Bull. Korean Math. Soc. **50** (2013), No. 6, pp. 1905–1914
http://dx.doi.org/10.4134/BKMS.2013.50.6.1905

STRONG MORI MODULES OVER AN INTEGRAL DOMAIN

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Abstract. Let D be an integral domain with quotient field K, M a torsion-free D-module, X an indeterminate, and $N_v = \{f \in D[X] \mid c(f)_v = D\}$. Let $q(M) = M \otimes_D K$ and $M_{w_D} = \{x \in q(M) \mid xJ \subseteq M$ for a nonzero finitely generated ideal J of D with $J_v = D\}$. In this paper, we show that $M_{w_D} = M[X]_{N_v} \cap q(M)$ and $(M[X])_{w_{D[X]}} \cap q(M)[X] = M_{w_D}[X] = M[X]_{N_v} \cap q(M)[X]$. Using these results, we prove that M is a strong Mori D-module if and only if $M[X]$ is a strong Mori $D[X]$-module if and only if $M[X]_{N_v}$ is a Noetherian $D[X]_{N_v}$-module. This is a generalization of the fact that D is a strong Mori domain if and only if $D[X]$ is a strong Mori domain if and only if $D[X]_{N_v}$ is a Noetherian domain.

0. Introduction

Let R be a commutative ring with identity. For any R-module A, let

$$A[X] = \{m_0 + m_1X + \cdots + m_kX^k \mid m_i \in A\}$$

be the set of all polynomials in X with coefficients in A. Then $A[X] = A \otimes_R R[X]$. For all $f = m_0 + m_1X + \cdots + m_kX^k$ and $g = n_0 + n_1X + \cdots + n_lX^l$ in $A[X]$ with $k \le l$ and $h = a_0 + a_1X + \cdots + a_nX^n$ in $R[X]$, define the addition of f, g and the scalar product of f and h by the obvious way:

$$\begin{aligned} f + g &= (m_0 + n_0) + (m_1 + n_1)X + \cdots + (m_k + n_k)X^k \\ &\quad + n_{k+1}X^{k+1} + \cdots + n_lX^l, \\ gf &= \sum_{i=1}^{k+n} c_iX^i, \text{ where } c_i = \sum_{j+s=i} a_jm_s. \end{aligned}$$

It is routine to check that $A[X]$ is an $R[X]$-module [2, Exercise 6, page 32]. Let $f = m_0 + m_1X + \cdots + m_kX^k \in A[X]$. As in the case of polynomial rings, we define "$f = 0 \Leftrightarrow m_0 = m_1 = \cdots = m_k = 0$", and we also say that if $m_k \neq 0$, then m_k is the leading coefficient of f and k is the degree of f denoted by $\deg(f)$ (For convenience, we let $\deg(0) = -\infty$). The *content* of a

Received May 6, 2012; Revised May 4, 2013.
2010 *Mathematics Subject Classification.* 13A15.
Key words and phrases. polynomial module, Noetherian module, strong Mori module.

polynomial $f \in A[X]$, denoted by $c(f)$, is the R-submodule of A generated by the coefficients of f, i.e., $c(f) = \sum_{i=0}^{k} Rm_i$.

Let S_1 be a (saturated) multiplicative subset of R with $0 \notin S_1$. The localization A_{S_1} of A with respect to S_1 is defined obviously, and so A_{S_1} is an R_{S_1}-module. If P is a prime ideal of R, we write A_P for $A_{R\setminus P}$. We say that A is an S_1*-torsion-free module* if $sa = 0$ for $s \in S_1$ and $a \in A$ implies $a = 0$. Clearly, if A is torsion-free, then A is S-torsion-free for any multiplicative subset S of R with $0 \notin S$. Note that the set $\{m \in A \mid sm = 0 \text{ for some } s \in S_1\}$ is the kernel of the canonical R-module homomorphism $\alpha : A \to A_{S_1}$ given by $m \mapsto \frac{m}{1}$. So if A is an S_1-torsion-free module, then α is injective, and hence A can be considered as an R-submodule of A_{S_1}; in this case, we write $m = s \cdot \frac{m}{s} \in A$ and "$\frac{m}{s} = \frac{m'}{s'} \Leftrightarrow sm' = s'm$" for any $s, s' \in S_1$ and $m, m' \in A$.

Let D be an integral domain with quotient field K. For any nonzero fractional ideal I of D, define $I_v = (I^{-1})^{-1}$, where $I^{-1} = \{x \in K \mid xI \subseteq D\}$, and let $I_t = \cup\{J_v \mid J \subseteq I$ is a nonzero finitely generated ideal of $D\}$ and $I_w = \{x \in K \mid xJ \subseteq I$ for a nonzero finitely generated ideal J of D with $J^{-1} = D\}$. Let $* = v, t$ or w. We say that I is a $*$-ideal if $I = I_*$. Clearly, $I_w \subseteq I_t \subseteq I_v$, and hence v-ideals are t-ideals and t-ideals are w-ideals. Let $*$-Max(D) denote the set of $*$-ideals maximal among proper integral $*$-ideals of D. It is known that w-Max$(D) = t$-Max(D) and $I_w = \cap_{P\in t\text{-Max}(D)} ID_P$ [1, Corollary 2.13]. Also, t-Max$(D) \neq \emptyset$ if D is not a field. For any undefined concepts and notations, see [2], [4] or [9].

Let M be a torsion-free D-module and $q(M) = M \otimes_D K$, which is the injective envelope of M. Let $GV(D)$ be the set of nonzero finitely generated ideals J of D with $J_v = D$. As in [10, Definition 3], we define $M_{w_D} = \{x \in q(M) \mid Jx \subseteq M$ for some $J \in GV(D)\}$ (If there is no confusion, we simply denote w_D by w). Then M_w is a D-submodule of $q(M)$ and $(M_w)_w = M_w$ [10, Section 2]. We say that M is a w_D-module (simply, w-module) if $M = M_w$; so M_w is a w-module. A w-module M is called a *strong Mori module* if M satisfies the ascending chain condition on w-submodules of M, while D is a *strong Mori domain* if D is a strong Mori module. Clearly, Noetherian modules are strong Mori modules. Conversely, if each maximal ideal of D is a t-ideal (e.g., one-dimensional integral domain), then $N_w = N$ for all D-submodules N of M, and thus a strong Mori module is a Noetherian module. Also, M is a strong Mori D-module if and only if every w-submodule N of M is of finite type, i.e., $N = (\sum_{i=1}^{k} Dm_i)_w$ for some $m_1, \ldots, m_k \in N$ [10, Theorem 4.4]. It is known that A is a Noetherian R-module if and only if $A[X]$ is a Noetherian $R[X]$-module [2, Exercise 10, page 85]. Also, if we set $S = \{f \in R[X] \mid c(f) = R\}$, then S is a regular multiplicative subset of $R[X]$ [4, Proposition 33.1]. So if R is a Noetherian ring, then $R[X]$, and thus $R[X]_S$ is a Noetherian ring. Conversely, if I is an ideal of R, then $IR[X]_S \cap R = I$ [4, Proposition 33.1]; so I is finitely generated when $IR[X]_S$ is finitely generated. Thus, R is a Noetherian ring if and only if $R[X]_S$ is a Noetherian ring. The purpose of this paper is to extend these results to strong Mori modules.

More precisely, let X be an indeterminate and $N_v = \{f \in D[X] \mid c(f)_v = D\}$. In Section 1, we show that A is a Noetherian R-module if and only if $A[X]_S$ is a Noetherian $R[X]_S$-module, where $S = \{f \in R[X] \mid c(f) = R\}$. Next, in Section 2, we show that $M_{w_D} = M[X]_{N_v} \cap q(M)$ and $(M[X])_{w_{D[X]}} \cap q(M)[X] = M_{w_D}[X] = M[X]_{N_v} \cap q(M)[X]$. Then we use these results to prove that M is a strong Mori D-module if and only if $M[X]$ is a strong Mori $D[X]$-module if and only if $M[X]_{N_v}$ is a Noetherian $D[X]_{N_v}$-module. As a corollary, we have that if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of torsion-free D-modules, then M is a strong Mori module if and only if M' and M'' are strong Mori modules. Also, we show that if M is a strong Mori D-module, then a D-module homomorphism $\phi : M \to M$ is surjective if and only if ϕ is bijective.

1. Hilbert basis theorem and Noetherian modules

Throughout this section, R denotes a commutative ring with identity, A is an R-module, X is an indeterminate, and $S = \{f \in R[X] \mid c(f) = R\}$.

The Hilbert basis theorem states that R is a Noetherian ring if and only if $R[X]$ is a Noetherian ring. As the module analog, it is known that A is a Noetherian R-module if and only if $A[X]$ is a Noetherian $R[X]$-module [2, Exercise 10, page 85], which implies the Hilbert basis theorem because R (resp., $R[X]$) is an R-module (resp., $R[X]$-module). Also, R is a Noetherian ring if and only if $R[X]_S$ is a Noetherian ring. In this section, we extend this result to Noetherian modules.

Lemma 1.1. *Let S_1 be a multiplicative subset of R, and assume that A is an S_1-torsion-free module.*

(1) *If N is an R_{S_1}-submodule of A_{S_1}, then $N = (N \cap A)_{S_1}$.*
(2) *If A is a Noetherian R-module, then A_{S_1} is a Noetherian R_{S_1}-module.*

Proof. (1) Clearly, $(N \cap A)_{S_1} \subseteq N$. For the reverse, let $x \in N \subseteq A_{S_1}$, and so $x = \frac{m}{s}$ for some $m \in A$ and $s \in S_1$. As we noted in the introduction, A can be considered as an R-submodule of A_{S_1}; so $sx = m \in N \cap A$. Hence $x \in (N \cap A)_{S_1}$. Thus $N \subseteq (N \cap A)_{S_1}$.

(2) Let N be an R_{S_1}-submodule of A_{S_1}; then $N = (N \cap A)_{S_1}$ by (1). Since $N \cap A$ is an R-submodule of A, there exist some $n_1, \dots, n_k \in N \cap A$ such that $N \cap A = \sum_{i=1}^{k} Rn_i$ [2, Proposition 6.2], and hence $N = (N \cap A)_{S_1} = \sum_{i=1}^{k} R_{S_1} n_i$. Thus A_{S_1} is a Noetherian R_{S_1}-module [2, Proposition 6.2]. □

Lemma 1.2. *Let N be an $R[X]$-submodule of $A[X]$ and let N_k be the set of leading coefficients of polynomials of degree $\leq k$ in N. Then N_k is an R-submodule of A.*

Proof. Let $a, b \in N_k$, and let $f, g \in A[X]$ such that $\deg(f) \leq \deg(g) \leq k$ and a, b are the leading coefficients of f, g, respectively. Clearly, $N_k \subseteq A$. Since

$A[X]$ is an $R[X]$-module, we have $h := X^{\deg(g)-\deg(f)}f \pm g \in A[X]$ and $\deg(h) \leq \deg(g) \leq k$. If $a \pm b = 0$, then $a \pm b \in N_{-\infty} \subseteq N_k$. If $a \pm b \neq 0$, then $a \pm b$ is the leading coefficient of h, and hence $a \pm b \in N_k$. Thus N_k is a subgroup of A. Next, if $r \in R$, then $rf \in A[X]$. If $ra = 0$, then $ra \in N_k$. If $ra \neq 0$, then ra is the leading coefficient of rf and $\deg(rf) = \deg(f) \leq k$; so $ra \in N_k$. Thus N_k is an R-submodule of A. □

The Dedekind-Mertens lemma states that if $f, g \in R[X]$ with $\deg(g) = m$, then $c(f)^{m+1}c(g) = c(f)^m c(fg)$ (see, for example, [4, Theorem 28.1]). The next result is the module analog, whose proof is the same as that of [4, Theorem 28.1].

Proposition 1.3 ([9, Theorem 1.8.11])**.** *If $f \in R[X]$ and $g \in A[X]$, then $c(f)^{m+1}c(g) = c(f)^m c(fg)$ for some integer $m \geq 1$.*

We next give the main result of this section. For easy reference of the reader, we also give the proof of the fact that A is a Noetherian R-module if and only if $A[X]$ is a Noetherian $R[X]$-module.

Theorem 1.4. *For any R-module A, the following statements are equivalent.*

(1) *A is a Noetherian R-module.*
(2) *$A[X]$ is a Noetherian $R[X]$-module.*
(3) *$A[X]_S$ is a Noetherian $R[X]_S$-module, where $S = \{f \in R[X] \mid c(f) = R\}$.*

Proof. (1) ⇒ (2) It suffices to show that every submodule of $A[X]$ is finitely generated. Let B be an $R[X]$-submodule of $A[X]$, and let N_k be the set of leading coefficients of polynomials in B of degree $\leq k$. Then N_k is an R-submodule of A by Lemma 1.2 and $N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq A$. Let $N = \cup_{k\geq 1} N_k$. Since A is a Noetherian R-module, all the N_k and N are finitely generated R-submodules of A. Let $f_1, \ldots, f_n \in B$ such that $f_i = a_i X^{r_i}+$(lower terms) and $N = \sum_{i=1}^n Ra_i$. Let $r = \max\{r_1, \ldots, r_n\}$. For each j from 1 to $r-1$, pick $g_{j1}, \ldots, g_{jk_j} \in B$ such that $g_{ji} = b_{ji}X^{r_{ji}}+$(lower terms) and $N_j = \sum_{i=1}^{k_j} Rb_{ji}$.

Let $f = aX^m+$(lower terms) $\in B$; then $a \in N$, and hence $a = u_1a_1 + u_2a_2 + \cdots + u_na_n$ for some $u_i \in R$. If $m \geq r$, then $f - \sum_{i=1}^n u_iX^{m-r_i}f_i \in B$ and has degree $\leq m-1$. Repeating this process, we have polynomials $h_i \in R[X]$ such that $h^{(0)} := f - \sum_{i=1}^n h_if_i \in B$ and $\deg(h^{(0)}) = k_{h^{(0)}} \leq r-1$. Next, note that the leading coefficient of $h^{(0)}$ is in $N_{k_{h^{(0)}}}$; so there are polynomials $h_i' \in R[X]$ such that $h^{(1)} := h^{(0)} - \sum_i h_i' g_{k_{h^{(0)}}i} \in B$ and $\deg(h^{(1)}) \leq k_{h^{(0)}} - 1$. This process continues until $h^{(i)} = 0$, because $\deg(h^{(0)}) > \deg(h^{(1)}) > \deg(h^{(2)}) > \cdots \geq 0$ cannot contain more than $\deg(h^{(0)})$ integers. Hence $f \in \sum_{i=1}^n R[X]f_i + \sum_{j,i} R[X]g_{ji}$. Thus we have $B = \sum_{i=1}^n R[X]f_i + \sum_{j,i} R[X]g_{ji}$.

(2) ⇒ (3) Let $f \in S$ and $g \in A[X]$ such that $fg = 0$. Then, since $c(f) = R$, by Proposition 1.3 we have $c(g) = c(fg) = (0)$; so $g = 0$. Hence $A[X]$ is S-torsion-free, and thus by Lemma 1.1(2), $A[X]_S$ is a Noetherian $R[X]_S$-module.

(3) $\Rightarrow$ (1) Suppose that N is an R-submodule of A. Then $N[X]$ is an $R[X]$-submodule of $A[X]$, and hence $N[X]_S$ is an $R[X]_S$-submodule of $A[X]_S$. Let $f_1, \ldots, f_k \in N[X]$ such that $N[X]_S = \sum_{i=1}^k R[X]_S f_i$. Note that $N \subseteq N[X]_S$; so if $a \in N$, then there exist polynomials $h_1, \ldots, h_k \in R[X]$ and $g_1, \ldots, g_k \in S$ such that $a = \sum_{i=1}^k \frac{h_i}{g_i} f_i \in (\sum_{i=1}^k c(f_i))[X]_S \subseteq N[X]_S$. So $a \in \sum_{i=1}^k c(f_i)$. Hence $N \subseteq \sum_{i=1}^k c(f_i)$, and thus $N = \sum_{i=1}^k c(f_i)$. Thus N is finitely generated, so A is a Noetherian R-module [2, Proposition 6.2]. $\square$

2. Strong Mori modules

Let D denote an integral domain with quotient field K, M a torsion-free D-module, $q(M) = M \otimes_D K$, X an indeterminate, $N_v = \{f \in D[X] \mid c(f)_v = D\}$.

Note that, since M is torsion-free, $\frac{m}{s} = \frac{m'}{s'}$ if and only if $s'm = sm'$ for $m, m' \in M$ and $s, s' \in D \setminus \{0\}$. Also, if $f = m_0 + m_1X + \cdots + m_kX^k \in M[X]$ with $m_k \neq 0$ and $h = a_0 + a_1X + \cdots + a_nX^n \in D[X]$ with $a_n \neq 0$, then $hf = a_n m_k X^{n+k}+$ (lower terms), and since M is torsion-free, we have $a_n m_k \neq 0$, and hence $hf \neq 0$. Thus, $M[X]$ is a torsion-free $D[X]$-module.

Lemma 2.1. (1) $q(M) = M_{D\setminus\{0\}}$.

(2) *If S is a multiplicative subset of D, then M_S is a D_S-module.*

(3) *M is a D-submodule of $q(M)$.*

(4) *$M[X]_{N_v}$ is a $D[X]_{N_v}$-module.*

(5) *If $S \subseteq T$ are multiplicative subsets of D, then $(M_S)_T = M_T \subseteq q(M)$.*

Proof. (1) [7, Theorem 4.4]. (2) See [7, page 25]. (3) If we let $S = D \setminus \{0\}$, then $M_S = q(M)$ by (1), and since M is S-torsion-free, M is a D-submodule of $q(M)$. (4) This is an immediate consequence of (2) because N_v is a multiplicative subset of $D[X]$. (5) Clear. $\square$

Lemma 2.2. *Let M be a w-module and $P \in t$-$Max(D)$. If N is a D_P-submodule of M_P, then*

(1) *$N = (N \cap M)_P$ and*

(2) *$N \cap M$ is a w-module.*

Proof. (1) This follows directly from Lemma 1.1(1) because M is a torsion-free D-module.

(2) Let $x \in (N \cap M)_w$. Let $J \in GV(D)$ such that $Jx \subseteq N \cap M$; then $J \nsubseteq P$. Choose $a \in J \setminus P$. Then $ax \in Jx \subseteq N \cap M$, and hence $x = \frac{ax}{a} \in (N \cap M)_P = N$. So $(N \cap M)_w \subseteq N \cap M$, and thus $(N \cap M)_w = N \cap M$. $\square$

Proposition 2.3. *If M is a strong Mori D-module, then M_P is a Noetherian D_P-module for all $P \in t$-$Max(D)$.*

Proof. Let $N_1 \subseteq N_2 \subseteq \cdots$ be an ascending chain of D_P-submodules of M_P. Then $N_1 \cap M \subseteq N_2 \cap M \subseteq \cdots$ is an ascending chain of w-submodules of M over D by Lemma 2.2(2). So there exists a positive integer k such that

$N_k \cap M = N_{k+i} \cap M$ for $i = 1, 2, 3, \ldots$. Thus by Lemma 2.2(1), we have $N_k = (N_k \cap M)_P = (N_{k+i} \cap M)_P = N_{k+i}$ for $i = 1, 2, 3, \ldots$. $\square$

It is known that $I_w = ID[X]_{N_v} \cap K = \cap_{P \in t\text{-Max}(D)} ID_P$ for all nonzero fractional ideals I of D. Our next result is a torsion-free module analog.

Lemma 2.4. (1) *$M_P = (M_{w_D})_P$ for all $P \in t$-Max(D).*

(2) *$M_{w_D} = M[X]_{N_v} \cap q(M) = \cap_{P \in t\text{-}Max(D)} M_P$.*

(3) *$M[X]_{N_v} = M_{w_D}[X]_{N_v}$.*

(4) *$(M[X])_{w_{D[X]}} \cap q(M)[X] = M_{w_D}[X]$.*

(5) *$M_{w_D}[X] = M[X]_{N_v} \cap q(M)[X]$.*

Proof. (1) [10, page 1297].

(2) It is known that $M_w = \cap_{P \in t\text{-Max}(D)} M_P$ [1, p. 2463]; so it suffices to show that $M_w = M[X]_{N_v} \cap q(M)$:

($\subseteq$) Let $0 \neq x \in M_w$. Then there exists an $I \in GV(D)$ such that $xI \subseteq M$. So if we choose an $f \in D[X]$ with $c(f) = I$, then $f \in N_v$, and thus $x = \frac{xf}{f} \in M[X]_{N_v} \cap q(M)$.

($\supseteq$) Let $m \in M[X]_{N_v} \cap q(M)$. Then $m = \frac{g}{f}$ for some $g \in M[X]$ and $f \in N_v$, so $fm = g$. Hence $(c(f))m = c(fm) = c(g) \subseteq M$, and thus $m \in M_w$ because $c(f) \in GV(D)$.

(3) This follows directly from (2).

(4) Put $R = D[X]$. Let t be an indeterminate over R, $N = \{f \in R[t] \mid c(f)_v = R\}$, and $N_v(t) = \{g \in D[t] \mid c(g)_v = D\}$. Recall that $(ID[X])_v = I_v D[X]$ for a nonzero ideal I of D [5, Proposition 4.3]; so $N_v(t) \subseteq N$, and hence $M_w = M[t]_{N_v(t)} \cap q(M) \subseteq (M[X])[t]_N \cap q(M[X]) = (M[X])_w$ by (2). Thus $M_w[X] \subseteq (M[X])_w \cap q(M)[X]$.

Conversely, let $g \in (M[X])_w \cap q(M)[X]$, and let $J = (f_1, \ldots, f_s)$ be a finitely generated ideal of $D[X]$ such that $J_v = D[X]$ and $Jg \subseteq M[X]$. Let m be a positive integer such that $c(f_i)^{m+1}c(g) = c(f_i)^m c(f_i g)$ for $i = 1, \ldots, s$ by Proposition 1.3. Then $(c(f_1)^{m+1} + \cdots + c(f_s)^{m+1})c(g) = c(f_1)^m c(f_1 g) + \cdots + c(f_s)^m c(f_s g) \subseteq M$. Since $(\sum c(f_i))_v = D$ [5, Lemma 4.2], $(c(f_1)^{m+1} + \cdots + c(f_s)^{m+1})_v = D$, and hence $c(g) \subseteq M_w$. Thus $g \in M_w[X]$.

(5) Since $M_w \subseteq M[X]_{N_v} \cap q(M)$ by (2), $M_w[X] \subseteq M[X]_{N_v} \cap q(M)[X]$. For the reverse containment, let $f = \frac{h}{g} \in M[X]_{N_v} \cap q(M)[X]$, where $g \in N_v$ and $h \in M[X]$. Then $fg = h$, and hence $c(g)^{m+1}c(f) = c(g)^m c(fg) = c(g)^m c(h) \subseteq M$ for some $m \geq 1$ by Proposition 1.3. Note that $c(g)^{m+1}$ is finitely generated and $(c(g)^{m+1})_v = ((c(g)_v)^{m+1})_v = D$; hence $c(g)^{m+1} \in GV(D)$, and thus $c(f) \in M_w$ or $f \in M_w[X]$. Thus $M[X]_{N_v} \cap q(M)[X] \subseteq M_w[X]$. $\square$

We next give the main result of this paper, which generalizes the fact that D is a strong Mori domain if and only if $D[X]$ is a strong Mori domain if and only if $D[X]_{N_v}$ is a Noetherian domain [3, Theorem 2.2] (Note that an integral domain R is a strong Mori (resp., Noetherian) domain if and only if R is a strong Mori (resp., Noetherian) R-module).

Theorem 2.5. *The following statements are equivalent for a w-module M.*

(1) M *is a strong Mori* D*-module.*
(2) $M[X]$ *is a strong Mori* $D[X]$*-module.*
(3) $M[X]_{N_v}$ *is a Noetherian* $D[X]_{N_v}$*-module.*

Proof. (1) $\Rightarrow$ (3) Suppose that M is a strong Mori D-module, and let B' be a $D[X]_{N_v}$-submodule of $M[X]_{N_v}$. Then $B' = B_{N_v}$, where $B = B' \cap M[X]$, by Lemma 1.1, so it suffices to show that B_{N_v} is a finitely generated $D[X]_{N_v}$-module. Let N_k be the set of the leading coefficients of polynomials in B of degree $\leq k$; then N_k is a D-submodule of M by Lemma 1.2 and $N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots$. Set $N = \cup_{k \geq 0}(N_k)_w$. Since M is a strong Mori module, all the $(N_k)_w$ are finite type w-submodules of M and $N = (N_s)_w$ for some s.

Let $f_1, \ldots, f_k \in B$ such that $f_i = m_i X^{r_i} +$(lower terms) and $N = (\sum_{i=1}^{k} D m_i)_w$. Let $r = \max\{r_1, \ldots, r_k\}$. For each j from 1 to $r-1$, choose $g_{j1}, \ldots, g_{jk_j} \in B$ such that $g_{ji} = b_{ji} X^{r_{ji}} +$(lower terms) and $(N_j)_w = (\sum_{i=1}^{k_j} D b_{ji})_w$. We claim that $B_{N_v} = \sum_i D[X]_{N_v} f_i + \sum_{j,m} D[X]_{N_v} g_{jm}$.

Let $A = \sum_i D[X]_{N_v} f_i + \sum_{j,m} D[X]_{N_v} g_{jm}$, and let Q be a maximal ideal of $D[X]_{N_v}$. Then $A = A_{N_v}$ and $Q = P[X]_{N_v}$ for some $P \in t$-Max(D). Note that M_P is a Noetherian D_P-module by Proposition 2.3; so $M_P[X]$ is a Noetherian $D_P[X]$-module by Theorem 1.4. Also, since $N_P = \sum_{i=1}^{k} D_P m_i$ and $(N_j)_P = \sum_{i=1}^{k_j} D_P b_{ji}$ by Lemma 2.4(1), the proof of (1) $\Rightarrow$ (2) of Theorem 1.4 shows that $B_{D\setminus P} = A_{D\setminus P}$. Thus by Lemma 2.1(5),

$$\begin{aligned}(B_{N_v})_Q &= (B_{N_v})_{P[X]_{N_v}} = B_{P[X]} = (B_{D\setminus P})_{PD_P[X]} \\ &= (A_{D\setminus P})_{PD_P[X]} = A_{P[X]} = (A_{N_v})_{P[X]_{N_v}} = (A_{N_v})_Q.\end{aligned}$$

Since Q is an arbitrary maximal ideal of $D[X]_{N_v}$, we conclude $B_{N_v} = A_{N_v} = A$.

(3) $\Rightarrow$ (2) Let t be an indeterminate over $D[X]$, $\mathcal{M} = M[t]$, and $R = D[t]$. By replacing t with X, it suffices to show that $\mathcal{M}$ is a strong Mori R-module. Note that $(M[X]_{N_v})[t] = \mathcal{M}[X]_{N_v}$ and $(D[X]_{N_v})[t] = R[X]_{N_v}$; so $\mathcal{M}[X]_{N_v}$ is a Noetherian $R[X]_{N_v}$-module by (3) and Theorem 1.4. Let $N = \{g \in R[X] | c(g)_v = R\}$; then $N_v \subseteq N$ (see the proof of Lemma 2.4(4)), and hence $(\mathcal{M}[X]_{N_v})_N = \mathcal{M}[X]_N$ and $(R[X]_{N_v})_N = R[X]_N$ by Lemma 2.1(5). Hence, by Theorem 1.4, $\mathcal{M}[X]_N$ is a Noetherian $R[X]_N$-module.

Let $M_1 \subseteq M_2 \subseteq \cdots$ be an ascending chain of w-submodules of $\mathcal{M}$ over R. Then $M_1[X]_N \subseteq M_2[X]_N \subseteq \cdots$ is an ascending chain of $R[X]_N$-submodules of $\mathcal{M}[X]_N$. So there exists a positive integer k such that $M_k[X]_N = M_{k+i}[X]_N$ for $i = 1, 2, 3, \ldots$. Thus by Lemma 2.4(2), we have $M_k = M_k[X]_N \cap q(\mathcal{M}) = M_{k+i}[X]_N \cap q(\mathcal{M}) = M_{k+i}$ for $i = 1, 2, 3, \ldots$. Thus $\mathcal{M}$ is a strong Mori R-module.

(2) $\Rightarrow$ (1) Let $M_1 \subseteq M_2 \subseteq \cdots$ be an ascending chain of w-submodules of M. Then $(M_1[X])_w \subseteq (M_2[X])_w \subseteq \cdots$ is an ascending chain of w-submodules of $(M[X])_w$ over $D[X]$. So there exists a positive integer k such that $(M_k[X])_w =$

$(M_{k+i}[X])_w$ for $i = 1, 2, 3, \ldots$. Thus by Lemma 2.4(4), we have

$$M_k = (M_k[X])_w \cap q(M) = (M_{k+i}[X])_w \cap q(M) = M_{k+i} \text{ for } i = 1, 2, 3, \ldots. \square$$

Corollary 2.6 ([10, Theorem 4.5]). *D is an SM domain if and only if every finite type w-module M over D is a strong Mori module.*

Proof. Suppose that D is an SM domain, and let $M = N_w$ for some finitely generated D-submodule N of M; so $M[X]_{N_v} = N[X]_{N_v}$ by Lemma 2.4(3). Hence $M[X]_{N_v}$ is a finitely generated $D[X]_{N_v}$-module, and since $D[X]_{N_v}$ is Noetheian [3, Theorem 2.2], $M[X]_{N_v}$ is a Noetherian $D[X]_{N_v}$-module [2, Proposition 6.5]. Thus M is a strong Mori module by Theorem 2.5. The converse follows because D is a finite type w-module over D itself. $\square$

Let R be a commutative ring with identity. It is well known that if $M_1, \ldots, M_k$ are Noetherian R-modules, then $\bigoplus_{i=1}^k M_i$ is also a Noetherian R-module. This follows directly from the fact that if $0 \to L \to M \to N \to 0$ is an exact sequence of R-modules, then M is Noetherian if and only if L and N are Noetherian [2, Proposition 6.3]. We next generalize this result to strong Mori module, which shows that if D is a strong Mori domain, then $D^n = \bigoplus_{i=1}^n D_i$, where $D_i = D$, is a strong Mori D-module for any positive integer n (see Corollary 2.8).

Corollary 2.7 (cf. [8, Proposition 3.5(2)]). *Let $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ be an exact sequence of torsion-free D-modules. If M', M, and M'' are w-modules, then M is a strong Mori module if and only if M' and M'' are strong Mori modules.*

Proof. For each $f = m_0 + m_1X + \cdots + m_kX^k \in M'[X]$, define

$$\alpha'(f) = \sum_{i=0}^{k} \alpha(m_i)X^i.$$

It is obvious that the map $\alpha' : M'[X] \to M[X]$, given by $f \mapsto \alpha'(f)$, is an $D[X]$-module homomorphism. Also, the map $\beta' : M[X] \to M''[X]$, defined by $\beta'(\sum_{i=0}^k m_iX^i) = \sum_{i=0}^k \beta(m_i)X^i$, is a $D[X]$-module homomorphism. Moreover, since the sequence $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ is sequence, it follows that $0 \to M'[X] \xrightarrow{\alpha'} M[X] \xrightarrow{\beta'} M''[X] \to 0$ is an exact sequence of $D[X]$-modules; hence $0 \to M'[X]_{N_v} \xrightarrow{(N_v)^{-1}\alpha'} M[X]_{N_v} \xrightarrow{(N_v)^{-1}\beta'} M''[X]_{N_v} \to 0$ is an exact sequence [2, Proposition 3.3]. Thus by Theorem 2.5 and [2, Proposition 6.3], we have that M is a strong Mori module $\Leftrightarrow$ $M[X]_{N_v}$ is a Noetherian module $\Leftrightarrow$ $M'[X]_{N_v}$ and $M''[X]_{N_v}$ are Noetherian $\Leftrightarrow$ M' and M'' are strong Mori modules. $\square$

Corollary 2.8 ([8, Corollary 3.2] or [11, Proposition 4.5]). *If M_i $(i = 1, \ldots, n)$ are strong Mori D-modules, then $\bigoplus_{i=1}^n M_i$ is also a strong Mori D-module.*

Proof. This can be proved by induction on n and Corollary 2.7 applying to the exact sequence $0 \to M_n \longrightarrow \bigoplus_{i=1}^{n} M_i \longrightarrow \bigoplus_{i=1}^{n-1} M_i \to 0$. □

The following lemma is a variant of [6, Corollary 2.15].

Lemma 2.9. *Let M be a w-module. If $\varphi : M \to M$ is a module homomorphism, then the kernel of φ is a w-module.*

Proof. Let N be the kernel of φ. If $x \in N_w$, there exists a $J \in GV(D)$ such that $Jx \in N$. Choose $0 \neq a \in J$; then $ax \in N$, and hence $0 = \varphi(ax) = a\varphi(x)$. Since M is torsion-free, we have $\varphi(x) = 0$, and hence $x \in N$. Thus $N_w \subseteq N$. □

Theorem 2.10. *Let M be a strong Mori D-module and $\varphi : M \to M$ be a D-module homomorphism. If φ is surjective, then φ is an isomorphism.*

Proof. It suffices to prove that φ is injective. Let $\varphi^2 = \varphi \circ \varphi$ and $\varphi^n = \varphi^{n-1} \circ \varphi$ for all integers $n \geq 2$. Clearly, φ^n is a D-module homomorphism from M onto itself. Hence $ker(\varphi^n)$, the kernel of φ^n, is a w-module by Lemma 2.9, and since $ker(\varphi) \subseteq ker(\varphi^2) \subseteq \cdots$, there exists a positive integer k such that $ker(\varphi^k) = ker(\varphi^{k+i})$ for $i = 1, 2, \ldots$. Let $u \in ker(\varphi)$. Then $u \in ker(\varphi^k)$. Since φ^k is onto, there exists $v \in M$ such that $u = \varphi^k(v)$. Then $0 = \varphi^k(u) = \varphi^{2k}(v)$; so $v \in ker(\varphi^{2k}) = ker(\varphi^k)$, and hence $u = \varphi^k(v) = 0$. Thus φ is injective. □

Remark 2.11. As in [8], we say that M is *of finite type* if there is a finitely generated submodule B of M such that $M_P = B_P$ for all $P \in w\text{-Max}(D)$; M is *w-Noetherian* if every submodule of M is of finite type; and a sequence $A \to B \to C$ of modules is w-exact if the sequence $A_P \to B_P \to C_P$ is exact for any maximal w-ideal P of D. Wang proved that if $0 \to A \to B \to C \to 0$ is w-exact, then B is w-Noetherian if and only if A and C are w-Noetherian [8, Proposition 3.5(2)] and that $\bigoplus_{i=1}^{n} M_i$ is w-Noetherian if and only if each M_i is w-Noetherian [8, Corollary 3.2] (The definitions and results of [8] are given in a more general setting of commutative rings with zero divisors). By Lemma 2.4(1) and (2), if M is a w-module, then M is w-Noetherian if and only if M is a strong Mori module. Thus, Corollaries 2.7 and 2.8 follow directly from [8, Proposition 3.5(2) and Corollary 3.2].

Acknowledgements. The author would like to thank the referees for their several valuable comments and suggestions (including an information on the paper [8]). This work was supported by the University of Incheon Research Fund in 2012 (2012-0045).

References

[1] D. D. Anderson and S. J. Cook, *Two star-operations and their induced lattices*, Comm. Algebra **28** (2000), no. 5, 2461–2475.

[2] M. F. Atiyah and I. G. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley, 1969.

[3] G. W. Chang, *Strong Mori domains and the ring $D[X]_{N_v}$*, J. Pure Appl. Algebra **197** (2005), no. 1-3, 293–304.

[4] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, New York, 1972.
[5] J. R. Hedstrom and E. G. Houston, *Some remarks on star-operations*, J. Pure Appl. Algebra **18** (1980), no. 1, 37–44.
[6] H. Kim, *Module-theoretic characterizations of t-linkative domains*, Comm. Algebra **36** (2008), no. 5, 1649–1670.
[7] H. Matsumura, *Commutative Ring Theory*, Cambridge Univ. Press, 1986.
[8] F. Wang, *Finitely presented type modules and w-coherent rings*, J. Sichuan Normal Univ. **33** (2010), 1–9.
[9] ______, *Foundations of Commutative Ring Theory*, unpublished book.
[10] F. Wang and R. L. McCasland, *On w-modules over strong Mori domains*, Comm. Algebra **25** (1997), no. 4, 1285–1306.
[11] H. Yin, F. Wang, X. Zhu, and Y. Chen, *w-modules over commutative rings*, J. Korean Math. Soc. **48** (2011), no. 1, 207–222.

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