Bull. Korean Math. Soc. **50** (2013), No. 6, pp. 1873–1886
http://dx.doi.org/10.4134/BKMS.2013.50.6.1873

INNER UNIFORM DOMAINS AND THE APOLLONIAN INNER METRIC

Yaxiang Li and Xiantao Wang

Abstract. In this paper, we characterize inner uniform domains in $\mathbb{R}^n$ in terms of Apollonian inner metric and the metric j'_D when D are Apollonian. As an application, a new characterization for A-uniform domains is obtained.

1. Introduction and main results

Throughout the paper, we assume that D is a proper subdomain of the Euclidean n-space $\mathbb{R}^n$, $n \geq 2$, $[x, y]$ denotes the closed segment between x and y, and $B^n(x, r)$ stands for the open ball centered at x with radius $r > 0$, i.e., $B^n(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}$. In particular, we use $\mathbb{B}^n$ to denote the unit ball $B^n(0, 1)$. For $x, y \in D$, the Apollonian distance is defined by

$$\alpha_D(x, y) = \sup_{a,b \in \partial D} \Big\{ \log \frac{|a - x||b - y|}{|a - y||b - x|} \Big\},$$

where ∂D means the boundary of D. If one of a, b equals to ∞, we understand that $\frac{|\infty - x|}{|\infty - y|} = 1$. We note that this metric is invariant under Möbius transformations and equals the hyperbolic distance in balls and half spaces (cf. [2]). It is in fact a metric if and only if the complement of D is not contained in a hyperplane as was noted in [2, Theorem 1.1] (see also [9]). In this paper, these domains are called to be *Apollonian*. This metric was introduced in [2] and considered in [1, 5, 8, 9, 10, 11, 12, 19, 20].

Let $\gamma : [0, 1] \to D$ be a path, i.e., a continuous function. If d is a metric in D, then the d-length of γ is defined by

$$d(\gamma) = \sup \Big\{ \sum_{i=0}^{k-1} d(\gamma(t_i), \gamma(t_{i+1})) \Big\},$$

Received April 6, 2012; Revised May 18, 2013.
2010 *Mathematics Subject Classification.* Primary 30C65, 30F45; Secondary 30C20.
Key words and phrases. inner uniform, Apollonian metric, inner metric.
The research was partly supported by NSF of China (No. 11071063) and Hunan Provincial Innovation Foundation For Postgraduate.

where the supremum is taken over all $k < \infty$ and all sequences $\{t_i\}$ satisfying $0 = t_0 < t_1 < \cdots < t_k = 1$. All the paths in this paper are assumed to be rectifiable, that is, they have the finite Euclidean arc length. The inner metric of d is defined by the formula

$$\widetilde{d}(x, y) = \inf_{\gamma}\{d(\gamma)\},$$

where the infimum is taken over all paths connecting x and y in D. Particularly, we use $\widetilde{\alpha}_D$ to denote the inner metric of the Apollonian metric α_D and call it the *Apollonian inner metric*. Also we use $\lambda_D(x, y)$ to denote $\widetilde{d}(x, y)$ when $d(\gamma)$ is the Euclidean arc length.

In [8, Theorem 1.2], Hästö proved that $\widetilde{\alpha}_D$ is a metric if and only if the complement of D is not contained in an $(n-2)$-dimensional hyperplane in $\mathbb{R}^n$. Further, in [8], Hästö showed:

Theorem A ([8, Theorem 1.5]). *Let D be Apollonian. Then for x, $y \in D$, there exists a path γ in D connecting x and y such that*

$$\alpha_D(\gamma) = \widetilde{\alpha}_D(x, y).$$

And further, in [13], the authors got the following.

Theorem B ([13, Lemma 2.4]). *Let $x, y \in D$ and let $\gamma \subset D$ be a path such that $\widetilde{\alpha}_D(x, y) = \alpha_D(\gamma)$. Then for each z, $w \in \gamma$, we have*

$$\widetilde{\alpha}_D(z, w) = \alpha_D(\gamma[z, w]),$$

where $\gamma[z, w]$ denotes the part of γ between z and w.

Definition 1. A domain D is called *inner c-uniform* provided there exists a positive constant c such that each pair of points z_1, z_2 in D can be joined by a rectifiable arc γ in D satisfying (cf. [24])

(1) $\min\{\ell(\gamma[z_1, z])\, \ell(\gamma[z_2, z])\} \leq c\, d_D(z)$ for all $z \in \gamma$; and
(2) $\ell(\gamma) \leq c\, \lambda_D(z_1, z_2)$,

where $d_D(z)$ denotes the distance from z to the boundary ∂D of D.

If $\lambda_D(z_1, z_2)$ is replaced by $|z_1 - z_2|$ in Definition 1, then D is said to be *c-uniform*.

Obviously, uniformity implies inner uniformity.

Definition 2. A domain D is called to be a *c-John domain* provided there exists a positive constant c such that each pair of points z_1, z_2 in D can be joined by a rectifiable arc γ in D satisfying (cf. [18])

$$\min\{\mathrm{diam}(\gamma[z_1, w]), \mathrm{diam}(\gamma[w, z_2])\} \leq c d_D(w).$$

In [24], Väisälä showed the following two theorems.

Theorem C ([24, Theorem 3.3 and Theorem 3.4]). *Suppose that $D \subset \mathbb{R}^n$ is an inner c-uniform domain. Then for $x, y \in D$, we have*

$$\lambda_D(x, y) \leq \nu_1 \varrho_D(x, y),$$

where $\nu_1 \geq 6c$ *is a constant depending on* c *and* n*, and* $\varrho_D(x,y)$ *denotes the inner diameter metric, defined by*

$$\varrho_D(x,y) = \inf_{\gamma}\{\operatorname{diam}(\gamma)\}$$

over all arcs γ *joining* x *and* y *in* D*.*

Theorem D ([24, Theorem 3.11])**.** *For a domain* $D \subset \mathbb{R}^n$*, the following conditions are quantitatively equivalent*:

(1) D *is inner* c*-uniform.*
(2) *Each pair of points* $z_1, z_2 \in D$ *can be joined by an arc* γ *such that for* $w \in \gamma$*,*

$$\min\{\operatorname{diam}(\gamma[z_1,w]),\ \operatorname{diam}(\gamma[z_2,w])\} \leq \nu_2 d_D(w) \ \ \textit{and}\ \ \operatorname{diam}(\gamma) \leq \nu_2 \varrho_D(z_1,z_2),$$

where the constants c *and* ν_2 *depend on each other and* n*.*

Let D be a domain and $x, y \in D$. We write

$$j_D(x,y) = \log\Big(1 + \frac{|x-y|}{\min\{d_D(x), d_D(y)\}}\Big).$$

Kim [14] (see also [24]) introduced the following version of the j-metric:

$$j'_D(x,y) = \log\Big(1 + \frac{\varrho_D(x,y)}{\min\{d_D(x), d_D(y)\}}\Big),$$

and the quasihyperbolic metric [7] is defined by

$$k_D(x,y) = \inf_{\gamma} \int_{\gamma} \frac{|dz|}{d_D(z)},$$

where the infimum is taken over all paths γ joining x and y in D.

We easily know from the proof of [22, Lemma 2.2] that for x, $y \in D$,

$$j_D(x,y) \leq j'_D(x,y) \leq k_D(x,y). \tag{1.1}$$

Further, we have:

Theorem E. *For* $x, y \in D$*, the following hold true.*

(1) ([2, Corollary 3.2]) $\big|\log \frac{d_D(x)}{d_D(y)}\big| \leq \alpha_D(x,y) \leq 2 j_D(x,y)$;
(2) ([9, Lemma 5.3]) $\widetilde{j}_D(x,y) = k_D(x,y)$;
(3) ([9, Corollary 5.4]) $\widetilde{\alpha}_D(x,y) \leq 2k_D(x,y)$.

In [6], Gehring and Osgood got a characterization of uniform domains in terms of k_D and j_D.

Theorem F ([6, Corollary 1])**.** *A domain* D *is* μ*-uniform if and only if there exists a constant* μ_1 *such that*

$$k_D(z_1,z_2) \leq \mu_1 j_D(z_1,z_2)$$

for all $z_1, z_2 \in D$*, where the constants* μ *and* μ_1 *depend only on each other.*

As a matter of fact, the above inequality appearing in [6] in a form with an additive constant on the right hand side: it was shown by Vuorinen [25, 2.50] that the additive constant can be chosen to be 0. Moreover, in [13], the authors proved the following.

Theorem G ([13, Theorem 1.2]). *A domain $D \subset \mathbb{R}^n$ is μ-uniform if and only if there exists a constant μ_2 such that $\widetilde{\alpha}_D(x,y) \leq \mu_2 j_D(x,y)$ for any $x, y \in D$, where the constants μ and μ_2 depend only on each other.*

See [3, 4, 6, 13, 15, 16, 17, 24, 21, 23] for more details on uniform domains and inner uniform domains.

By Theorem G, one may ask that if we can characterize inner uniform domains in terms of $\widetilde{\alpha}_D$ and j'_D. The main aim of this paper is to consider this problem. Our result shows that the answer to this problem is affirmative. Combining with [15, Theorem 2.1] and Theorem D, we state our result in the following form.

Theorem 1. *Let D be a proper subdomain of $\mathbb{R}^n$. If D is Apollonian, then the followings are quantitatively equivalent.*

(1) *D is an inner c-uniform domain*;
(2) *There exists a constant c_1 such that*
$$k_D(x,y) \leq c_1 j'_D(x,y) \quad \forall x, y \in D;$$
(3) *There exists a constant c_2 such that*
$$\widetilde{\alpha}_D(x,y) \leq c_2 j'_D(x,y) \quad \forall x, y \in D; \tag{1.2}$$
(4) *Each pair of points $x, y \in D$ can be joined by an arc γ such that for $w \in \gamma$, $\min\{\operatorname{diam}(\gamma[x,w]), \operatorname{diam}(\gamma[y,w])\} \leq c_3 d_D(w)$ and $\operatorname{diam}(\gamma) \leq c_3 \varrho_D(x,y)$,*

where c, c_1, c_2 and c_3 are constants greater than 1, and depend on each other and n.

In [9], Hästö proved the following result.

Theorem H ([9, Proposition 6.6]). *Let $D \subset \mathbb{R}^n$ be a domain. The following conditions are quantitatively equivalent*:

(1) *D is A-uniform with coefficient K, that is, there exist some constant K such that for $x, y \in D$, $k_D(x,y) \leq K \alpha_D(x,y)$*;
(2) *D is μ-uniform and has the comparison property with some constant L*;
(3) *D is μ_3-quasi-isotropic and $\widetilde{\alpha}_D \leq \mu_4 \alpha_D$,*

where the constants K, L, μ, μ_3 and μ_4 depend only on each other.

Here we say that a domain $D \subset \mathbb{R}^n$ has the *comparison property* if there exists a constant L such that
$$j_D/L \leq \alpha_D \leq 2 j_D,$$

and D is μ_3-quasi-isotropic if

$$\limsup_{r\to 0} \frac{\sup\{\alpha_D(x,z) : |x-z| = r\}}{\inf\{\alpha_D(x,y) : |x-y| = r\}} \le \mu_3$$

for every $x \in D$ (see [9]).

As an application of Theorem 1, we get a new characterization for A-uniform domains.

Corollary 1. *Let $D \subset \mathbb{R}^n$ be an Apollonian domain. The following conditions are quantitatively equivalent*:

(1) *D is A-uniform with coefficient K*;

(2) *D is c-inner uniform and $j'_D(x,y) \le \mu_5 \alpha_D(x,y)$ for all $x, y \in D$,*

where the constants c, K and μ_5 depend on each other and n.

In the next section, we will prove Theorem 1 and Corollary 1.

2. Proofs of Theorem 1 and Corollary 1

2.1. Proof of Theorem 1

The implication (1) $\Rightarrow$ (2) follows from [15, Theorem 2.1] and Theorem C, and Theorem E shows that (2) $\Rightarrow$ (3) is true. The implication (4)$\Rightarrow$ (1) follows from Theorem D. Hence to finish the proof of Theorem 1, it remains only one implication (3) $\Rightarrow$ (4) to be checked.

Suppose that the assertion (3) in the theorem holds. To prove the truth of the assertion of (4) in the theorem, we let x, $y \in D$. Without loss of generality, we assume that $d_D(x) \le d_D(y)$. We consider the case where $|x-y| < d_D(x)$ and the case where $|x-y| \ge d_D(x)$, separately.

Case 1. $|x-y| < d_D(x)$.

Let $\gamma = [x,y]$ be the Euclidean line segment joining x and y. Clearly, $\gamma \subset D$,

$$\operatorname{diam}(\gamma) = |x-y| = \varrho_D(x,y)$$

and

$$\min\{\operatorname{diam}(\gamma[x,w]),\ \operatorname{diam}(\gamma[y,w])\} \le d_D(w)$$

for $w \in \gamma$. Thus the assertion (4) in the theorem is true in this case.

Case 2. $|x-y| \ge d_D(x)$.

By Theorem A there exists a path $\gamma \subset D$ connecting x and y such that

$$\widetilde{\alpha}_D(x,y) = \alpha_D(\gamma).$$

By compactness we see that there is a point z_0 in γ which is the first point along the direction from x to y satisfying

$$d_D(z_0) = \sup_{w\in\gamma}\{d_D(w)\}.$$

Let $m \geq 0$ be the integer such that

$$2^m d_D(x) \leq d_D(z_0) < 2^{m+1} d_D(x),$$

and let x_0 be the first point of $\gamma[x, z_0]$ from x to z_0 with

$$d_D(x_0) = 2^m d_D(x). \tag{2.1}$$

Then we have

$$d_D(x_0) \leq d_D(z_0) < 2 d_D(x_0). \tag{2.2}$$

Let $x_1 = x$, and let $x_2, \dots, x_{m+1}$ be the points such that for each $i \in \{2, \dots, m+1\}$, x_i denotes the first point in $\gamma[x, z_0]$ along the direction from x to z_0 satisfying

$$d_D(x_i) = 2^{i-1} d_D(x_1).$$

Apparently, $x_{m+1} = x_0$. If $x_0 \neq z_0$, we denote z_0 by x_{m+2}. By the choice of x_i, we know that for each $i \in \{1, 2, \dots, m\}$,

$$d_D(x_{i+1}) = 2 d_D(x_i), \tag{2.3}$$

and so

$$\varrho_D(x_i, x_{i+1}) \geq d_D(x_{i+1}) - d_D(x_i) = d_D(x_i). \tag{2.4}$$

For each $i \in \{1, 2, \dots, m\}$ and $w \in \gamma[x_i, x_{i+1}]$, it easily follows that

$$d_D(w) \leq d_D(x_{i+1}) = 2 d_D(x_i). \tag{2.5}$$

Let w_0 be the first point of γ along the direction from y to x satisfying

$$d_D(w_0) = \sup_{w \in \gamma} \{d_D(w)\}.$$

Obviously, $d_D(w_0) = d_D(z_0)$. It is possible that $w_0 = z_0$. A similar argument as above shows that there are points $\{y_j\}_{j=1}^{s+1}$ in $\gamma[y, w_0]$ such that for each $j \in \{1, \dots, s+1\}$, y_j denotes the first point in $\gamma[y, w_0]$ along the direction from y to w_0 satisfying

$$d_D(y_j) = 2^{j-1} d_D(y_1),$$

where $y_1 = y$ and $d_D(y_{s+1}) = 2^s d_D(y_1)$. We also use y_0 to denote y_{s+1}. If $y_0 \neq w_0$, we use y_{s+2} to denote w_0.

Lemma 1. *For each $i \in \{1, 2, \dots, m\}$, we have*

1. $\operatorname{diam}(\gamma[x_i, x_{i+1}]) \leq b_1 \varrho_D(x_i, x_{i+1})$ *with* $b_1 = 24c_2'$ *and* $c_2' = [c_2] + 1$. *Here and in the following,* $[\cdot]$ *always denotes the greatest integer part*;
2. $\varrho_D(x_i, x_{i+1}) \leq b_2 d_D(x_i)$ *with* $b_2 = (1 + b_1)^2$;
3. $d_D(x_i) \leq b_3 d_D(w)$ *for all* $w \in \gamma$ *with* $b_3 = (1 + b_2)^{\frac{c_2}{2}}$,

where c_2 is the same constant as in the inequality (1.2).

Proof. We now prove the first assertion in the lemma. Suppose on the contrary that there is some $i \in \{1, \dots, m\}$ satisfying

$$\operatorname{diam}(\gamma[x_i, x_{i+1}]) > b_1 \varrho_D(x_i, x_{i+1}).$$

Let $u_{i,1} = x_i$, and take the points $u_{i,2}, u_{i,3}, \dots, u_{i,c_2'+1}$ in γ such that for each $t \in \{2, \dots, c_2' + 1\}$, $u_{i,t}$ is the first point of γ from x_i to x_{i+1} satisfying

$$|x_i - u_{i,t}| = 6(t-1)\varrho_D(x_i, x_{i+1}).$$

Then for each $t \in \{1, \dots, c_2'\}$, we have

$$|u_{i,t} - u_{i,t+1}| \geq |x_i - u_{i,t+1}| - |u_{i,t} - x_i| \geq 6\varrho_D(x_i, x_{i+1}). \tag{2.6}$$

Let $p \in \partial D$ be such that $d_D(u_{i,t+1}) = |u_{i,t+1} - p|$. Then (2.4), (2.5) and (2.6) yield

$$\begin{aligned} |u_{i,t} - p| &\geq |u_{i,t} - u_{i,t+1}| - d_D(u_{i,t+1}) \\ &\geq 6\varrho_D(x_i, x_{i+1}) - 2d_D(x_i) \\ &\geq 2\varrho_D(x_i, x_{i+1}) + 2d_D(x_i). \end{aligned} \tag{2.7}$$

Similarly, for $q \in \partial D$ with $d_D(u_{i,t}) = |u_{i,t} - q|$, we get

$$|u_{i,t+1} - q| \geq 2\varrho_D(x_i, x_{i+1}) + 2d_D(x_i). \tag{2.8}$$

Thus we infer from (2.5), (2.7) and (2.8) that

$$\alpha_D(u_{i,t}, u_{i,t+1}) \geq \log\left(\frac{|u_{i,t} - p|}{d_D(u_{i,t+1})}\frac{|u_{i,t+1} - q|}{d_D(u_{i,t})}\right) \geq 2\log\left(1 + \frac{\varrho_D(x_i, x_{i+1})}{d_D(x_i)}\right),$$

which together with Theorem B show that

$$\begin{aligned} \widetilde{\alpha}_D(x_i, x_{i+1}) &= \alpha_D(\gamma[x_i, x_{i+1}]) \\ &\geq \sum_{t=1}^{c_2'} \alpha_D(u_{i,t}, u_{i,t+1}) \\ &\geq 2c_2' \log\left(1 + \frac{\varrho_D(x_i, x_{i+1})}{d_D(x_i)}\right) \\ &\geq 2c_2 \log\left(1 + \frac{\varrho_D(x_i, x_{i+1})}{d_D(x_i)}\right) \\ &= 2c_2 j_D'(x_i, x_{i+1}), \end{aligned}$$

which contradicts with (1.2). Hence (1) is true.

Then we come to prove the second assertion. Suppose on the contrary that there is some $i \in \{1, 2, \dots, m\}$ satisfying

$$\varrho_D(x_i, x_{i+1}) > b_2 d_D(x_i). \tag{2.9}$$

Obviously, there exists some point $v \in \gamma[x_i, x_{i+1}]$ such that $|x_i - v| \geq \frac{1}{2}\varrho_D(x_i, x_{i+1})$. We let $v_{i,1} = x_i$, and let $v_{i,2}, \dots, v_{i,\frac{b_1}{12}+1}$ be the points in γ

such that for each $h \in \{2, \ldots, \frac{b_1}{12}+1\}$, $v_{i,h}$ is the first point of γ from x_i to x_{i+1} satisfying

$$|x_i - v_{i,h}| = \frac{6(h-1)}{b_1}\varrho_D(x_i, x_{i+1}).$$

Then

$$|v_{i,h} - v_{i,h+1}| \geq |v_{i,h+1} - x_i| - |v_{i,h} - x_i| \geq \frac{6}{b_1}\varrho_D(x_i, x_{i+1}). \tag{2.10}$$

Let $p \in \partial D$ satisfy $d_D(v_{i,h+1}) = |v_{i,h+1} - p|$. Then it follows from (2.4), (2.5), (2.9) and (2.10) that

$$\begin{aligned} |v_{i,h} - p| &\geq |v_{i,h} - v_{i,h+1}| - d_D(v_{i,h+1}) \\ &\geq \frac{6}{b_1}\varrho_D(x_i, x_{i+1}) - 2d_D(x_i) \\ &> \frac{2}{b_1}\varrho_D(x_i, x_{i+1}) + 2d_D(x_i). \end{aligned} \tag{2.11}$$

Similarly, for $q \in \partial D$ with $d_D(v_{i,h}) = |v_{i,h} - q|$, we know

$$|v_{i,h+1} - q| \geq \frac{2}{b_1}\varrho_D(x_i, x_{i+1}) + 2d_D(x_i). \tag{2.12}$$

Thus we infer from (2.5), (2.9), (2.11) and (2.12) that

$$\begin{aligned} \alpha_D(v_{i,h}, v_{i,h+1}) &\geq \log\left(\frac{|v_{i,h} - p|}{d_D(v_{i,h+1})}\frac{|v_{i,h+1} - q|}{d_D(v_{i,h})}\right) \\ &\geq 2\log\left(1 + \frac{\varrho_D(x_i, x_{i+1})}{b_1 d_D(x_i)}\right) \\ &> \frac{12c_2}{b_1}\log\left(1 + \frac{\varrho_D(x_i, x_{i+1})}{d_D(x_i)}\right). \end{aligned}$$

Whence Theorem B yields

$$\begin{aligned} \widetilde{\alpha}_D(x_i, x_{i+1}) &= \alpha_D(\gamma[x_i, x_{i+1}]) \\ &\geq \sum_{h=1}^{\frac{b_1}{12}} \alpha_D(v_{i,h}, v_{i,h+1}) \\ &> c_2 \log\left(1 + \frac{\varrho_D(x_i, x_{i+1})}{d_D(x_i)}\right) \\ &= c_2 j'_D(x_i, x_{i+1}), \end{aligned}$$

which is the desired contradiction.

To finish the proof of Lemma 1, it remains to check (3). Let $w \in \gamma$. Then (2) in the lemma, (1.2), Theorems B and E lead to

$$\begin{aligned} 2\log\frac{d_D(x_i)}{d_D(w)} &< \alpha_D(x_i, w) + \alpha_D(w, x_{i+1}) \\ &\leq \alpha_D(\gamma[x_i, x_{i+1}]) \end{aligned}$$

$$\begin{aligned} &= \widetilde{\alpha}_D(x_i, x_{i+1}) \\ &\leq c_2 \log\left(1 + \frac{\varrho_D(x_i, x_{i+1})}{d_D(x_i)}\right) \\ &\leq c_2 \log(1 + b_2), \end{aligned}$$

which shows that

$$d_D(x_i) \leq (1 + b_2)^{\frac{c_2}{2}} d_D(w),$$

which shows that (3) is true by taking $b_3 = (1 + b_2)^{\frac{c_2}{2}}$. Hence the proof of Lemma 1 is complete. □

Similarly, we know that:

Lemma 2. *For each $j \in \{1, \dots, s\}$, we have*

(1) $\operatorname{diam}(\gamma[y_j, y_{j+1}]) \leq b_1 \varrho_D(y_j, y_{j+1})$;
(2) $\varrho_D(y_j, y_{j+1}) \leq b_2 d_D(y_j)$;
(3) $d_D(y_j) \leq b_3 d_D(w)$ *for all* $w \in \gamma$.

Suppose $x_0 \neq y_0$. Then:

Lemma 3. *For $w \in \gamma[x_0, y_0]$, we have*

$$d_D(x_0) \leq b_3^2 d_D(w) \ \ \textit{and} \ \operatorname{diam}(\gamma[x_0, y_0]) \leq b_4 d_D(w),$$

where $b_4 = b_1 b_2 b_3^2$.

Proof. We note by (2.2) that

$$\frac{1}{2} d_D(y_0) < d_D(x_0) < 2 d_D(y_0), \tag{2.13}$$

and for $w \in \gamma[x_0, y_0]$, we have

$$d_D(w) < d_D(z_0) < 2 d_D(x_0). \tag{2.14}$$

We prove this lemma by considering the case where $\varrho_D(x_0, y_0) \geq d_D(x_0)$ and the case where $\varrho_D(x_0, y_0) < d_D(x_0)$, separately.

Suppose first that $\varrho_D(x_0, y_0) \geq d_D(x_0)$. Then by (2.13), (2.14) and a similar argument as in the proof of Lemma 1, we get for each $w \in \gamma[x_0, y_0]$,

$$d_D(x_0) \leq b_3^2 d_D(w) \tag{2.15}$$

and

$$\operatorname{diam}(\gamma[x_0, y_0]) \leq b_4 d_D(w), \tag{2.16}$$

where $b_4 = b_1 b_2 b_3^2$.

Suppose next that $\varrho_D(x_0, y_0) < d_D(x_0)$. In this case, we need the following claim.

Claim 1. *For $w \in \gamma[x_0, y_0]$, we have $|w - x_0| \leq (3^{c_2} + 1) d_D(x_0)$.*

Obviously, to prove this claim, it suffices to consider the case $|x_0 - w| \geq 2d_D(x_0)$. Let $z \in \partial D$ satisfy $|x_0 - z| = d_D(x_0)$. Then it follows from (1.2), (2.13), Theorems B and E that

$$\begin{aligned} \log\left(\frac{|w-x_0|}{d_D(x_0)} - 1\right) &\leq \log\frac{|z-w|}{d_D(x_0)} \\ &\leq \alpha_D(x_0, w) \\ &\leq \alpha_D(\gamma[x_0, y_0]) \\ &= \widetilde{\alpha}_D(x_0, y_0) \\ &\leq c_2 \log\left(1 + \frac{\varrho_D(x_0, y_0)}{\min\{d_D(x_0), d_D(y_0)\}}\right) \\ &\leq c_2 \log 3, \end{aligned} \tag{2.17}$$

from which the claim easily follows.

By Claim 1, we get

$$\operatorname{diam}(\gamma[x_0, y_0]) \leq 2(3^{c_2} + 1)d_D(x_0). \tag{2.18}$$

Moreover, by Theorem E and a similar argument as in (2.17), we also have

$$\log\frac{d_D(x_0)}{d_D(w)} \leq \alpha_D(x_0, w) \leq c_2 \log 3,$$

and so

$$d_D(x_0) \leq 3^{c_2} d_D(w), \tag{2.19}$$

which together with (2.18) show that

$$\operatorname{diam}(\gamma[x_0, y_0]) \leq 2(3^{c_2} + 1)^2 d_D(w) \leq b_4 d_D(w). \tag{2.20}$$

The inequalities (2.15), (2.16), (2.19) and (2.20) imply that the lemma is true. $\square$

Now we come to prove that the first part of (4) in Theorem 1 holds with constant $2b_4$, i.e., for $w \in \gamma$,

$$\min\{\operatorname{diam}(\gamma[x, w]), \operatorname{diam}(\gamma[y, w])\} \leq 2b_4 d_D(w). \tag{2.21}$$

Let $w \in \gamma$. We divide the discussions into three cases.

Case 3. $w \in \gamma[x, x_0]$.

Clearly, there exists an integer $k \in \{1, 2, \ldots, m\}$ such that $w \in \gamma[x_k, x_{k+1}]$. By Lemma 1 and (2.3) we have

$$\begin{aligned} \operatorname{diam}(\gamma[x, w]) &\leq \sum_{i=1}^{k} \operatorname{diam}(\gamma[x_i, x_{i+1}]) \leq b_1 b_2 \sum_{i=1}^{k} d_D(x_i) \\ &\leq 2b_1 b_2 d_D(x_k) \leq 2b_1 b_2 b_3 d_D(w) < b_4 d_D(w). \end{aligned} \tag{2.22}$$

Case 4. $w \in \gamma[y, y_0]$.

By Lemma 2, we see from a similar argument as in the proof of Case 3 that

$$\operatorname{diam}(\gamma[y,w]) < b_4 d_D(w). \tag{2.23}$$

Case 5. *If* $w \in \gamma[x_0, y_0]$.

It follows from Lemmas 1 and 3 that

$$\begin{aligned} \operatorname{diam}(\gamma[x,w]) &\le \sum_{i=1}^{m} \operatorname{diam}(\gamma[x_i, x_{i+1}]) + \operatorname{diam}(\gamma[x_0, y_0]) \\ &\le b_1 b_2 \sum_{i=1}^{m} d_D(x_i) + b_4 d_D(w) \\ &\le b_1 b_2 d_D(x_{m+1}) + b_4 d_D(w) \\ &\le 2 b_4 d_D(w). \end{aligned} \tag{2.24}$$

The proof for (2.21) easily follows from the combination of (2.22), (2.23) and (2.24).

Next we prove the second part of (4) in Theorem 1 with constant $b_6 = 2^{b_5+2} b_4$, where $b_5 = c_2 \log_2(8b_4+1)+1$, i.e.,

$$\operatorname{diam}(\gamma[x,y]) \le b_6 \varrho_D(x,y). \tag{2.25}$$

We first prove a lemma.

Lemma 4. $\varrho_D(x,y) \ge 2^{m-b_5} d_D(x)$.

Proof. If $m \le b_5$, then it is obvious from the assumption "$|x-y| \ge d_D(x)$". So we assume that $m > b_5$. In this case, we prove the lemma by contradiction. Suppose that

$$\varrho_D(x,y) < 2^{m-b_5} d_D(x). \tag{2.26}$$

Then

$$\begin{aligned} d_D(y) &\le \varrho_D(x,y) + d_D(x) \\ &< (2^{m-b_5}+1) d_D(x) \\ &\le \frac{2^m}{(8b_4+1)^{c_2}} d_D(x) \\ &< \frac{2^m}{(8b_4+1)^{\frac{c_2}{2}}} d_D(x). \end{aligned} \tag{2.27}$$

By (2.1) and (2.13) we have

$$d_D(y_0) \ge \frac{1}{2} d_D(x_0) = 2^{m-1} d_D(x) > \frac{2^m}{(8b_4+1)^{\frac{c_2}{2}}} d_D(x), \tag{2.28}$$

then we obtain from (2.27), (2.28) and the easy fact

$$\frac{2^m}{(8b_4+1)^{\frac{c_2}{2}}} = 2^{m-\frac{c_2}{2}\log_2(8b_4+1)} > 1$$

that there exist $w_1 \in \gamma[x, x_0]$ and $w_2 \in \gamma[y, y_0]$ such that

$$d_D(w_1) = d_D(w_2) = \frac{2^m}{(8b_4+1)^{\frac{c_2}{2}}} d_D(x). \tag{2.29}$$

On one hand, we obtain from (1.2), (2.1), (2.29), Theorems B and E that

$$\begin{aligned} c_2 \log\left(1 + \frac{\varrho_D(w_1, w_2)}{d_D(w_1)}\right) &\geq \widetilde{\alpha}_D(w_1, w_2) \\ &= \alpha_D(\gamma[w_1, w_2]) \\ &\geq \alpha_D(w_1, x_{m+1}) + \alpha_D(x_{m+1}, w_2) \\ &\geq 2\log\frac{d_D(x_{m+1})}{d_D(w_1)} \\ &\geq c_2 \log(1 + 8b_4), \end{aligned}$$

which imply that

$$\varrho_D(w_1, w_2) \geq 8b_4 d_D(w_1). \tag{2.30}$$

On the other hand, by (2.22), (2.23), (2.26) and (2.29) we obtain

$$\begin{aligned} \varrho_D(w_1, w_2) &\leq \varrho_D(w_1, x) + \varrho_D(x, y) + \varrho_D(y, w_2) \\ &\leq \operatorname{diam}(\gamma[x, w_1]) + \varrho_D(x, y) + \operatorname{diam}(\gamma[y, w_2]) \\ &< 2b_4 d_D(w_1) + \varrho_D(x, y) \\ &\leq (2b_4 + \frac{2^{m-b_5}(8b_4+1)^{\frac{c_2}{2}}}{2^m}) d_D(w_1) \\ &< (2b_4 + 1) d_D(w_1), \end{aligned}$$

which is contradict with (2.30). Hence the proof of the lemma is complete. □

Now we are ready to conclude the proof of (2.25). It follows from (2.1), (2.13), (2.22), (2.23), (2.28), Lemmas 3 and 4 that

$$\begin{aligned} \operatorname{diam}(\gamma[x, y]) &\leq \operatorname{diam}(\gamma[x, x_0]) + \operatorname{diam}(\gamma[x_0, y_0]) + \operatorname{diam}(\gamma[y_0, y]) \\ &\leq 4b_4 d_D(x_{m+1}) \\ &= 2^{m+2} b_4 d_D(x) \\ &\leq 2^{b_5+2} b_4 \varrho_D(x, y). \end{aligned}$$

Hence the proof of (4) of Theorem 1 is complete by taking $c_3 = 2^{b_5+2} b_4$.

2.2. The proof of Corollary 1

(1)⇒ (2). Suppose (1) holds. Then by Theorem H, we see that D is a μ-uniform, so it is obvious inner uniform.

For the second part of (2), we can obtain easily from (1.1) and the definition of A-uniform (see Theorem H). Hence (2) is true.

(2) ⇒ (1). Suppose (2) holds. Then by Theorem 1, we know that for all $x, y \in D$,

$$k_D(x,y) \le c_1 j'_D(x,y) \le c_1 \mu_5 \alpha_D(x,y),$$

which shows that D is A-uniform with coefficient $K = c_1\mu_5$.

Acknowledgements. The authors thank the referees who have made valuable comments on this manuscript.

References

[1] D. Barbilian, *Einordnung von Lobayschewskys Massenbestimmung in einer gewissen allgemeinen Metrik der Jordansche Bereiche*, Casopsis Mathematiky a Fysiky **64** (1934-35), 182–183.

[2] A. F. Beardon, *The Apollonian metric of a domain in* $\mathbb{R}^n$, in: Quasiconformal Mappings and Analysis, edited by Peter Duren, Juha Heinonen, Brad Osgood, and Bruce Palka (Springer-Verlag, New York), 91–108, 1998.

[3] M. Bonk, J. Heinonen, and P. Koskela, *Uniformizing Gromov hyperbolic spaces*, Asterisque **270** (2001), 1–99.

[4] F. W. Gehring, *Uniform domains and the ubiquitous quasidisks*, Jahresber. Deutsch. Math.-Verein. **89** (1987), no. 2, 88–103.

[5] F. W. Gehring and K. Hag, *The Apollonian metric and quasiconformal mappings*, In the tradition of Ahlfors and Bers (Stony Brook, NY, 1998), 143–163, Contemp. Math., 256, Amer. Math. Soc., Providence, RI, 2000.

[6] F. W. Gehring and B. G. Osgood, *Uniform domains and the quasihyperbolic metric*, J. Analyse Math. **36** (1979), 50–74.

[7] F. W. Gehring and B. P. Palka, *Quasiconformally homogeneous domains*, J. Analyse Math. **30** (1976), 172–199.

[8] P. Hästö, *The Apollonian inner metric*, Comm. Anal. Geom. **12** (2004), no. 4, 927–947.

[9] ———, *The Apollonian metric: Uniformity and quasiconvexity*, Ann. Acad. Sci. Fenn. Math. **28** (2003), no. 2, 385–414.

[10] ———, *The Apollonian metric: limits of the comparison and bilipschitz properties*, Abstr. Appl. Anal. **2003** (2003), no. 20, 1141–1158.

[11] ———, *The Apollonian metric: quasi-isotropy and Seittenranta's metric*, Comput. Methods Funct. Theory **4** (2004), no. 2, 249–273.

[12] ———, *The Apollonian metric: the comparison property, bilipschitz mappings and thick sets*, J. Appl. Anal. **12** (2006), no. 2, 209–232.

[13] M. Huang, S. Ponnusamy, X. Wang, and S. K. Sahoo, *The Apollonian inner metric and uniform domain*, Math. Nachr. **283** (2010), no. 9, 1277–1290.

[14] K. W. Kim, *The quasihyperbolic metric and analogues of the Hardy-Littlewood property for* $\alpha = 0$ *in uniformity John domains*, Bull. Korean. Math. Soc. **43** (2006), no. 2, 395–410.

[15] ———, *Inner uniform domains, the quasihyperbolic metric and weak bloch functions*, Bull. Korean. Math. Soc. **49** (2012), no. 1, 11–24.

[16] O. Martio, *Definitions for uniform domains*, Ann. Acad. Sci. Fenn. Ser. A I Math. **5** (1980), no. 1, 197–205.

[17] O. Martio and J. Sarvas, *Injectivity theorems in plane and space*, Ann. Acad. Sci. Fenn. Ser. A I Math. **4** (1978), 383–401.

[18] R. Näkki and J. Väisälä, *John disks*, Exposition. Math. **9** (1991), no. 1, 3–43.

[19] A. G. Rhodes, *An upper bound for the hyperbolic metric of a convex domain*, Bull. London Math. Soc. **29** (1997), no. 5, 592–594.

[20] P. Seittenranta, *Möbius-invariant metrics*, Math. Proc. Cambridge Philos. Soc. **125** (1999), no. 3, 511–533.

[21] J. Väisälä, *Uniform domains*, Tohoku Math. J. **40** (1988), no. 1, 101–118.

[22] ______, *Free quasiconformality in Banach spaces. I*, Ann. Acad. Sci. Fenn. Ser. A I Math. **15** (1990), no. 2, 355–379.

[23] ______, *Free quasiconformality in Banach spaces. II*, Ann. Acad. Sci. Fenn. Ser. A I Math. **16** (1991), no. 2, 255–310.

[24] ______, *Relatively and inner uniform domains*, Conform. Geom. Dyn. **2** (1998), 56–88.

[25] M. Vuorinen, *Conformal invariants and quasiregular mappings*, J. Anal. Math. **45** (1985), 69–115.

Yaxiang Li
College of Science
Central South University of Forestry and Technology
Changsha, Hunan 410004, P. R. China
E-mail address: yaxiangli@163.com

Xiantao Wang
Department of Mathematics
Hunan Normal University
Changsha, Hunan 410081, P. R. China
E-mail address: xtwang@hunnu.edu.cn