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ON THE ADMISSIBILITY OF THE SPACE $L_0(\mathcal{A}, X)$ OF VECTOR-VALUED MEASURABLE FUNCTIONS

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Abstract. We prove the admissibility of the space $L_0(\mathcal{A}, X)$ of vector-valued measurable functions determined by real-valued finitely additive set functions defined on algebras of sets.

The notion of admissibility introduced by Klee [7] guarantees that a compact mapping into an admissible Hausdorff topological vector space E can be approximated by compact finite dimensional mappings. This notion is very important in degree theory and fixed point theory. It is known that locally convex spaces are admissible (see [10]). There are some classes of nonlocally convex spaces which are admissible. Riedrich in [13] proved the admissibility of the space $S(0, 1)$ of measurable functions and in [12] the admissibility of the space $L_p(0, 1)$ for $0 < p < 1$. The admissibility of other function spaces has been proved by Mach [6] and Ishii [8]. In [14] it is proved the admissibility of spaces of Besov-Triebel-Lizorkin type.

Definition 1 ([7]). Let E be a Haudorff topological vector space. A subset Z of E is said to be admissible if for every compact subset K of Z and for every neighborhood V of zero in E there exists a continuous mapping $H : K \to Z$ such that dim(span $[H(K)]$)$< +\infty$ and $x - Hx \in V$ for every $x \in K$. If $Z = E$ we say that the space E is admissible.

In this paper we deal with spaces of vector-valued measurable functions and, as a major fact, instead of σ-additive measures we consider finitely additive set functions defined on algebras of sets.

Let X be a Banach space, Ω a nonempty set, $\mathcal{A}$ a subalgebra of the power set $\mathcal{P}(\Omega)$ of Ω and $\mu : \mathcal{A} \to \mathbb{R}$ a finitely additive set function. We prove

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the admissibility of the space $L_0(\mathcal{A}, X)$ of all X-valued μ-measurable functions defined on Ω (see [3, Chp III]).

It is important to notice that in [2] Cauty provides an example of a metric linear space in which the admissibility fails. Moreover, it is known that in general $L_0(\mathcal{A}, X)$ is not homeomorphic to the classical space $L_0([0,1], X)$ of all Lebesgue measurable functions from $[0,1]$ to X endowed with the topology generated by the convergence in measure, and to our knowledge the question if all the spaces $L_0(\mathcal{A}, X)$ are homeomorphic or not to an Hilbert space is open. Some results in this latter direction have appeared in [11, Theorem 4.9] in the case where μ is a finite nonatomic measure.

1. Preliminaries and notations

Let $(X, \|\cdot\|_X)$ be a real or complex Banach space, Ω a nonempty set, $\mathcal{A}$ a subalgebra of the power set $\mathcal{P}(\Omega)$ of Ω and $\mu : \mathcal{A} \to \mathbb{R}$ a finitely additive set function. Then for every $A \in \mathcal{A}$ the total variation $|\mu|(A)$ of μ on A is defined by $|\mu|(A) = \sup \sum_{i=1}^{n} |\mu(A_i)|$ where the supremum is taken over all finite sequences (A_i) of disjoint sets in $\mathcal{A}$ with $A_i \subseteq A$. Then $|\mu|$ induces the submeasure $\eta : \mathcal{P}(\Omega) \to [0, +\infty[$ defined by $\eta(E) = \inf\{|\mu|(A) : A \in \mathcal{A} \text{ and } E \subseteq A\}$ for $E \subseteq \Omega$. We denote by

$$S(\mathcal{A}, X) = \Big\{ \sum_{i=1}^{n} x_i \chi_{A_i} : \ n \in \mathbb{N}, \ x_i \in X, \ A_i \in \mathcal{A} \Big\},$$

the space of all X-valued simple functions on Ω; where χ_A denotes the characteristic function of the set A defined on Ω. Let X^Ω denote the set of all functions f from Ω to X. For a function $f \in X^\Omega$ we set

$$\|f\|_0 = \inf\{\alpha \geq 0 : \eta(\{\|f\|_X \geq \alpha\}) \leq \alpha\},$$

where $\|f\|_X$ denotes the function $t \to \|f(t)\|_X$ and $\{\|f\|_X \geq \alpha\} = \{t \in \Omega : \|f(t)\|_X \geq \alpha\}$, with the convention $\inf \emptyset = +\infty$. Then $\|\cdot\|_0$ has the following properties:

$$\begin{aligned} &\|0\|_0 = 0, \\ &\|f+g\|_0 \leq \|f\|_0 + \|g\|_0, \\ &\|f\|_X \leq \|g\|_X \text{ implies } \|f\|_0 \leq \|g\|_0 \text{ for } f, g \in X^\Omega, \\ &\|y\chi_A\|_0 = \min\{\eta(A), \|y\|_X\} \text{ for } A \subseteq \Omega, \ y \in X \text{ and } f \in X^\Omega. \end{aligned} \tag{1}$$

A function $f \in X^\Omega$ is said to be a *μ-null function* if $\eta(\{\|f\|_X \geq a\}) = 0$ for any $a > 0$. Then by $L_0(\mathcal{A}, X)$ we denote the F-normed space (in the sense of [9]) given by the closure of the space $S(\mathcal{A}, X)$ in $(X^\Omega, \|\cdot\|_0)$, where it is understood that we identify functions differing by a μ-null function.

We briefly recall the definitions of integrable function and integral for an integrable function, with respect to μ, of a function f of $L_0(\mathcal{A}, X)$ as introduced in [3]. Let $s = \sum_{i=1}^{n} x_i \chi_{A_i}$ be a simple function in $S(\mathcal{A}, X)$ and $A \in \mathcal{A}$, then the integral over A of f is defined by

$$\int_A f(t)\ \mu(dt) = \sum_{i=1}^{n} x_i\ \mu(A_i).$$

Let $L_1(X)$ denote the Lebesgue space of all functions $f \in L_0(\mathcal{A}, X)$ for which there is a sequence (s_n) in $S(\mathcal{A}, X)$ converging to f with respect to $\|\cdot\|_0$ such that

$$\lim_{m,n} \int_\Omega \|s_m(t) - s_n(t)\|_X\ |\mu|(dt) = 0.$$

The sequence (s_n) is said to be a determining sequence for f and the integral over A of f is defined by

$$\int_A f(t)\ \mu(dt) = \lim_{n\to\infty} \int_A s_n(t)\mu(dt), \quad A \in \mathcal{A}.$$

For each $f \in L_1(X)$, $\|f\|_1 = \int_\Omega \|f(t)\|_X |\mu|(dt)$ and we also have

$$\|f\|_1 = \lim_{n\to\infty} \int_\Omega \|s_n(t)\|_X |\mu|(dt).$$

Obviously $\eta(A) = |\mu|(A) = \|\chi_A\|_1$ for $A \in \mathcal{A}$. In the sequel we will use the property given in the following lemma.

Lemma 1. *Let $f \in L_1(X)$. Then $\|f\|_0 \le \|f\|_1^{1/2}$.*

Proof. Let $s \in S(\mathcal{A}, X) \setminus \{0\}$. Assume on the contrary that $\|s\|_0 > \|s\|_1^{1/2}$. Take $\alpha = \|s\|_1^{1/2}$, then

$$\begin{aligned} \|s\|_1 &= \int_\Omega \|s(t)\|_X\ |\mu|(dt) \ge \int_{\{\|s\|_X \ge \alpha\}} \|s(t)\|_X\ |\mu|(dt) \ge \alpha\eta(\{\|s\|_X \ge \alpha\}) \\ &> \|s\|_1, \end{aligned}$$

which is a contradiction.

Next let $f \in L_1(X)$ and (s_n) a sequence in $S(\mathcal{A}, X)$ determining f. Then we have both $\lim_{n\to\infty} \|s_n\|_0 = \|f\|_0$ and $\lim_{n\to\infty} \|s_n\|_1 = \|f\|_1$, which imply the assert. □

Let $B_a(X)$ denote the closed ball of radius $a > 0$. We denote by ρ_a the radial projection of X onto $B_a(X)$ defined by

$$\rho_a(x) = \begin{cases} x & \text{if } \|x\|_X \le a, \\ a\frac{x}{\|x\|_X} & \text{if } \|x\|_X > a. \end{cases}$$

Then we define the mapping $T_a :\ L_0(\mathcal{A}, X) \to X^\Omega$ by setting

$$(T_a f)(t) = \rho_a(f(t)), \quad t \in \Omega.$$

The function $T_a s$ is a simple function for each simple function $s \in S(\mathcal{A}, X)$, and moreover it can be easily seen that $T_a(L_0(\mathcal{A}, X)) \subseteq L_0(\mathcal{A}, X)$. The projection

ρ_a is Lipschitz with constant 2 (cf. [4]), thus, since X is a Banach space, by (1) for $f, g \in L_0(\mathcal{A}, X)$ we have

$$\|T_a f - T_a g\|_0 \le 2\|f - g\|_0. \tag{2}$$

Lemma 2. *Let K be a subset of $L_0(\mathcal{A}, X)$. Then*
(i) *$T_a(K) \subseteq L_1(X)$,*
(ii) *the $\|\cdot\|_0$-topology and the $\|\cdot\|_1$-topology coincide on $T_a(K)$.*

Proof. (i) Since $\|T_a f\|_X \le a\chi_\Omega$ for $f \in K$, by [3, Theorem III.2.22] we have $T_a f \in L_1(X)$.

(ii) As $T_a(K) \subseteq \{f \in L_1(X) : \|f\|_X \le a\chi_\Omega\}$ the assert follows from [3, Theorem III.3.6]. □

Next we introduce the operator P_π which will be used for the proof of our main result. Given a partition $\pi = \{A_1, \ldots, A_n\}$ of Ω with $\eta(A_i) > 0$ for $i = 1, \ldots, n$ we consider $P_\pi : L_1(X) \to S(\mathcal{A}, X)$ the linear operator defined by setting

$$P_\pi f = \sum_{i=1}^{n} \frac{\int_{A_i} f(t)\mu(dt)}{\eta(A_i)} \chi_{A_i}.$$

Then for each $f \in L_1(X)$ we have

$$\|P_\pi f\|_1 \le \|f\|_1. \tag{3}$$

Indeed if, for each i, we put $s_i(f) = \int_{A_i} f(t)\mu(dt)/\eta(A_i)$ applying Jensen's inequality, we have

$$\|s_i(f)\|_X \le \frac{\int_{A_i} \|f(t)\|_X \; |\mu|(dt)}{\eta(A_i)}.$$

Consequently we get

$$\begin{aligned} \|P_\pi f\|_1 &= \int_\Omega \|\sum_{i=1}^{n} s_i(f)\chi_{A_i}\|_X |\mu|(dt) = \sum_{i=1}^{n} \int_{A_i} \|s_i(f)\|_X |\mu|(dt) \\ &= \sum_{i=1}^{n} \eta(A_i)\|s_i(f)\|_X \le \sum_{i=1}^{n} \int_{A_i} \|f(t)\|_X \; |\mu|(dt) \\ &= \int_\Omega \|f(t)\|_X |\mu|(dt) = \|f\|_1. \end{aligned}$$

2. Admissibility of $L_0(\mathcal{A}, X)$

We recall that for a bounded subset A of X the Hausdorff measure of noncompactness $\gamma(A)$ of A is the infimum of all $\varepsilon > 0$ such that A has an ε-net in X ([5]). Moreover for each bounded subsets K of $L_0(\mathcal{A}, X)$ we consider the quantitative characteristic $\sigma(K)$, introduced in [1], defined by setting $\sigma(K) = \inf\{\epsilon > 0 : \exists M \subseteq X$ with $\gamma(M) \le \epsilon$ such that, $\forall f \in K$, there exists $D_f \subseteq \Omega$ with $\eta(D_f) \le \epsilon$ and $f(\Omega \setminus D_f) \subseteq M\}$. In order to prove the admissibility of $L_0(\mathcal{A}, X)$ we need the following two lemmas.

Lemma 3. *Let K be a bounded subset of $L_0(\mathcal{A}, X)$. If $\sigma(K) = 0$, then for all $\varepsilon > 0$ there is $a > 0$ such that*

$$\|f - T_a f\|_0 \le \varepsilon \quad \text{for each} \;\; f \in K.$$

Proof. Let $\varepsilon > 0$ be given. Since $\sigma(K) = 0$ there is a subset M of X, with $\gamma(M) \le \varepsilon/2$, such that for all $f \in K$ there is $D_f \subseteq \Omega$ with $\eta(D_f) \le \varepsilon/2$ and $f(\Omega \setminus D_f) \subseteq M$. Fix $y_1, \ldots, y_m \in X$ such that $M \subseteq \cup_{j=1}^m (y_j + B_{\varepsilon/2}(X))$. Then for each $f \in K$ and $t \in \Omega \setminus D_f$ there exists $j \in \{1, \ldots, m\}$ such that $f(t) \in y_j + B_{\varepsilon/2}(X)$. Therefore

$$\|f(t)\|_X \le \|f(t) - y_j\|_X + \|y_j\|_X \le \frac{\varepsilon}{2} + \|y_j\|_X.$$

Set $a = \varepsilon/2 + \max_j \|y_j\|_X$. Then $f(\Omega \setminus D_f) \subseteq B_a(X)$, which implies $f = T_a f$ on $\Omega \setminus D_f$. Since $\|T_a f\|_X \le \|f\|_X$, by (1), we have $\|(T_a f)\chi_{D_f}\|_0 \le \|f\chi_{D_f}\|_0$. Moreover $\|f\chi_{D_f}\|_0 \le \eta(D_f)$, and thus we find

$$\|f - T_a f\|_0 = \|(f - T_a f)\chi_{D_f}\|_0 \le \|f\chi_{D_f}\|_0 + \|(T_a f)\chi_{D_f}\|_0 \le 2\|f\chi_{D_f}\|_0 \le \varepsilon,$$

which gives the result. □

Lemma 4. *Let $\pi = \{A_1, \ldots, A_n\}$ be a finite partition of Ω. Then the subspace*

$$S(\pi) = \Big\{ s \in S(\mathcal{A}, X) : s = \sum_{i=1}^n x_i \chi_{A_i}, \quad x_i \in X \Big\}$$

of $L_0(\mathcal{A}, X)$ is admissible.

Proof. Let W be a compact subset of $S(\pi)$ and $\varepsilon > 0$ be given. For each $u \in W$ we can write

$$u = \sum_{i=1}^n x_i(u)\chi_{A_i}$$

for suitable elements $x_i(u)$ of X. For any fixed $i = 1, \ldots, n$, the set $C_i = \{x_i(u) : \; u \in W\}$ is a compact subset of X, and consequently $C = \cup_{i=1}^n C_i$ is also a compact subset of X.

Let $\delta = \varepsilon/l$. Then by the admissibility of the Banach space X, there exist a finite dimensional space $Z_\varepsilon = \text{span}[z_1, \ldots, z_m]$ in X and a continuous mapping $H_\varepsilon : \; C \to Z_\varepsilon$ such that

$$\|x - H_\varepsilon(x)\|_X \le \delta \;\; \text{for all} \;\; x \in C. \tag{4}$$

Then for each $i \in \{1, \ldots, n\}$ and for suitable $x_j^i(u) \in X$, with $j = 1, \ldots, m$, we can write

$$H_\varepsilon(x_i(u)) = \sum_{j=1}^m x_j^i(u) z_j.$$

As no confusion can arise, we denote again by H_ε the continuous mapping $H_\varepsilon : W \to S(\pi)$ defined by

$$H_\varepsilon u = \sum_{i=1}^{n} H_\varepsilon(x_i(u))\chi_{A_i} = \sum_{i=1}^{n}\Big(\sum_{j=1}^{m} x_j^i(u)z_j\Big)\chi_{A_i}.$$

Then $H_\varepsilon(W) \subseteq \operatorname{span}[\chi_{A_i} z_j;\ i = 1, \dots, n;\ j = 1, \dots, m]$ and

$$\dim(\operatorname{span}[H_\varepsilon(W)]) < +\infty.$$

On the other hand, for each $u \in W$ we have

$$\|u - H_\varepsilon u\|_0 = \Big\| \sum_{i=1}^{n} x_i(u)\chi_{A_i} - \sum_{i=1}^{n}\Big(\sum_{j=1}^{m} x_j^i(u)z_j\Big)\chi_{A_i}\Big\|_0 \le \sum_{i=1}^{n}\Big\|\Big(x_i(u) - \sum_{j=1}^{m} x_j^i(u)z_j\Big)\chi_{A_i}\Big\|_0. \tag{5}$$

Next by (4) we have

$$\Big\|\Big(x_i(u) - \sum_{j=1}^{m} x_j^i(u)z_j\Big)\chi_{A_i}\Big\|_X \le\ \delta\chi_{A_i},$$

hence for a fixed $y \in X$ with $\|y\|_X = 1$ we find

$$\Big\|\Big(x_i(u) - \sum_{j=1}^{m} x_j^i(u)z_j\Big)\chi_{A_i}\Big\|_X \le \|\delta y\chi_{A_i}\|_X.$$

Consequently

$$\Big\|\Big(x_i(u) - \sum_{j=1}^{m} x_j^i(u)z_j\Big)\chi_{A_i}\Big\|_0 \le \|\delta y\chi_{A_i}\|_0 = \min\{\eta(A_i),\ \delta\} \le \delta.$$

From (5) we get $\|u - H_\varepsilon u\|_0 \le \varepsilon$ which completes the proof. □

Now we are in the position to prove our main result.

Theorem 1. *The space $L_0(\mathcal{A}, X)$ is admissible.*

Proof. Fix K a compact set in $L_0(\mathcal{A}, X)$, and $\varepsilon > 0$. Since K is compact, by [1, Theorem 2.1 and Proposition 2.1] we have $\sigma(K) = 0$. Thus by Lemma 3 there is $a > 0$ such that

$$\|f - T_a f\|_0 \le \frac{\varepsilon}{3}. \tag{6}$$

Next we show that there is a partition π of Ω such that

$$\|g - P_\pi g\|_0 \le \frac{\varepsilon}{3} \quad \text{for each } g \in T_a(K). \tag{7}$$

Let $\delta > 0$ be given. Since by (2) T_a is continuous with respect to $\|\cdot\|_0$, we have that $T_a(K)$ is compact in $(L_1(X), \|\cdot\|_0)$. Moreover by Lemma 2, the $\|\cdot\|_0$-topology and the $\|\cdot\|_1$-topology coincide on $T_a(K)$. So $T_a(K)$ is compact in $(L_1(X), \|\cdot\|_1)$. Hence we can choose $g_1, \ldots, g_n$ in $T_a(K)$ such that $T_a(K) \subseteq \bigcup_{i=1}^n \big(g_i + B_{\delta/3}(X)\big)$. For each fixed $i = 1, \ldots, n$ let s_i, say $s_i = \sum_{j=1}^{k_i} x_j \chi_{A_j}$, be a simple function such that $\|g_i - s_i\|_1 \leq \delta/6$. Set $\pi(g_i) = \{A_1, \ldots, A_{k_i}\}$, then $P_{\pi(g_i)} s_i = s_i$, therefore having in mind (3) we find

$$\|g_i - P_{\pi(g_i)} g_i\|_1 \leq \|g_i - s_i\|_1 + \|P_{\pi(g_i)} s_i - P_{\pi(g_i)} g_i\|_1 \leq \frac{\delta}{3}.$$

Denote π the partition generated by all $\pi(g_i)$ $(i = 1, \ldots, n)$. Let $g \in T_a(K)$, then there exists $i \in \{1, \ldots, n\}$ such that $g = g_i + h$ and $\|h\|_1 < \frac{\delta}{3}$. Therefore

$$\|g - P_\pi g\|_1 \leq \|g_i - P_\pi g_i\|_1 + \|h - P_\pi h\|_1 \leq \frac{\delta}{3} + 2\|h\|_1 \leq \delta.$$

By Lemma 1, for $\delta = (\varepsilon/3)^2$ the assert (7) follows.

Assume $\pi = \{A_1, \ldots, A_l\}$, then the set $W = P_\pi(T_a(K))$ is a compact set included in $S(\pi) = \Big\{s \in S(\mathcal{A}, X) : s = \sum_{i=1}^l x_i \chi_{A_i}, \quad x_i \in X\Big\}$.

Hence by Lemma 4 there is $H_\varepsilon : W \to L_0(\mathcal{A}, X)$ such that $\operatorname{span}[H_\varepsilon(W)]$ is finite dimensional and

$$\|u - H_\varepsilon u\|_0 \leq \frac{\varepsilon}{3} \quad \text{for each } u \in W. \tag{8}$$

Then the continuous mapping $H : K \to L_0(\mathcal{A}, X)$ defined by $H = H_\varepsilon \circ P_\pi \circ T_a$ satisfies $\operatorname{span}[H(K)] < +\infty$. Moreover by (6), (7) and (8) we have

$$\|f - Hf\|_0 \leq \|f - T_a f\|_0 + \|T_a f - P_\pi T_a f\|_0 + \|P_\pi T_a f - Hf\|_0 \leq \varepsilon$$

and the admissibility of $L_0(\mathcal{A}, X)$ is proved. □

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