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# MCCOY CONDITION ON IDEALS OF COEFFICIENTS

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Abstract. We continue the study of McCoy condition to analyze zero-dividing polynomials for the constant annihilators in the ideals generated by the coefficients. In the process we introduce the concept of ideal-$\pi$-McCoy rings, extending known results related to McCoy condition. It is shown that the class of ideal-$\pi$-McCoy rings contains both strongly McCoy rings whose non-regular polynomials are nilpotent and 2-primal rings. We also investigate relations between the ideal-$\pi$-McCoy property and other standard ring theoretic properties. Moreover we extend the class of ideal-$\pi$-McCoy rings by examining various sorts of ordinary ring extensions.

## 1. Ideal-$\pi$-McCoy rings

Throughout this note every ring is associative with identity unless otherwise stated. Let $R$ be a ring and we use $R[x]$ to denote the polynomial ring with an indeterminate $x$ over $R$. Denote the $n$ by $n$ full matrix ring over $R$ by $\mathrm{Mat}_n(R)$ and the $n$ by $n$ upper (resp. lower) triangular matrix ring over $R$ by $U_n(R)$ (resp. $L_n(R)$). Use $E_{ij}$ for the matrix with $(i,j)$-entry 1 and elsewhere 0. $\mathbb{Z}$ and $\mathbb{Z}_n$ denote the set of integers and the ring of integers modulo $n$, respectively. Note $\mathrm{Mat}_n(R)[x] \cong \mathrm{Mat}_n(R[x])$ and $U_n(R)[x] \cong U_n(R[x])$. We will apply these isomorphisms freely. $N_*(R)$ and $N(R)$ denote the prime radical and the set of all nilpotent elements in $R$, respectively.

McCoy [21, Theorem 2] showed the following fact in 1942:

$$f(x)g(x) = 0 \text{ implies } f(x)r = 0 \text{ for some nonzero } r \in R,$$

where $f(x)$ and $0 \neq g(x)$ are polynomials over a commutative ring $R$. Based on this result, Nielsen [22] in 2006 called a (possibly noncommutative) ring $R$ (possibly without identity) *right McCoy* when the equation $f(x)g(x) = 0$ implies $f(x)r = 0$ for some nonzero $r \in R$, where $f(x), 0 \neq g(x)$ are polynomials in $R[x]$. Left McCoy rings are defined similarly. If a ring is both left and right McCoy, then the ring is called a *McCoy* ring. Nielsen [22, Section 3 and Section 4] showed that the McCoy condition is not left-right symmetric.

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It is well-known that the set of nilpotent elements forms an ideal in any commutative ring. This property is also possessed by the following kinds of noncommutative rings. Given a ring $R$, Shin [24, Proposition 1.11] proved that $N_*(R) = N(R)$ if and only if every minimal prime ideal of $R$ is completely prime. Birkenmeier et al. [6] called a ring $R$ *2-primal* when $N_*(R) = N(R)$. Note that $R$ is 2-primal if and only if $R/N_*(R)$ is reduced. A well-known property between the commutativity and the 2-primal condition is the *insertion-of-factors-property* (or simply *IFP*). Due to Bell [5], a ring $R$ is called *IFP* if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. IFP rings are 2-primal by [24, Theorem 1.5]. Following Cohn [8], a ring $R$ is called *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Anderson and Camillo [2], observing the rings whose zero products commute, used the term $ZC_2$ for what is called reversible. A ring $R$ is called *reduced* if $N(R) = 0$. Reduced rings are reversible via a simple computation, and commutative rings are clearly reversible. Every reversible ring is McCoy by [22, Theorem 2], and hence both commutative rings and reduced rings are McCoy. It is evident that reversible rings are IFP. However IFP rings need not be right McCoy by [22, Section 3]. There exist right McCoy rings that are not IFP with the help of [19, Theorem 2] and [17, Example 1.3].

According to Hong et al. [12], a ring $R$ (possibly without identity) is called *strongly right McCoy* provided that $f(x)g(x) = 0$ implies $f(x)r = 0$ for some nonzero $r$ in the right ideal of $R$ generated by the coefficients of $g(x)$, where $f(x)$ and $g(x)$ are nonzero polynomials in $R[x]$. Strongly left McCoy rings are defined similarly. If a ring is both strongly left and strongly right McCoy, then the ring is called a *strongly McCoy* ring. Reversible rings are strongly McCoy by [12, Theorem 1.6] or the proof of Nielsen [22, Theorem 2] and strongly McCoy rings are clearly McCoy. A ring is usually called *right duo* if each right ideal is two-sided. It is easily checked that right duo rings are IFP. Right duo rings are strongly right McCoy by the proof of [7, Theorem 8.2], and moreover we can find more concrete shape of right annihilators in the proof of [12, Theorem 1.11].

Due to Jeon et al. [15], a ring $R$ (possibly without identity) is called *$\pi$-McCoy* if $f(x)g(x) \in N(R[x])$ implies $f(x)r \in N(R[x])$ for some nonzero $r \in R$, where $f(x)$ and $g(x)$ are nonzero polynomials in $R[x]$. The class of $\pi$-McCoy rings contains McCoy rings by [15, Proposition 1.1(2)] and 2-primal rings (hence reversible rings, right duo rings and IFP rings) by [15, Proposition 1.7]. Next we consider the $\pi$-McCoy in conjunction with the nilpotency.

Let $R$ be a ring and $f(x), 0 \neq g(x) \in R[x]$ and $J$ be the ideal of $R$ generated by the coefficients of $g(x)$. In this situation consider the following condition:

(∗) $f(x)g(x) \in N(R[x])$ implies $f(x)a \in N(R[x])$ for some nonzero $a \in J$.

Recall that an element is called *right* (resp. *left*) *regular* if its right (resp. left) annihilator is zero. An element is called *regular* if it is both left and right regular.

**Proposition 1.1.** (1) *The condition* $(*)$ *is left-right symmetric.*

(2) *Let* $R$ *be a ring over which non-regular polynomials are nilpotent. If* $R$ *is strongly McCoy, then* $R$ *satisfies the condition* $(*)$*.*

*Proof.* (1) Let $R$ be a ring satisfying the condition $(*)$, and we will show that $R$ satisfies the left version of the condition $(*)$. Say $f(x)g(x) \in N(R[x])$ for $f(x), g(x) \in R[x]$ with $f(x) \neq 0$. From $f(x)g(x) \in N(R[x])$ we also get $g(x)f(x) \in N(R[x])$. Since $R$ satisfies the condition $(*)$, $g(x)b \in N(R[x])$ for some nonzero $b$ in the ideal of $R$ generated by the coefficients of $f(x)$. This yields $bg(x) \in N(R[x])$. The converse can be proved by changing the roles of $f(x)$ and $g(x)$.

(2) Suppose that $R$ is a strongly McCoy ring whose non-regular polynomials are nilpotent. Let $f(x)g(x) \in N(R[x])$ for $f(x), g(x) \in R[x]$. If $f(x) = 0$ or $g(x) = 0$, then $R$ satisfies both the condition $(*)$ and the left version of the condition $(*)$. So we suppose that $f(x) \neq 0$ and $g(x) \neq 0$. We apply the proof of [15, Proposition 1.1(2)]. Let $(f(x)g(x))^n = 0$ and $(f(x)g(x))^{n-1} \neq 0$ for some $n \geq 1$. Let $I$ and $J$ be the ideals of $R$ generated by the coefficients of $f(x)$ and $g(x)$, respectively.

`Case 1`. $f(x)g(x) = 0$ and $g(x)f(x) = 0$

Since $R$ is strongly McCoy, there exist $0 \neq a \in J$ and $0 \neq b \in I$ such that $f(x)a = 0$ and $bg(x) = 0$.

`Case 2`. $f(x)g(x) = 0$ and $g(x)f(x) \neq 0$

Since $R$ is strongly McCoy, there exist $0 \neq a \in J$ and $0 \neq b \in I$ such that $f(x)a = 0$ and $bg(x) = 0$.

`Case 3`. $f(x)g(x) \neq 0$ and $g(x)f(x) = 0$

Since $R$ is strongly McCoy, there exist $0 \neq a \in J$ and $0 \neq b \in I$ such that $af(x) = 0$ (hence $f(x)a \in N(R[x])$) and $g(x)b = 0$ (hence $bg(x) \in N(R[x])$).

`Case 4`. $f(x)g(x) \neq 0$ (then $n \geq 2$) and $g(x)f(x) \neq 0$

Recall $(f(x)g(x))^{n-1} \neq 0$. From $(f(x)g(x))(f(x)g(x))^{n-1} = 0$, there exists $0 \neq b \in I \cap J$ such that $f(x)g(x)b = 0$ since $R$ is strongly right McCoy. So $g(x)b \in N(R[x])$ by the hypothesis, entailing $bg(x) \in N(R[x])$.

By Cases 1, 2, 3 and 4, we have that

$$bg(x) \in N(R[x]) \text{ for some } 0 \neq b \in I.$$

Thus $R$ satisfies the condition $(*)$ by (1). □

As we see in the proof of Proposition 1.1(2), the condition $(*)$ is satisfied automatically when $f(x) = 0$ or $g(x) = 0$. So we will examine the condition $(*)$, assuming that $f(x) \neq 0$ and $g(x) \neq 0$.

The strongly McCoy condition is not left-right symmetric by help of [16, Example 1.8]. Based on Proposition 1.1, a ring will be called *ideal-$\pi$-McCoy* if it satisfies the condition $(*)$. Ideal-$\pi$-McCoy rings are clearly $\pi$-McCoy but the converse need not hold by Example 1.4(3) below. Note that there exists an ideal-$\pi$-McCoy ring which is not strongly McCoy by Example 1.3(2),(3) to follow.

Given a ring $R$ and $n \geq 2$, consider the subrings $D_n(R) = \{(m_{ij}) \in U_n(R) \mid m_{11} = \cdots = m_{nn}\}$ and $V_n(R) = \{m = (m_{ij}) \in D_n(R) \mid m_{st} = m_{(s+1)(t+1)}$ for $s = 1, \ldots, n-2$ and $t = 2, \ldots, n-1\}$ of $U_n(R)$. For any set $M$ of matrices over a ring $R$, $M^T$ denotes the set of all transposes of matrices in $M$.

**Lemma 1.2.** (1) *Let $R$ be a ring with an essential ideal $I$ of $R$ such that $I[x] \subseteq N(R[x])$. Then $R$ is ideal-$\pi$-McCoy.*

(2) *Let $R$ be a ring with a nilpotent essential ideal. Then* $\mathrm{Mat}_n(R)$ *is ideal-$\pi$-McCoy for $n \geq 1$.*

(3) *Let $R$ be a ring with an essential ideal whose finitely generated subrings are nilpotent. Then* $\mathrm{Mat}_n(R)$ *is ideal-$\pi$-McCoy for $n \geq 1$.*

(4) *Both $U_n(A)$ and $L_n(A)$ are ideal-$\pi$-McCoy for any ring $A$ and $n \geq 2$.*

(5) *Both $D_n(A)$ and $D_n(A)^T$ are ideal-$\pi$-McCoy for any ring $A$ and $n \geq 2$.*

(6) *Both $V_n(A)$ and $V_n(A)^T$ are ideal-$\pi$-McCoy for any ring $A$ and $n \geq 2$.*

(7) *The factor ring $R[x]/\langle x^n \rangle$ is ideal-$\pi$-McCoy for any $n \geq 2$, where $\langle x^n \rangle$ is a two-sided ideal of $R[x]$ generated by $x^n$.*

(8) *Let $A$, $B$ be any rings and ${}_AM_B$ be an $(A, B)$-bimodule such that $aM \neq 0$ and $Mb \neq 0$ for $0 \neq a \in A$ and $0 \neq b \in B$. Then $\left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ is ideal-$\pi$-McCoy.*

*Proof.* (1) Let $f(x)g(x) \in N(R[x])$ for nonzero polynomials $f(x), g(x)$ in $R[x]$ and let $J$ be the ideal of $R$ generated by the coefficients of $g(x)$. Since $I$ is an essential ideal, we have $I \cap J \neq 0$. If $f(x) \in (I \cap J)[x]$, then $f(x)r \in N(R[x])$ for all $r \in R$. If $f(x) \notin (I \cap J)[x]$, then $f(x)s \in N(R[x])$ for all nonzero $s \in I \cap J$. Thus $R$ is ideal-$\pi$-McCoy.

(2) Let $I$ be a nilpotent essential ideal of $R$. Then $\mathrm{Mat}_n(I)$ is a nilpotent essential ideal of $\mathrm{Mat}_n(R)$. Clearly $\mathrm{Mat}_n(I)[x] \subseteq N(\mathrm{Mat}_n(R)[x])$. Thus $\mathrm{Mat}_n(R)$ is ideal-$\pi$-McCoy by (1).

(3) Let $I$ be an essential ideal whose finitely generated subrings are nilpotent. Then $\mathrm{Mat}_n(I)$ is also an essential ideal of $\mathrm{Mat}_n(R)$. Let $f(x) = (a(0)_{ij}) + (a(1)_{ij})x + \cdots + (a(n)_{ij})x^n \in \mathrm{Mat}_n(I)[x]$ and $S$ be the subring of $I$ generated by $a(k)_{ij}$'s for $k = 0, \ldots, n$ and $i, j = 1, \ldots, n$. By hypothesis, $S$ is nilpotent and this yields that $f(x)$ is nilpotent. Thus $\mathrm{Mat}_n(R)$ is ideal-$\pi$-McCoy by (1).

(4) Let $A$ be any ring and $R = U_n(A)$ for $n \geq 2$. Then

$$I = \{(m_{ij}) \in R \mid m_{ii} = 0 \text{ for all } i = 1, \ldots, n\}$$

is an essential ideal of $R$ such that $I^n = 0$. Thus $R$ is ideal-$\pi$-McCoy by (2).

The proofs of (5), (6), (8) are almost same as (4), noting that $\left(\begin{smallmatrix} 0 & M \\ 0 & 0 \end{smallmatrix}\right)$ is a nilpotent essential ideal of $\left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$.

(6)$\Leftrightarrow$(7) follows from the well-known fact that $V_n(R) \cong R[x]/\langle x^n \rangle$. $\square$

**Question.** Is a ring $R$ ideal-$\pi$-McCoy when $\text{Mat}_n(R)$ ($U_n(R)$) is ideal-$\pi$-McCoy?

Given a ring $R$ and an $(R,R)$-bimodule $M$, the *trivial extension* of $R$ by $M$ is the ring $T(R,M) = R \oplus M$ with the usual addition and the following multiplication: $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This is isomorphic to the ring of all matrices $\left(\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right)$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

Hence, the trivial extension $T(R,R)$ of any ring $R$ is an ideal-$\pi$-McCoy ring by the similar argument to Lemma 1.2(4).

We will use freely the fact that ideal-$\pi$-McCoy rings are $\pi$-McCoy.

**Example 1.3.** (1) Let $R$ be a reduced ring. Then $\text{Mat}_n(R)$ is not $\pi$-McCoy when $n \geq 2$ by [15, Theorem 1.4]. Thus $\text{Mat}_n(R)$ cannot be ideal-$\pi$-McCoy over a reduced ring $R$.

(2) There exist many kinds of rings that satisfy the hypothesis of Lemma 1.2(2). For example, $\mathbb{Z}_{p^m} (m \geq 2)$ for a prime $p$ has an essential nilpotent ideal $p\mathbb{Z}_{p^m}$. So $R = \text{Mat}_n(\mathbb{Z}_{p^m})$ ($n \geq 1$, $m \geq 2$) is ideal-$\pi$-McCoy by Lemma 1.2(2). Note that $R$ is neither left nor right McCoy by [15, Proposition 1.6] and so $R$ is not strongly McCoy.

(3) By Lemma 1.2(4), we can always construct an ideal-$\pi$-McCoy ring that is neither left nor right McCoy. $U_n(A)$ (for any ring $A$ and $n \geq 2$) is ideal-$\pi$-McCoy but neither left nor right McCoy by [15, Example 1.3] and hence $R$ is not strongly McCoy.

Recall that the class of $\pi$-McCoy rings contains McCoy rings and ideal-$\pi$-McCoy rings. Moreover the properties of the McCoy and the ideal-$\pi$-McCoy are independent of each other by the following.

**Example 1.4.** (1) (non-semiprime case) There exist ideal-$\pi$-McCoy rings that are not one-sided McCoy by Example 1.3(2),(3).

(2) (semiprime case) There exists an ideal-$\pi$-McCoy ring that is not right McCoy, applying the proof of [14, Theorem 2.2]. Let $S$ be a reduced ring, $n$ be a positive integer and $R_n$ be the $2^n$ by $2^n$ upper triangular matrix ring over $S$. Define a map $\sigma : R_n \to R_{n+1}$ by $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$, then $R_n$ can be considered as a subring of $R_{n+1}$ via $\sigma$ (i.e., $A = \sigma(A)$ for $A \in R_n$). Notice that $D = \{R_n, \sigma_{nm}\}$, with $\sigma_{nm} = \sigma^{m-n}$ whenever $n \leq m$, is a direct system over $\{1, 2, \ldots\}$. Set $R = \varinjlim R_n$ be the direct limit of $D$. Then $R$ is a semiprime ring by [14, Theorem 2.2], and $R$ is neither left nor right McCoy by [15, Example 1.3]. Moreover $R$ is an ideal-$\pi$-McCoy ring by Theorem 2.3 to follow since every $R_n$ is ideal-$\pi$-McCoy by Lemma 1.2(4). As another proof, consider the ideal $I = \{A \in R \mid$ the diagonal entries of $A$ are zero$\}$ of $R$. Then $I$ is an essential ideal of $R$ whose finitely generated subrings are nilpotent since the subring is contained in $U_{2^k}(S)$ for some $k \geq 1$. So $R$ is ideal-$\pi$-McCoy by Lemma 1.2(3).

(3) There exists a McCoy ring that is not ideal-$\pi$-McCoy. Let $I$ be an infinite indexing set and $R_i$ be a ring for all $i \in I$. Let $R = \sum_{i \in I} R_i$ be the direct

sum of $R_i$'s. Suppose that $R_j$ is not ideal-$\pi$-McCoy for some $j \in I$. Then by Proposition 2.1(2), to follow, $R$ is not ideal-$\pi$-McCoy. But $R$ is McCoy by [7, Proposition 4.3] and thus $R$ is $\pi$-McCoy by [15, Proposition 1.1(2)].

McCoy rings are $\pi$-McCoy, but $\pi$-McCoy rings need not be ideal-$\pi$-McCoy by Example 1.4(3). We will see another $\pi$-McCoy ring but not ideal-$\pi$-McCoy in the following.

**Example 1.5.** Let $R = \mathrm{Mat}_2(\mathbb{Z}) \bigoplus \mathrm{Mat}_2(\mathbb{Z}_4)$. Then $\mathrm{Mat}_2(\mathbb{Z})$ is not $\pi$-McCoy by [15, Theorem 1.4] but $\mathrm{Mat}_2(\mathbb{Z}_4)$ is non-semiprime with $N_*(\mathrm{Mat}_2(\mathbb{Z}_4)) = \mathrm{Mat}_2(2\mathbb{Z}_4)$; hence $R$ is $\pi$-McCoy by [15, Proposition 2.7]. Note that $\mathrm{Mat}_2(\mathbb{Z})$ is not ideal-$\pi$-McCoy by Example 1.3(1), and so $R$ is not ideal-$\pi$-McCoy by Proposition 2.1(1) to follow.

For any polynomial $f(x)$ in $R[x]$, let $C_{f(x)}$ denote the set of all coefficients of $f(x)$. In [23], Rege and Chhawchharia called a ring $R$ *Armendariz* if whenever any polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$, then $ab = 0$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$. This nomenclature was used by them since it was Armendariz [4, Lemma 1] who initially showed that a reduced ring always satisfies this condition. The class of 2-primal rings and the class of Armendariz rings don't imply each other by [23, Example 3.2] and [3, Example 4.8].

**Proposition 1.6.** (1) *Armendariz rings are ideal-$\pi$-McCoy.*
(2) *2-primal rings are ideal-$\pi$-McCoy.*

*Proof.* (1) Note that $R$ is Armendariz if and only if $R[x]$ is, by [1, Theorem 2]. Let $R$ be Armendariz. Suppose that $f(x)g(x) \in N(R[x])$ for nonzero polynomials $f(x), g(x)$ in $R[x]$. By [3, Corollary 5.2] and [3, Proposition 2.7], we have $ab \in N(R)$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$ and so $f(x)b' \in N(R)[x] = N(R[x])$ for some $0 \neq b' \in C_{g(x)}$ because $g(x) \neq 0$. Thus $R$ is ideal-$\pi$-McCoy.

(2) Note that $R[x]$ over a 2-primal ring $R$ is 2-primal by [6, Proposition 2.6], and hence $N(R[x]) = N(R)[x]$. Let $R$ be a 2-primal ring. Then $R[x]/N(R)[x] \cong (R/N(R))[x]$ is reduced and $N(R[x]) = N(R)[x]$. Let $f(x)g(x) \in N(R[x]) = N(R)[x]$ for nonzero polynomials $f(x), g(x)$ in $R[x]$. Since $(R/N(R))[x]$ is reduced and so Armendariz, $ab \in N(R)$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$. By the same argument to the proof of (1), we can show that $R$ is ideal-$\pi$-McCoy. $\square$

The converse of Proposition 1.6(1) does not hold with the help of Example 1.7 to follow. Proposition 1.6(2) implies that 2-primal rings are $\pi$-McCoy [15, Proposition 1.7], since ideal-$\pi$-McCoy rings are $\pi$-McCoy, and the converse of Proposition 1.6(2) need not hold. Let $R$ be the ring in Example 1.4(2). Then $R$ is an ideal-$\pi$-McCoy ring, but not 2-primal by [13, Example 1.2].

As a generalization of Armendariz rings, Antoine [3] called a ring $R$ *nil-Armendariz* if $ab \in N(R)$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$ whenever two polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) \in N(R)[x]$. Nil-Armendariz rings contain both 2-primal rings and Armendariz rings, but each converse does not hold by [3, Example 4.8 and Example 4.9]. Hence, we raise the following question:

**Question.** Is a ring $R$ ideal-$\pi$-McCoy if $R$ is nil-Armendariz?

Recall that a ring $R$ is called (*von Neumann*) *regular* if for each $a \in R$ there exists $x \in R$ such that $a = axa$. Observe that a regular ring $R$ is Armendariz if and only if $R$ is nil-Armendariz if and only if $R$ is 2-primal by [18, Theorem 20].

A ring is called *Abelian* if every idempotent is central. Ideal-$\pi$-McCoy rings need not be Abelian as can be seen by $U_n(A)$ for $n \geq 2$ and any ring $A$. Abelian rings are not $\pi$-McCoy by [15, Example 1.9] and so not ideal-$\pi$-McCoy. Note that a ring $R$ is Abelian regular if and only if $R$ is reduced regular [10, Theorem 3.2]. Consequently a regular ring $R$ is reduced if and only if $R$ is reversible if and only if $R$ is Abelian if and only if $R$ is 2-primal. Since the class of ideal-$\pi$-McCoy rings contains both Armendariz rings and 2-primal rings, one may conjecture that regular ideal-$\pi$-McCoy rings are Abelian (hence reduced). However the following provides a negative answer.

**Example 1.7.** We use the ring in [15, Example 1.10]. Let $S$ be a regular ring, $n$ be a positive integer, and $R_n$ be the $2^n$ by $2^n$ full matrix ring over $S$. Define a map $\sigma : R_n \to R_{n+1}$ by $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$, then $R_n$ can be considered as a subring of $R_{n+1}$ via $\sigma$ (i.e., $A = \sigma(A)$ for $A \in R_n$). Notice that $D = \{R_n, \sigma_{nm}\}$, with $\sigma_{nm} = \sigma^{m-n}$ whenever $n \leq m$, is a direct system over $I = \{1, 2, \ldots\}$. Set $R = \varinjlim R_n$ be the direct limit of $D$. Then $R$ is regular since every $R_n$ is regular, but not reduced.

Let $f(x)g(x) \in N(R[x])$ for $0 \neq f(x), 0 \neq g(x) \in R[x]$. Then there exists $k \geq 1$ such that $f(x), g(x) \in R_k[x]$. Since $g(x) \neq 0$, there exists a nonzero coefficient of $g(x)$, say $(a_{ij}) \neq 0$ with $a_{pq} \neq 0$. So the ideal of $R$ generated by the coefficients of $g(x)$ contains the matrix

$$(b_{st}) = E_{1p}(a_{ij})E_{q2^{k+1}}$$

in $R_{k+1}$ such that the $(1, 2^{k+1})$-entry of $(b_{st})$ is $a_{pq}$ and other entries of $(b_{st})$ are all zero. Thus $(f(x)(b_{st}))^{2^{k+1}} = 0$ and this implies that $R$ is ideal-$\pi$-McCoy.

A ring $R$ is called *directly finite* if $ab = 1$ implies $ba = 1$ for $a, b \in R$. Clearly Abelian rings are directly finite. Both strongly McCoy rings and 2-primal rings are directly finite by help of [6, Proposition 2.10] and [7, Theorem 5.2]. So one may conjecture that ideal-$\pi$-McCoy rings are directly finite. However the following erases the possibility.

**Example 1.8.** We use the ring and argument in [15, Example 1.8]. Let $F$ be a field and $\mathbb{V}$ be an infinite dimensional vector space over $F$ with a basis $\{v_1, v_2, \ldots\}$. Consider the endomorphism ring $R = \mathrm{End}_F(\mathbb{V})$ and define $f, g \in R$ such that $fv_1 = 0, fv_j = v_{j-1}$ for $j = 2, 3, \ldots$ and $gv_i = v_{i+1}$ for $i = 1, 2, \ldots$. Then $fg = 1$ but $gf \neq 1$. Now consider $U_n(R)$ for $n \geq 2$. Then $U_n(R)$ is ideal-$\pi$-McCoy by Lemma 1.2(4). Take $a = (a_{ij})$ and $b = (b_{ij})$ in $U_n(R)$ such that $a_{ii} = f$ for all $i$, elsewhere zero, and $b_{ii} = g$ for all $i$, elsewhere zero. Then $ab = 1$ but $ba \neq 1$; hence $U_n(R)$ is not directly finite.

The following is a similar result to [16, Proposition 2.1].

**Proposition 1.9.** *Let $R$ be an ideal-$\pi$-McCoy ring and suppose that $f_1(x), \ldots, f_n(x) \in R[x]$ are such that $f_1(x) \cdots f_n(x) \in N(R[x])$ and $f_i(x) \neq 0$ for all $i \in \{2, \ldots, n\}$. Then there exists a nonzero $r_i$ in the ideal of $R$ generated by the coefficients of $f_i(x)$ in $R$ such that*

$$f_1(x) \cdots f_{n-1}(x) r_n, \ldots, f_1(x) f_2(x) r_3 \cdots r_n, f_1(x) r_2 r_3 \cdots r_n \in N(R[x])$$

*for $i = 2, \ldots, n$.*

*Proof.* Suppose that $R$ is ideal-$\pi$-McCoy. Since $(f_1(x) \cdots f_{n-1}(x)) f_n(x) \in N(R[x])$, there exists a nonzero $r_n$ in the ideal of $R$ generated by the coefficients of $f_n(x)$ such that $(f_1(x) \cdots f_{n-1}(x)) r_n \in N(R[x])$. Hence we get $r_n(f_1(x) \cdots f_{n-1}(x)) \in N(R[x])$. Since $r_n(f_1(x) \cdots f_{n-1}(x)) \in N(R[x])$, there exists a nonzero $r_{n-1}$ in the ideal of $R$ generated by the coefficients of $f_{n-1}(x)$, entailing $r_{n-1} r_n (f_1(x) \cdots f_{n-2}(x)) \in N(R[x])$.

Proceeding in this manner, we finally obtain $r_2 \cdots r_{n-1} r_n f_1(x) \in N(R[x])$ and this yields

$$f_1(x) r_2 \cdots r_{n-1} r_n \in N(R[x]),$$

where $r_i$ is a nonzero element in the ideal of $R$ generated by the coefficients of $f_i(x)$ for $i = 2, \ldots, n$. □

When we apply this proposition we should proceed our computation for each of $\{r_n, \ldots, r_3 \cdots r_n, r_2 r_3 \cdots r_n\}$ to be nonzero.

## 2. Examples of ideal-$\pi$-McCoy rings

In this section we examine the ideal-$\pi$-McCoy property in various kinds of ordinary ring extensions.

**Proposition 2.1.** *Let $\Gamma$ be an index set and $R_\gamma$ be a ring for each $\gamma \in \Gamma$.*

(1) *The direct product $R = \prod_{\gamma \in \Gamma} R_\gamma$ is ideal-$\pi$-McCoy if and only if $R_\gamma$ is ideal-$\pi$-McCoy for all $\gamma \in \Gamma$.*

(2) *The direct sum $R = \sum_{\gamma \in \Gamma} R_\gamma$ (possibly without identity) is ideal-$\pi$-McCoy if and only if $R_\gamma$ is ideal-$\pi$-McCoy for all $\gamma \in \Gamma$.*

(3) *Let $R$ be the subring of $\prod_{\gamma \in \Gamma} R_\gamma$ generated by $\sum_{\gamma \in \Gamma} R_\gamma$ and $1_{\prod_{\gamma \in \Gamma} R_\gamma}$. Then $R$ is ideal-$\pi$-McCoy if and only if $R_\gamma$ is ideal-$\pi$-McCoy for all $\gamma \in \Gamma$.*

(4) *Let $R$ be an ideal-$\pi$-McCoy ring. If $I$ is a proper right or left ideal of $R$, then $I$ is an ideal-$\pi$-McCoy ring (without identity).*

(5) *The class of ideal-$\pi$-McCoy rings is not closed under subrings.*

(6) *The class of ideal-$\pi$-McCoy rings is not closed under homomorphic images.*

*Proof.* (1) Let $f(x)g(x) \in N(R[x])$ for

$$f(x) = \sum_{i=0}^{m}(a(i)_\gamma)x^i,\ 0 \neq g(x) = \sum_{j=0}^{n}(b(j)_\gamma)x^j \in R[x].$$

Letting $f_\gamma(x) = \sum_{i=0}^{m} a(i)_\gamma x^i$ and $g_\gamma(x) = \sum_{j=0}^{n} b(j)_\gamma x^j$, we can write $f(x) = (f_\gamma(x))$ and $g(x) = (g_\gamma(x))$. From $f(x)g(x) \in N(R[x])$, we get $f_\gamma(x)g_\gamma(x) \in N(R_\gamma[x])$ for all $\gamma \in \Gamma$. Suppose that each ring $R_\gamma$ is ideal-$\pi$-McCoy. Since $g(x) \neq 0$ there exists some index $k \in \Gamma$ such that $g_k(x) \neq 0$. Then since $R_k$ is ideal-$\pi$-McCoy, there exists some nonzero $r_k$ in the ideal of $R_k$ generated by the coefficients of $g_k(x)$ such that $f_k(x)r_k \in N(R_k[x])$. Say $r_k = \sum_{\lambda\in\Lambda} s_\lambda d_\lambda t_\lambda$ where $s_\lambda, t_\lambda \in R_k$, $d_\lambda \in \{b \mid b \text{ is a coefficient of } g_k(x)\}$, and $\Lambda$ is finite. Let $r = (r_\gamma) \in R$ be the sequence with $r_\gamma = r_k$ for $\gamma = k$ and $r_\gamma = 0$ for $\gamma \neq k$; and $(u(\lambda)_\gamma)$ be a coefficient of $g(x)$ for $\lambda \in \Lambda$ such that $u(\lambda)_\gamma = d_\lambda$ for $\gamma = k$. Then $r = \sum_{\lambda\in\Lambda}(s(\lambda)_\gamma)(u(\lambda)_\gamma)_\lambda(t(\lambda)_\gamma)$ where $s(\lambda)_\gamma = s_\lambda$, $t(\lambda)_\gamma = t_\lambda$ for $\gamma = k$ and $s(\lambda)_\gamma = 0$, $t(\lambda)_\gamma = 0$ for $\gamma \neq k$. This yields $f(x)r \in N(R[x])$, and so $R$ is ideal-$\pi$-McCoy.

Conversely, let $R$ be ideal-$\pi$-McCoy, and assume on the contrary that $R_{\gamma_0}$ is not ideal-$\pi$-McCoy for some $\gamma_0 \in \Gamma$. Then there exist $f_{\gamma_0}(x), 0 \neq g_{\gamma_0}(x)$ in $R_{\gamma_0}[x]$ such that $f_{\gamma_0}(x)g_{\gamma_0}(x) \in N(R_{\gamma_0}[x])$ but $f_{\gamma_0}(x)r_{\gamma_0} \notin N(R_{\gamma_0}[x])$ for all $0 \neq r_{\gamma_0}$ in the ideal of $R_{\gamma_0}$ generated by the coefficients of $g_{\gamma_0}(x)$. Taking $f(x) = (f_\gamma(x)), g(x) = (g_\gamma(x))$ such that $f(x)$ and $g(x)$ are the sequences in $R[x]$ such that $f_\gamma(x) = f_{\gamma_0}(x)$ for $\gamma = \gamma_0$, $f_\gamma(x) = 0$ for $\gamma \neq \gamma_0$, and $g_\gamma(x) = g_{\gamma_0}(x)$ for $\gamma = \gamma_0$, $g_\gamma(x) = 0$ for $\gamma \neq \gamma_0$. Then $f(x)g(x) \in N(R[x])$ from $f_{\gamma_0}(x)g_{\gamma_0}(x) \in N(R_{\gamma_0}[x])$. But since $R$ is ideal-$\pi$-McCoy, there exists a nonzero $s = (s_\gamma)$ in the ideal of $R$ generated by the coefficients of $g(x)$ such that $f(x)s \in N(R[x])$. Note that $s_\gamma \neq 0$ for $\gamma = \gamma_0$ and $s_\gamma = 0$ for $\gamma \neq \gamma_0$ and that $s_{\gamma_0}$ is in the ideal of $R_{\gamma_0}$ generated by the coefficients of $g_{\gamma_0}(x)$. This yields $f_{\gamma_0}(x)s_{\gamma_0} \in N(R_{\gamma_0}[x])$, a contradiction.

The proofs of (2) and (3) are much the same as (1).

(4) Let $f(x)g(x) \in N(I[x])$ for polynomials $f(x), 0 \neq g(x)$ in $I[x]$. Since $f(x)g(x) \in N(I[x]) \subseteq N(R[x])$ and $R$ is ideal-$\pi$-McCoy, $f(x)r \in N(R[x])$ for some $0 \neq r$ in the ideal of $R$ generated by the coefficients of $g(x)$. Since $I$ is an ideal of $R$, $r \in I$ and $f(x)r \in N(I[x])$. So $I$ is an ideal-$\pi$-McCoy ring without identity.

(5) Let $R = \mathrm{Mat}_n(S)$ for a reduced ring $S$ and $n \geq 2$. Then $R$ is not ideal-$\pi$-McCoy by Example 1.3(1). But $U_2(S)$ is ideal-$\pi$-McCoy by Lemma 1.2(4).

(6) Let $R$ be the ring of quaternions with integer coefficients. Then $R$ is a domain and clearly ideal-$\pi$-McCoy. However for any odd prime integer $q$, the ring $R/qR$ is isomorphic to $\mathrm{Mat}_2(\mathbb{Z}_q)$ by the argument in [11, Exercise 2A]. Thus $R/qR$ is not ideal-$\pi$-McCoy by Example 1.3(1). □

The construction in Example 1.7 also provides an ideal-$\pi$-McCoy ring which has a non-ideal-$\pi$-McCoy subring. Let $S$ be a division ring in Example 1.7.

Then every $R_n = \mathrm{Mat}_{2^n}(S)$ is not ideal-$\pi$-McCoy by Example 1.3(1). But $R$ is ideal-$\pi$-McCoy by the computation in Example 1.7. Note that every $R_n$ is a subring of $R$.

We find a kind of subring that inherits the ideal-$\pi$-McCoy property against Proposition 2.1(5).

**Corollary 2.2.** *Let $e$ be a central idempotent of a ring $R$. Then $R$ is ideal-$\pi$-McCoy if and only if $eR$ and $(1-e)R$ are both ideal-$\pi$-McCoy.*

*Proof.* The proof is obtained from Proposition 2.1(1) since $R = eR \oplus (1-e)R$. □

Concerning the preceding corollary, we write an actual computation to show the sufficiency. Suppose that $eR$ and $(1-e)R$ are both ideal-$\pi$-McCoy. Let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ be such that $f(x)g(x) \in N(R[x])$ and $g(x) \neq 0$. Next set $f_1(x) = ef(x)$, $f_2(x) = (1-e)f(x)$ and $g_1(x) = eg(x)$, $g_2(x) = (1-e)g(x)$. Then $f_1(x)g_1(x) = ef(x)g(x) \in N(R[x])$ and $f_2(x)g_2(x) = (1-e)f(x)g(x) \in N(R[x])$. Since $g(x) \neq 0$, $eg(x) \neq 0$ or $(1-e)g(x) \neq 0$.

Assume $g_1(x) \neq 0$ and $g_2(x) \neq 0$. Since $eR$ (resp. $(1-e)R$) is ideal-$\pi$-McCoy, there exists $r_1 \neq 0$ (resp. $r_2 \neq 0$) in the ideal of $eR$ (resp. $(1-e)R$) generated by the coefficients of $g_1(x)$ (resp. $g_2(x)$) such that $f_1(x)r_1 \in N(eR[x])$ (resp. $f_2(x)r_2 \in N((1-e)R[x])$). Let $r = r_1 + r_2$. Then $r \neq 0$ since $eR \cap (1-e)R = 0$, and $r$ is contained in the ideal of $R$ generated by the coefficients of $g(x)$. Moreover we have

$$f(x)r = (f_1(x) + f_2(x))(r_1 + r_2) = f_1(x)r_1 + f_2(x)r_2 \in N(R[x]).$$

The computations of the cases of $(g_1(x) \neq 0, g_2(x) = 0)$ and $(g_1(x) = 0, g_2(x) \neq 0)$ are similar. These imply that $R$ is ideal-$\pi$-McCoy.

**Theorem 2.3.** *The class of ideal-$\pi$-McCoy rings is closed under direct limits with injective maps.*

*Proof.* Let $D = \{R_i, \alpha_{ij}\}$ be a direct system of ideal-$\pi$-McCoy rings $R_i$ for $i \in I$ and ring homomorphisms $\alpha_{ij} : R_i \to R_j$ for each $i \leq j$ satisfying $\alpha_{ij}(1) = 1$, where $I$ is a directed partially ordered set. Set $R = \varinjlim R_i$ be the direct limit of $D$ with $\iota_i : R_i \to R$ and $\iota_j \alpha_{ij} = \iota_i$, where every $\iota_i$ is injective. We will show that $R$ is an ideal-$\pi$-McCoy ring. Take $a, b \in R$. Then $a = \iota_i(a_i)$, $b = \iota_j(b_j)$ for some $i, j \in I$ and there is $k \in I$ such that $i \leq k$, $j \leq k$. Define

$$a + b = \iota_k(\alpha_{ik}(a_i) + \alpha_{jk}(b_j)) \text{ and } ab = \iota_k(\alpha_{ik}(a_i)\alpha_{jk}(b_j)),$$

where $\alpha_{ik}(a_i)$ and $\alpha_{jk}(b_j)$ are in $R_k$. Then $R$ forms a ring with $0 = \iota_i(0)$ and $1 = \iota_i(1)$.

Now let $f(x), g(x) \in R[x]$ be nonzero polynomials such that $f(x)g(x) \in N(R[x])$. There is $k \in I$ such that $f(x), g(x) \in R_k[x]$ via $\iota_i$'s and $\alpha_{ij}$'s; hence we get $f(x)g(x) \in N(R_k[x])$. Since $R_k$ is ideal-$\pi$-McCoy, there exists $0 \neq c_k$ in the ideal of $R_k$ generated by the coefficients of $g(x)$ such that $f(x)c_k \in N(R_k[x])$.

Put $c = \iota_k(c_k)$. Then $f(x)c \in N(R[x])$ with a nonzero $c$ in the ideal of $R$ generated by the coefficients of $g(x)$, entailing $R$ being ideal-$\pi$-McCoy. $\Box$

Write $R$ and $R_n$ as in Example 1.7. Then the direct limit $R$ of $R_n$'s is ideal-$\pi$-McCoy, but $R_n$ need not ideal-$\pi$-McCoy with the help of [15, Theorem 1.4].

The class of ideal-$\pi$-McCoy rings is not closed under subrings by Proposition 2.1(5). This is comparable with the following.

**Proposition 2.4.** *Let $R$ be a ring. If $R[x]$ is ideal-$\pi$-McCoy over a ring $R$, then so is $R$.*

*Proof.* Suppose that $S[t]$ is the polynomial ring with an indeterminate $t$ over $S = R[x]$. We apply the proof of [15, Proposition 2.3]. Let $S$ be ideal-$\pi$-McCoy and suppose that $f(x)g(x) \in N(R[x])$ for nonzero polynomials $f(x), g(x)$ in $R[x]$. This can be converted to that $f(t)g(t) \in N(S[t])$ for nonzero polynomials $f(t), g(t)$ in $S[t]$. Since $S$ is ideal-$\pi$-McCoy, $f(t)h(x) \in N(S[t])$ for some $0 \neq h(x)$ in the ideal of $R[x]$ generated by the coefficients of $g(t)$, say $(f(t)h(x))^k = 0$. Note that $f(t) \in R[t]$. Here letting $h(x) = \sum_{i=0}^{n} a_i x^i$ (we can set $a_0 \neq 0$, dividing by $x$ if necessary), we get $f(t)a_0 \in N(R[t])$ from $0 = (f(t)h(x))^k = (f(t)a_0)^k + h_1 x + \cdots + h_{nk} x^{nk}$ with $h_1, \ldots, h_{nk} \in R[t]$. This implies that $f(x)a_0 \in N(R[x])$ and $0 \neq a_0$ in the ideal of $R$ generated by the coefficients of $g(x)$, showing that $R$ is ideal-$\pi$-McCoy. $\Box$

In fact, we do not know of any example of an ideal-$\pi$-McCoy ring whose polynomial ring is not ideal-$\pi$-McCoy.

**Question.** Does a ring $R$ being ideal-$\pi$-McCoy imply $R[x]$ being ideal-$\pi$-McCoy?

Recall that a regular element means a neither left nor right zero-divisor. A ring $R$ is called right *Ore* if given $a, b \in R$ with $b$ regular there exist $a_1, b_1 \in R$ with $b_1$ regular such that $ab_1 = ba_1$. It is well-known that $R$ is a right Ore ring if and only if the classical right quotient ring of $R$ exists.

**Theorem 2.5.** *Let $R$ be a right Ore ring with its classical right quotient ring $Q$. Let $R$ be an ideal-$\pi$-McCoy ring such that non-regular polynomials in $R[x]$ are nilpotent. Then $Q$ is ideal-$\pi$-McCoy.*

*Proof.* We will use [20, Proposition 2.1.16] freely, and apply the proof of [15, Theorem 2.1]. Denote the set of all regular elements in $R$ by $C(R)$. Suppose $F(x)G(x) \in N(Q[x])$ for $0 \neq F(x)$, $0 \neq G(x) \in Q[x]$. Say that $(F(x)G(x))^k = 0$ and $(F(x)G(x))^{k-1} \neq 0$ for some $k \geq 1$. We can write

$$F(x) = \sum_{i=0}^{m} a_i u^{-1} x^i = (\sum_{i=0}^{m} a_i x^i) u^{-1} \text{ and } G(x) = \sum_{j=0}^{n} b_j v^{-1} x^j = (\sum_{j=0}^{n} b_j x^j) v^{-1}$$

for some $a_i$'s, $b_j$'s in $R$ and $u, v \in C(R)$. Since $R$ is right Ore, there exist $u_1, v_1 \in C(R)$ for all $i$'s and $j$'s such that $u^{-1} b_j = b_j' u_1^{-1}$ and $v^{-1} a_i = a_i' v_1^{-1}$

for some $a'_i, b'_j \in R$. Next set

$$f(x) = \sum_{i=0}^{m} a_i x^i,\ f_1(x) = \sum_{i=0}^{m} a'_i x^i,$$
$$g(x) = \sum_{j=0}^{n} b_j x^j,\ g_1(x) = \sum_{j=0}^{n} b'_j x^j \text{ and}$$
$$v_2 = uv_1,\ u_2 = vu_1.$$

Then we have

$$u^{-1}g(x) = g_1(x)u_1^{-1},\ v^{-1}f(x) = f_1(x)v_1^{-1},$$
$$F(x)G(x) = f(x)g_1(x)u_2^{-1} \text{ and}$$
$$G(x)F(x) = g(x)f_1(x)v_2^{-1},$$

noting that $f(x) \neq 0$, $g(x) \neq 0$, $f_1(x) \neq 0$, and $g_1(x) \neq 0$. Let $I$ and $J$ be the ideals of $Q$ generated by the coefficients of $F(x)$ and $G(x)$, respectively. Since

$$a_i = a_i u^{-1} u,\ u^{-1} b_j u_1 = b'_j,\ b_j = b_j v^{-1} v,\ v^{-1} a_i v_1 = a'_i,$$

we have $f(x), f_1(x) \in I[x]$ and $g(x), g_1(x) \in J[x]$. Note that $\sum_{i=0}^{m} Ra_iR \subseteq I$, $\sum_{i=0}^{m} Ra'_iR \subseteq I$, $\sum_{j=0}^{n} Rb_jR \subseteq J$, $\sum_{j=0}^{n} Rb'_jR \subseteq J$, $\sum_{i=0}^{m} Ra'_iR \subseteq R$, and $\sum_{j=0}^{n} Rb'_jR \subseteq R$. We will freely use this fact in the following computation. Set $I_0 = \sum_{i=0}^{m} Ra'_iR$ and $J_0 = \sum_{j=0}^{n} Rb'_jR$.

`Case 1`. $F(x)G(x) = 0$ and $G(x)F(x) = 0$

Consider $F(x)G(x) = 0$. Then $f(x)g_1(x)u_2^{-1} = 0$ and so $f(x)g_1(x) = 0$. Since $R$ is ideal-$\pi$-McCoy and $f(x), g_1(x) \in R[x]$, there exists $0 \neq \alpha \in J_0$ such that $f(x)\alpha \in N(R[x]) \subseteq N(Q[x])$. This yields

$$F(x)u\alpha = f(x)u^{-1}u\alpha = f(x)\alpha \in N(Q[x]),$$

where $0 \neq u\alpha \in J$.

Consider $G(x)F(x) = 0$. Then $g(x)f_1(x)v_2^{-1} = 0$ and so $g(x)f_1(x) = 0$. Since $R$ is ideal-$\pi$-McCoy and $g(x), f_1(x) \in R[x]$, there exists $0 \neq \beta \in I_0$ such that $g(x)\beta \in N(R[x]) \subseteq N(Q[x])$ by the similar argument to above. This yields

$$G(x)v\beta = g(x)v^{-1}v\beta = g(x)\beta \in N(Q[x]),$$

where $0 \neq v\beta \in I$. Note $v\beta G(x) \in N(Q[x])$.

`Case 2`. $F(x)G(x) = 0$ and $G(x)F(x) \neq 0$

Note that $G(x)F(x)G(x) = 0$ and $F(x)G(x)F(x) = 0$. Letting $H(x) = G(x)F(x)$, we have $H(x)G(x) = 0$. Note $0 \neq H(x) \in (I \cap J)[x]$. Say $H(x) = \sum_{s=0}^{\ell} c_s w^{-1} x^s = (\sum_{s=0}^{\ell} c_s x^s)w^{-1}$ with $c_s$'s in $R$ and $w \in C(R)$.

Consider $H(x)G(x) = 0$. Since $R$ is right Ore, there exists $w_1 \in C(R)$ for all $\ell$'s such that $w^{-1}b_j = d_j w_1^{-1}$ for some $d_j \in R$. Letting $h(x) =$

$\sum_{s=0}^{\ell} c_s x^s$ and $g_2(x) = \sum_{j=0}^{n} d_j x^j$, we have $H(x)G(x) = h(x)g_2(x)(vw_1)^{-1}$ and so $h(x)g_2(x) = 0$. Note

$$w^{-1}G(x) = w^{-1}g(x)v^{-1} = g_2(x)w_1^{-1}v^{-1} = g_2(x)(vw_1)^{-1}.$$

Since $R$ is ideal-$\pi$-McCoy and $h(x), g_2(x) \in R[x]$, there exists $0 \neq \gamma \in \sum_{s=0}^{\ell} Rc_sR$ such that $\gamma g_2(x) \in N(R[x]) \subseteq N(Q[x])$. Note that $\gamma \in I \cap J$, $g_2(x)\gamma \in N(Q[x])$ and

$$w^{-1}G(x)vw_1\gamma = g_2(x)(vw_1)^{-1}vw_1\gamma = g_2(x)\gamma \in N(Q[x]).$$

This yields $G(x)vw_1\gamma w^{-1} \in N(Q[x])$ with $0 \neq vw_1\gamma w^{-1} \in I \cap J$.

Next since $F(x)G(x) = 0$, the same computation as in Case 1 is applicable to find nonzero $q \in J$ such that $F(x)q \in N(Q[x])$.

Case 3. $F(x)G(x) \neq 0$ and $G(x)F(x) = 0$

Note that $F(x)G(x)F(x) = 0$ and $G(x)F(x)G(x) = 0$. Letting $K(x) = F(x)G(x)$, we have $K(x)F(x) = 0$ with $0 \neq K(x) \in (I \cap J)[x]$. Note $0 \neq K(x) \in (I \cap J)[x]$. Say $K(x) = \sum_{s=0}^{t} e_s z^{-1} x^s = (\sum_{s=0}^{t} e_s x^s)z^{-1}$ with $e_s$'s in $R$ and $z \in C(R)$.

Since $R$ is right Ore, there exists $z_1 \in C(R)$ for all $i$'s such that $z^{-1}a_i = y_i z_1^{-1}$ for some $y_i \in R$. Let $k(x) = \sum_{s=0}^{t} e_s x^s$ and $f_2(x) = \sum_{i=0}^{m} y_i x^i$. Then $k(x) \neq 0$ and we have $K(x)F(x) = k(x)f_2(x)(uz_1)^{-1}$, entailing $k(x)f_2(x) = 0$. Note

$$z^{-1}F(x) = z^{-1}f(x)u^{-1} = f_2(x)z_1^{-1}u^{-1} = f_2(x)(uz_1)^{-1}.$$

Since $R$ is ideal-$\pi$-McCoy and $k(x), f_2(x) \in R[x]$, there exists $0 \neq \delta \in \sum_{s=0}^{t} Re_sR$ such that $\delta f_2(x) \in N(R[x]) \subseteq N(Q[x])$. Note that $\delta \in I \cap J$, $f_2(x)\delta \in N(Q[x])$ and

$$z^{-1}F(x)uz_1\delta = f_2(x)(uz_1)^{-1}uz_1\delta = f_2(x)\delta \in N(Q[x]).$$

This yields $F(x)uz_1\delta z^{-1} \in N(Q[x])$ with $0 \neq uz_1\delta z^{-1} \in I \cap J$.

Next since $G(x)F(x) = 0$, the same computation as in Case 1 is applicable to find nonzero $p \in I$ such that $pG(x) \in N(Q[x])$.

Case 4. $F(x)G(x) \neq 0$ and $G(x)F(x) \neq 0$

Suppose $F(x)G(x) \neq 0$ and $G(x)F(x) \neq 0$. Then there exists $k \geq 2$ such that $(F(x)G(x))^k = 0$ and $(F(x)G(x))^{k-1} \neq 0$.

Since $R$ is right Ore, $u_2^{-1}(F(x)G(x))^{k-1} = A(x)u_3^{-1}$ for some $A(x) \in R[x]$ and some $u_3 \in C(R)$. Here $A(x)$ is also nonzero because both $(F(x)G(x))^{k-1}$ is nonzero. Note that $A(x) \in (I \cap J)[x]$ and

$$f(x)g_1(x)A(x) = f(x)g_1(x)u_2^{-1}(F(x)G(x))^{k-1}u_3 = (F(x)G(x))^k u_3 = 0.$$

Since $R$ is ideal-$\pi$-McCoy, there exists nonzero $\beta$ in the ideal of $R$ generated by the coefficients of $A(x)$ such that $f(x)g_1(x)\beta \in N(R[x])$. Note $\beta \in I \cap J$. Moreover $g_1(x)\beta$ is nilpotent by hypothesis. Then, from

$$g_1(x)\beta = g_1(x)u_2^{-1}u_2\beta = u^{-1}g(x)v^{-1}u_2\beta = u^{-1}G(x)u_2\beta,$$

we have $G(x)u_2\beta u^{-1} \in N(Q[x])$ with $0 \neq u_2\beta u^{-1} \in I \cap J$.

By Cases 1, 2, 3 and 4, we have that

$$\beta' G(x) \in N(Q[x]) \text{ for some } 0 \neq \beta' \in I.$$

Therefore $Q$ is ideal-$\pi$-McCoy by help of Proposition 1.1(1). □

The following has a similar structure to the case of classical quotient rings.

**Proposition 2.6.** *Let $R$ be a ring and $\Delta$ be a multiplicatively closed subset of $R$ consisting of central regular elements. Then*
(1) *$R$ is ideal-$\pi$-McCoy if and only if $\Delta^{-1}R$ is.*
(2) *$R$ is strongly right McCoy if and only if $\Delta^{-1}R$ is.*

*Proof.* The necessity follows from the proof of Theorem 2.5. Let $S = \Delta^{-1}R$ and put $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ be nonzero polynomials in $R[x]$.

(1) Suppose that $S$ is ideal-$\pi$-McCoy and $f(x)g(x) \in N(R[x])$. Since $S$ is ideal-$\pi$-McCoy, $f(x)(cw^{-1}) \in N(S[x])$ for some nonzero $cw^{-1}$ in the ideal of $S$ generated by the coefficients of $g(x)$. Note that $c \neq 0$ and $c$ is also contained in the ideal of $S$ generated by the coefficients of $g(x)$. But since $w$ is central, we get $f(x)c \in N(R[x])$, concluding that $R$ is ideal-$\pi$-McCoy.

(2) is similar to the proof of (1). □

**Corollary 2.7.** *Let $R$ be a ring. Then $R[x]$ is ideal-$\pi$-McCoy (resp. strongly right McCoy) if and only if $R[x; x^{-1}]$ is.*

*Proof.* Note that $\Delta = \{1, x, x^2, \ldots\}$ is a multiplicatively closed subset of $R[x]$ consisting of central regular elements. So Proposition 2.6 gives the proof since $R[x; x^{-1}] = \Delta^{-1}R[x]$. □

Let $R$ be an algebra (with or without identity) over a commutative ring $S$. The *Dorroh extension* of $R$ by $S$ is the Abelian group $R \oplus S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1 r_2 + s_1 r_2 + s_2 r_1, s_1 s_2)$ for $r_i \in R$ and $s_i \in S$.

**Theorem 2.8.** *Let $R$ be an algebra over a commutative domain $S$, and $D$ be the Dorroh extension of $R$ by $S$. Then*
(1) *$R$ is ideal-$\pi$-McCoy if and only if $D$ is.*
(2) *$R$ is strongly right McCoy if and only if $D$ is.*
(3) *$R$ is $\pi$-McCoy if and only if $D$ is.*

*Proof.* Note that $s \in S$ is identified with $s1 \in R$ and so $R = \{r + s \mid (r, s) \in D\}$ and $S$ is considered as a subring of $R$. Let $F(x) = (f_1(x), f_2(x))$ and $G(x) = (g_1(x), g_2(x))$ be any nonzero polynomials in $D[x]$ where $f_1(x) = \sum_{i=1}^{m} a_i x^i$, $g_1(x) = \sum_{j=1}^{n} b_j x^j \in R[x]$ and $f_2(x) = \sum_{i=1}^{m} s_i x^i$, $g_2(x) = \sum_{j=1}^{n} t_j x^j \in S[x]$.

(1) Suppose that $R$ is ideal-$\pi$-McCoy. Let $(F(x)G(x))^k = 0$ and

$$(F(x)G(x))^{k-1} \neq 0$$

for some $k \geq 1$. Then $(f_2(x)g_2(x))^k = 0$ and so we have $f_2(x) = 0$ or $g_2(x) = 0$ since $S[x]$ is a domain.

(i) If $f_2(x) = 0$, then we have $f_1(x)(g_1(x) + g_2(x)) \in N(R[x])$ from $0 = (F(x)G(x))^k = (f_1(x)(g_1(x) + g_2(x)), 0)^k$. Since $R$ is ideal-$\pi$-McCoy, there exists a nonzero $r = \sum_{u,v} r_{u_v}(b_u+t_u)r'_{u_v} \in \sum_{j=1}^{n} R(b_j+t_j)R$ such that $f_1(x)r \in N(R[x])$. Then

$$F(x)(r, 0) \in N(D[x]),$$

where $0 \neq (r, 0) = \sum_{u,v}(r_{u_v}, 0)(b_u, t_u)(r'_{u_v}, 0) \in \sum_{j=1}^{n} D(b_j, t_j)D$, entailing that $D$ is ideal-$\pi$-McCoy.

(ii) Let $g_2(x) = 0$. By the similar argument to (i), we have $(f_1(x) + f_2(x))g_1(x) \in N(R[x])$. Since $R$ is ideal-$\pi$-McCoy, there exists a nonzero $r = \sum_{u,v} r_{u_v} b_u r'_{u_v} \in \sum_{j=0}^{n} Rb_jR$ such that $(f_1(x) + f_2(x))r \in N(R[x])$. Then $F(x)(r, 0) \in N(D[x])$, where $0 \neq (r, 0) = \sum_{u,v}(r_{u_v}, 0)(b_u, 0)(r'_{u_v}, 0) \in \sum_{j=0}^{n} D(b_j, 0)D$, showing that $D$ is ideal-$\pi$-McCoy.

By (i) and (ii), $D$ is ideal-$\pi$-McCoy.

Conversely, suppose that $D$ is ideal-$\pi$-McCoy. Let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x]$ be nonzero polynomials with $f(x)g(x) \in N(R[x])$, say $(f(x)g(x))^k = 0$ but $(f(x)g(x))^{k-1} \neq 0$ for some $k \geq 1$. We take $F(x) = (f(x), 0)$ and $G(x) = (g(x), 0)$ in $D[x]$. Then $(F(x)G(x))^k = 0$ and so $F(x)G(x) \in N(D[x])$. Since $D$ is ideal-$\pi$-McCoy, there exists a nonzero $(r, s) = \sum_{u,v}(r_{u_v}, s_{u_v})(b_u, 0)(r'_{u_v}, s'_{u_v}) \in \sum_{j=0}^{n} D(b_j, 0)D$ with $F(x)(r, s) \in N(D[x])$. Since $(r, s) = (\sum_{u,v}(r_{u_v} + s_{u_v})b_u(r'_{u_v} + s'_{u_v}), 0)$, we have $0 \neq r = \sum_{u,v}(r_{u_v} + s_{u_v})b_u(r'_{u_v} + s'_{u_v}) \in \sum_{j=0}^{n} Rb_jR$ and $f(x)r = 0$. Therefore $R$ is ideal-$\pi$-McCoy.

(2) and (3) are similar to the proof of (1). □

Finally, we characterize the class of noncommutative ideal-$\pi$-McCoy rings of minimal order. $|-|$ means the cardinality.

**Proposition 2.9.** *Let $R$ be an ideal-$\pi$-McCoy ring. If $R$ is a noncommutative ideal-$\pi$-McCoy ring of minimal order, then $R$ is of order 8 and is isomorphic to $U_2(\mathbb{Z}_2)$.*

*Proof.* Let $R$ be a noncommutative ideal-$\pi$-McCoy of minimal order. Then $|R| \geq 2^3$ by [9, Theorem]. If $|R| = 2^3$, then $R$ is isomorphic to $U_2(\mathbb{Z}_2)$ by [9, Proposition]. But $U_2(\mathbb{Z}_2)$ is an ideal-$\pi$-McCoy ring by Lemma 1.2(4). This yields that $R$ is of order 8 and is isomorphic to $U_2(\mathbb{Z}_2)$. □

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