutative ideal over  $U$ , then so is  $\mathcal{F}_A^*$ .*

*Proof.* If  $\mathcal{F}_A$  is a union-soft commutative ideal over  $U$ , then  $e(\mathcal{F}_A; \tau)$  is a commutative ideal of  $A$  for any  $\tau \subseteq U$ . Hence  $0 \in e(\mathcal{F}_A; \tau)$ , and so  $f_A^*(0) = f_A(0) \subseteq f_A(x) \subseteq f_A^*(x)$  for all  $x \in A$ . Let  $x, y, z \in A$ . If  $(x * y) * z \in e(\mathcal{F}_A; \tau)$  and  $z \in e(\mathcal{F}_A; \tau)$ , then  $x * (y * (y * x)) \in e(\mathcal{F}_A; \tau)$  and so

$$\begin{aligned} f_A^*(x * (y * (y * x))) &= f_A(x * (y * (y * x))) \\ &\subseteq f_A((x * y) * z) \cup f_A(z) = f_A^*((x * y) * z) \cup f_A^*(z). \end{aligned}$$

If  $(x * y) * z \notin e(\mathcal{F}_A; \tau)$  or  $z \notin e(\mathcal{F}_A; \tau)$ , then  $f_A^*((x * y) * z) = U$  or  $f_A^*(z) = U$ . Hence

$$f_A^*(x * (y * (y * x))) \subseteq U = f_A^*((x * y) * z) \cup f_A^*(z).$$

This shows that  $\mathcal{F}_A^*$  is a union-soft commutative ideal over  $U$ . □

**Theorem 6.12.** *Let  $(U, E) = (U, X)$  where  $X$  is a BCK-algebra. Then any commutative ideal of  $E$  can be realized as an exclusive commutative ideal of some union-soft commutative ideal over  $U$ .*

*Proof.* Let  $A$  be a commutative ideal of  $E$ . For any subset  $\tau \subsetneq U$ , let  $\mathcal{F}_A$  be a soft set over  $U$  defined by

$$f_A : E \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \tau & \text{if } x \in A, \\ U & \text{if } x \notin A. \end{cases}$$

Obviously,  $f_A(0) \subseteq f_A(x)$  for all  $x \in E$ . For any  $x, y, z \in E$ , if  $(x * y) * z \in A$  and  $z \in A$ , then  $x * (y * (y * x)) \in A$ . Hence

$$f_A((x * y) * z) \cup f_A(z) = \tau = f_A(x * (y * (y * x))).$$

If  $(x * y) * z \notin A$  or  $z \notin A$ , then  $f_A((x * y) * z) = U$  or  $f_A(z) = U$ . It follows that

$$f_A(x * (y * (y * x))) \subseteq U = f_A((x * y) * z) \cup f_A(z).$$

Therefore  $\mathcal{F}_A$  is a union-soft commutative ideal over  $U$ , and clearly  $e(\mathcal{F}_A; \tau) = A$ . This completes the proof. □

**Theorem 6.13** (Extension property). *Let  $(U, E) = (U, X)$  where  $X$  is a BCK-algebra. Given subalgebras  $A$  and  $B$  of  $E$ , let  $\mathcal{F}_A, \mathcal{F}_B \in S(U)$  such that*

- (i)  $\mathcal{F}_A \tilde{\subseteq} \mathcal{F}_B$ ,
- (ii)  $\mathcal{F}_B$  is a union-soft ideal over  $U$ .

*If  $\mathcal{F}_A$  is a union-soft commutative ideal over  $U$ , then so is  $\mathcal{F}_B$ .*

*Proof.* Let  $\tau \subseteq U$  be such that  $e(\mathcal{F}_B; \tau) \neq \emptyset$ . From the condition (ii) and Theorem 5.14, we know that  $e(\mathcal{F}_B; \tau)$  is an ideal. Assume that  $\mathcal{F}_A$  is a union-soft commutative ideal over  $U$ . Then  $e(\mathcal{F}_A; \tau)$  is a commutative ideal for every  $\tau \subseteq U$ . Let  $x, y \in E$  and  $\tau \subseteq U$  be such that  $x * y \in e(\mathcal{F}_B; \tau)$ . Since

$$(x * (x * y)) * y = (x * y) * (x * y) = 0 \in e(\mathcal{F}_A; \tau),$$

it follows from (a3), Theorem 2.1 and (i) that

$$(x * (y * (y * (x * (x * y))))) * (x * y)$$

$$= (x * (x * y)) * (y * (y * (x * (x * y)))) \in e(\mathcal{F}_A; \tau) \subseteq e(\mathcal{F}_B; \tau)$$

so that

$$(6.4) \quad x * (y * (y * (x * (x * y)))) \in e(\mathcal{F}_B; \tau)$$

since  $e(\mathcal{F}_B; \tau)$  is an ideal and  $x * y \in e(\mathcal{F}_B; \tau)$ . Note that  $x * (x * y) \leq x$ , and so

$$y * (y * (x * (x * y))) \leq y * (y * x)$$

by (a2). Hence

$$(6.5) \quad x * (y * (y * x)) \leq x * (y * (y * (x * (x * y)))).$$

Using (6.4) and (6.5), we get  $x * (y * (y * x)) \in e(\mathcal{F}_B; \tau)$ . Thus  $e(\mathcal{F}_B; \tau)$  is a commutative ideal by Theorem 2.1, and so  $\mathcal{F}_B$  is a union-soft commutative ideal over  $U$  by Theorem 6.10.  $\square$

**Acknowledgements.** The author wishes to thank the anonymous reviewer(s) for their valuable suggestions. The author is an Executive Research Worker of Educational Research Institute Teachers College in Gyeongsang National University.

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