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SKEW n-DERIVATIONS ON SEMIPRIME RINGS

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Abstract. For a ring R with an automorphism σ, an n-additive mapping $\Delta : R \times R \times \cdots \times R \to R$ is called a skew n-derivation with respect to σ if it is always a σ-derivation of R for each argument. Namely, if $n-1$ of the arguments are fixed, then Δ is a σ-derivation on the remaining argument. In this short note, from Brešar Theorems, we prove that a skew n-derivation ($n \geq 3$) on a semiprime ring R must map into the center of R.

Let R be a ring with an automorphism σ. Recall that an additive mapping $\mu : R \to R$ is called a σ-derivation if $\mu(xy) = \sigma(x)\mu(y) + \mu(x)y$ holds for all $x, y \in R$. An n-additive mapping

$$\Delta : R \times R \times \cdots \times R \to R$$

(i.e., additive in each argument) is called a skew n-derivation with respect to σ in the sense that if $n-1$ of the arguments are fixed, then Δ is a σ-derivation on the remaining argument. Namely, if $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in R$ are fixed, then for all $x_i, y_i \in R$, we have

$$\Delta(a_1, \ldots, x_i + y_i, \ldots, a_n) = \Delta(a_1, \ldots, x_i, \ldots, a_n) + \Delta(a_1, \ldots, y_i, \ldots, a_n)$$

and

$$\Delta(a_1, \ldots, x_i y_i, \ldots, a_n) = \Delta(a_1, \ldots, x_i, \ldots, a_n)y_i + \sigma(x_i)\Delta(a_1, \ldots, y_i, \ldots, a_n).$$

Note that the skew derivation is an ordinary derivation when σ is the identity map 1_R. Naturally a skew n-derivation with respect to the identity map 1_R is called an n-derivation.

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In order to illustrate the results in the literature focused on this area clearly we will introduce some additional concepts related to skew n-derivations although our results will mainly concern skew n-derivations on prime and semiprime rings. A skew n-derivation Δ is called permuting or symmetric if

$$\Delta(x_1, x_2, \ldots, x_n) = \Delta(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$$

holds for all $x_1, x_2, \ldots, x_n \in R$ and $\pi \in S_n$ the symmetric group of degree n. The function $\delta : R \to R$ defined by $\delta(x) = \Delta(x, x, \ldots, x)$ is called the trace of Δ. A skew 2-derivation with respect to the automorphism σ is also called a σ-biderivation. Naturally a 2-derivation is called a biderivation. Generalized n-derivations on rings can be defined similarly (see [10] for the definitions of generalized biderivations).

A ring R is called prime if $aRb \neq 0$ for all $a, b \in R\backslash\{0\}$. A ring R is called semiprime if $aRa \neq 0$ for all $0 \neq a \in R$. For a semiprime ring R, denote its extended centroid by C and its symmetric Martindale ring of quotients by Q_s (see [7] for reference). Note that the extended centroid of a prime ring is a field. Denote the center of R by $Z(R)$. An automorphism σ of a semiprime ring R is called X-inner if there exists an invertible element $p \in Q_s$ such that $\sigma(x) = pxp^{-1}$ holds for all $x \in R$. Otherwise σ is called X-outer. For $a, b \in R$, write the commutator $ab - ba$ of a and b by $[a, b]$. We will always use the commutator formulas $[a, bc] = b[a, c] + [a, b]c$ and $[ab, c] = a[b, c] + [a, c]b$ for $a, b, c \in R$. At last recall that for a ring R with a nonempty subset S a mapping $f : R \to R$ is called centralizing (resp. commuting) on S if $[f(x), x] \in Z(R)$ (resp. $[f(x), x] = 0$) for all $x \in S$.

The notion of a symmetric biderivation had been introduced by Maksa [21] in 1980. In 1989, Vukman [26] initiated the research of biderivations on prime and semiprime rings. He extended classical Posner Theorem [25] to symmetric biderivations in prime and semiprime rings. Thereafter many articles were focused on biderivations of prime and semiprime rings (see [1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 15, 16, 18, 19, 22, 27, 28, 29, 30] for reference). Among these papers, the most important results are due to Brešar [8, 9, 10]. He gave the construction of biderivations on semiprime rings.

Brešar Theorem ([8, Theorem 4.1])**.** *Let R be a semiprime ring, and let $B : R \times R \to R$ be a biderivation. Then there exist an idempotent $\varepsilon \in C$ and an element $\mu \in C$ such that the algebra $(1 - \varepsilon)R$ is commutative and $\varepsilon B(x, y) = \mu\varepsilon[x, y]$ for all $x, y \in R$. Particularly if R is a noncommutative prime ring, then there exists $\lambda \in C$ such that $B(x, y) = \lambda[x, y]$ for all $x, y \in R$.*

In [10] skew biderivations and inner generalized biderivations on prime rings were also characterized. So almost all results appearing in the articles listed above can be implied by the Brešar Theorems [8, 10].

In 2007, Jung and Park [17] considered permuting 3-derivations on prime and semiprime rings and obtained the following results:

Theorem A (Jung and Park, [17, Theorem 2.3])**.** *Let R be a noncommutative 3-torsion free semiprime ring and let I be a nonzero two-sided ideal of R. Suppose that there exists a permuting 3-derivation $\Delta : R \times R \times R \to R$ such that δ is centralizing on I, where δ is the trace of Δ. Then δ is commuting on I.*

Theorem B (Jung and Park, [17, Theorem 2.4])**.** *Let R be a noncommutative 6-torsion free prime ring and let I be a nonzero two-sided ideal of R. Suppose that there exists a nonzero permuting 3-derivation $\Delta : R \times R \times R \to R$ such that δ is centralizing on I, where δ is the trace of Δ. Then R is commutative.*

Park [23] obtained similar results for permuting 4-derivations on prime and semiprime rings. Furthermore in 2009, Park [24] considered permuting n-derivations on prime and semiprime rings.

In [14] Fošner introduced the notion of permuting skew 3-derivations of prime or semiprime rings and proved that under some certain conditions a prime ring with a nonzero permuting skew 3-derivation has to be commutative.

In this short note, from Brešar Theorems ([8, Theorems 3.1 and 4.1]), we prove that an arbitrary skew n-derivation ($n \geq 3$) on a semiprime ring R must map into the center of R. As a corollary, we obtain that an arbitrary skew n-derivation ($n \geq 3$) on a noncommutative prime ring R must be zero. These results can reveal the reason why Theorems A, B and results obtained in the literatures [23, 24] hold.

This short note depends heavily on Brešar Theorems [8, Theorems 3.1 and 4.1]. In view of their proofs, we give a very mild modification of these two theorems in order to apply them better. The proof of [8, Theorem 3.1] implies its following form.

Remark 1 (Brešar, [8, Theorem 3.1]). Let S be a set and R be a semiprime ring. If functions f and g of S into R satisfy that

$$f(s)xg(t) = \xi g(s)xf(t) \text{ for all } s,t \in S,\ x \in R,$$

where $\xi \in C$ is an invertible element, then there exist idempotents $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in C$ and an invertible element $\lambda \in C$ such that $\varepsilon_i\varepsilon_j = 0$, for $i \neq j$, $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1$, and

$$\varepsilon_1 f(s) = \lambda\varepsilon_1 g(s),\ \varepsilon_2 g(s) = 0,\ \varepsilon_3 f(s) = 0,\ (1-\xi)\varepsilon_1 f(s) = 0$$

holds for all $s \in S$.

Proof. It is a small modification of the proof of [8, Theorem 3.1]. Define $\varphi : E \to R$ by

$$\varphi\Big(\varepsilon_1\Big(\sum_{i=1}^{n} x_i f(s_i) y_i\Big) + (1-\varepsilon_1)r\Big) = \xi\varepsilon_1\Big(\sum_{i=1}^{n} x_i g(s_i) y_i\Big) + (1-\varepsilon_1)r,$$

and then add ξ in corresponding formulas of the proof of [8, Theorem 3.1]. At last $(1-\xi)\varepsilon_1 f(S) = 0$ can be deduced from

$$\Big((1-\xi)\varepsilon_1 f(t)\Big)R\Big((1-\xi)\varepsilon_1 f(t)\Big) = 0$$

for all $t \in S$. □

A modification of the proof for [8, Theorem 4.1] will give the following remark.

Remark 2. Let R be a semiprime ring with an automorphism σ, and let $B : R \times R \to R$ be a σ-biderivation. Then there exist idempotents $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in C$ and invertible elements $p \in Q_s$, $\lambda \in C$ such that

- $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1$, $\varepsilon_1\varepsilon_2 = \varepsilon_1\varepsilon_3 = \varepsilon_2\varepsilon_3 = 0$,
- $\varepsilon_1 B(x, y) = \varepsilon_1 p[x, y]$, $\varepsilon_2 B(x, y) = 0$, and $\varepsilon_3[x, y] = 0$ for all $x, y \in R$.

Proof. By [10, Lemma 2.3] we have that for all $x, y, z, u, v \in R$

$$B(x, y)z[u, v] = [\sigma(x), \sigma(y)]\sigma(z)B(u, v). \tag{1}$$

If σ is X-outer, then for fixed x, y, u, v we deduce that

$$B(x, y)z[u, v] = [\sigma(x), \sigma(y)]z_1 B(u, v)$$

holds for all $z, z_1 \in R$ by [20, Theorem 2]. Moreover $B(x, y)z[u, v] = 0$ holds for all $x, y, z, u, v \in R$. So by [7, Theorem 2.3.9 and Lemma 2.3.10] there exists an idempotent $\varepsilon \in C$ such that $\varepsilon B(x, y) = (1 - \varepsilon)[x, y] = 0$ holds for all $x, y \in R$. Setting $p = 1$, $\varepsilon_1 = 0$, $\varepsilon_2 = \varepsilon$, $\varepsilon_3 = 1 - \varepsilon$, we get the conclusion in this case. If σ is X-inner, then there exists an invertible element $p \in Q_s$ such that $\sigma(x) = pxp^{-1}$ for all $x \in R$. Now observing (1) we get that for all $x, y, z, u, v \in R$

$$p^{-1}B(x, y)z[u, v] = [x, y]zp^{-1}B(u, v).$$

Following the proof of [8, Theorem 4.1] we complete the proof. □

Remark 3. In Remark 2 for σ-biderivations $B_1, \ldots, B_t$, the elements $\lambda, \varepsilon_1, \varepsilon_2, \varepsilon_3$ are different for different σ-biderivations in general. However p is the same when σ is X-inner. We can set $p = 1$ when σ is X-outer. So p can be chose such that p is same for different σ-biderivations.

Now we need some lemmas. Lemmas 1 and 2 are used to prove Lemma 3. Lemma 3 is crucial in the proof of Theorem 1. The proofs are elementary computation.

Lemma 1. *Let R be a semiprime ring and $a \in R$. Then $[a, [a, x]] = 0$ holds for all $x \in R$ if and only if a^2, $2a \in Z(R)$.*

Proof. We only deal with the "only if" part because the other part is obvious. For all $x, y \in R$

$$\begin{aligned} 0 = \big[a, [a, xy]\big] &= x\big[a, [a, y]\big] + [a, x][a, y] + [a, x][a, y] + \big[a, [a, x]\big]y \\ &= 2[a, x][a, y]. \end{aligned} \tag{2}$$

Putting $x = yz$ in (2) and applying (2) we have $[2a, y]R[2a, y] = 0$ holds for all $y \in R$. Then $2a \in Z(R)$ since R is semiprime. Moreover for any $x \in R$, we obtain

$$0 = [a, [a, x]] = a^2x + xa^2 - 2axa = a^2x + xa^2 - x(2a^2) = a^2x - xa^2.$$

So $a^2 \in Z(R)$. $\square$

Lemma 2. *Let R be a semiprime ring with extended centroid C and $a, b \in R$. Then $[a,[b,x]] = 0$ holds for all $x \in R$ if and only if there exist idempotents $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in C$ and an invertible element $\lambda \in C$ such that*

- $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1$, $\varepsilon_1\varepsilon_2 = \varepsilon_1\varepsilon_3 = \varepsilon_2\varepsilon_3 = 0$ *and*
- $\varepsilon_1 a - \lambda\varepsilon_1 b,\ \varepsilon_2 a,\ \varepsilon_3 b,\ 2\varepsilon_1 b,\ \varepsilon_1 b^2 \in C$.

Proof. The "if" part can be checked by direct computation. Now we consider the "only if" part. For any $x, y \in R$

$$0 = [a,[b,xy]] = x\big[a,[b,y]\big] + [a,x][b,y] + [b,x][a,y] + \big[a,[b,x]\big]y \\ = [a,x][b,y] + [b,x][a,y]. \tag{3}$$

Putting $x = xz$ in (3) and applying (3) we have that

$$[a,x]z[b,y] + [b,x]z[a,y] = 0 \tag{4}$$

holds for all $x, y, z \in R$. By Remark 1 there exist idempotents $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in C$ and an invertible element $\lambda \in C$ such that

- $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1$, $\varepsilon_1\varepsilon_2 = \varepsilon_1\varepsilon_3 = \varepsilon_2\varepsilon_3 = 0$,
- $\varepsilon_1[a,x] = \lambda\varepsilon_1[b,x]$, $\varepsilon_2[a,x] = 0$ and $\varepsilon_3[b,x] = 0$ for all $x \in R$.

That is $\varepsilon_1 a - \lambda\varepsilon_1 b,\ \varepsilon_2 a,\ \varepsilon_3 b \in C$. Then for all $x \in R$ $[\varepsilon_1 b, [\varepsilon_1 b, x]] = 0$ since λ is invertible. By Lemma 1 we obtain $2\varepsilon_1 b,\ \varepsilon_1 b^2 \in C$. $\square$

Lemma 3. *Let R be a semiprime ring with extended centroid C and $a, b \in R$. Then $[[a,x],[b,x]] = 0$ holds for all $x \in R$ if and only if there exist an idempotent $\varepsilon \in C$ and an element $\zeta \in C$ such that $\varepsilon a - \zeta\varepsilon b, (1-\varepsilon)b \in C$.*

Proof. The "if" part is obvious. Now we deal with the "only if" part. Firstly, we will prove $[a,b] = 0$. For any $x, y \in R$, we get $\big[[a,x+y],[b,x+y]\big] = 0$. Then for any $x, y \in R$

$$\big[[a,x],[b,y]\big] + \big[[a,y],[b,x]\big] = 0. \tag{5}$$

Put $x = xb$ in (5). Then for any $x, y \in R$

$$\big[x[a,b] + [a,x]b, [b,y]\big] + \big[[a,y],[b,x]b\big] = 0.$$

That is for any $x, y \in R$

$$x\big[[a,b],[b,y]\big] + \big[x,[b,y]\big][a,b] + [a,x]\big[b,[b,y]\big] + \big[[a,x],[b,y]\big]b \\ + [b,x]\big[[a,y],b\big] + \big[[a,y],[b,x]\big]b = 0.$$

Then by (5) for any $x, y \in R$

$$x\big[[a,b],[b,y]\big] + \big[x,[b,y]\big][a,b] + [a,x]\big[b,[b,y]\big] + [b,x]\big[[a,y],b\big] = 0. \tag{6}$$

Put $y = b$ in (6). Then for any $x \in R$

$$[b,x]\big[[a,b],b\big] = 0. \tag{7}$$

Putting $x = xy$ in (7) and applying (7) we have that

$$[b, x]y\big[[a, b], b\big] = 0 \tag{8}$$

holds for all $x, y \in R$. Putting $x = -[a, b]$ in (8), we get that $\big[[a, b], b\big]y\big[[a, b], b\big] = 0$ holds for all $y \in R$. Then $\big[[a, b], b\big] = 0$ since R is a semiprime ring. Putting $y = a$ into (6) and applying $\big[[a, b], b\big] = 0$ we obtain that

$$\big[x, [b, a]\big][a, b] = 0 \tag{9}$$

holds for all $x \in R$. Putting $x = xy$ into (9) and applying (9) we get that $\big[x, [b, a]\big]y[a, b] = 0$ holds for all $x, y \in R$. Particularly $\big[x, [b, a]\big]y\big[x, [b, a]\big] = 0$ for any $x, y \in R$. Then $[a, b] \in Z(R)$ since R is semiprime. By $\big[[a, ab], [b, ab]\big] = 0$ and $[a, b] \in Z(R)$ we find $-[a, b]^3 = 0$. Then $[a, b] = 0$ since $[a, b] \in Z(R)$ and R is semiprime.

Thus, by (6) then for any $x, y \in R$

$$[a, x]\big[b, [b, y]\big] + [b, x]\big[[a, y], b\big] = 0. \tag{10}$$

Putting $x = xz$ in (10) and applying (10) we obtain that

$$[a, x]z\big[b, [b, y]\big] + [b, x]z\big[[a, y], b\big] = 0 \tag{11}$$

holds for all $x, y, z \in R$. Putting $x = [b, x]$ in (11) and applying $\big[[a, y], b\big] = -\big[a, [b, y]\big]$ (because of $[a, b] = 0$), we have that

$$\big[a, [b, x]\big]z\big[b, [b, y]\big] = \big[b, [b, x]\big]z\big[a, [b, y]\big]$$

holds for all $x, y, z \in R$. Then by Brešar Theorem [8, Theorem 3.1] there exist idempotents $\omega_1, \omega_2, \omega_3 \in C$ and an invertible element $\xi \in C$ such that

- $\omega_1 + \omega_2 + \omega_3 = 1$, $\omega_1\omega_2 = \omega_1\omega_3 = \omega_2\omega_3 = 0$,
- $\omega_1[a, [b, x]] = \xi\omega_1[b, [b, x]]$, $\omega_2[a, [b, x]] = 0$ and $\omega_3[b, [b, x]] = 0$ for all $x \in R$.

Putting $x = -[a, y]$ in (11) and then multiplying it by ω_3, we get that

$$\omega_3\big[[a, y], b\big]z\omega_3\big[[a, y], b\big] = 0$$

holds for all $y, z \in R$. So $\omega_3\big[a, [b, y]\big] = -\omega_3\big[[a, y], b\big] = 0$ since R is semiprime. Hence

$$\big[\omega_1 a - \xi\omega_1 b, [b, x]\big] = 0 \text{ and } \big[(1 - \omega_1)a, [b, x]\big] = 0$$

hold for all $x \in R$. Thus $\big[a - \xi\omega_1 b, [b, x]\big] = 0$ holds for all $x \in R$. Then by Lemma 2 there exist idempotents $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in C$ and an invertible element $\lambda \in C$ such that

- $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1$, $\varepsilon_1\varepsilon_2 = \varepsilon_1\varepsilon_3 = \varepsilon_2\varepsilon_3 = 0$ and
- $\varepsilon_1(a - \xi\omega_1 b) - \lambda\varepsilon_1 b = c_1,\ \varepsilon_2(a - \xi\omega_1 b) = c_2,\ \varepsilon_3 b,\ 2\varepsilon_1 b,\ \varepsilon_1 b^2 \in C.$

Then

$$(\varepsilon_1 + \varepsilon_2)a = (\lambda\varepsilon_1 + \xi\omega_1(\varepsilon_1 + \varepsilon_2))(\varepsilon_1 + \varepsilon_2)b + c_1 + c_2.$$

Setting $\varepsilon = \varepsilon_1 + \varepsilon_2$ and $\zeta = \lambda\varepsilon_1 + \xi\omega_1(\varepsilon_1 + \varepsilon_2)$, we complete the proof. □

Theorem 1. *A skew n-derivation $(n \geq 3)$ on a semiprime ring R must map into the center of R.*

Proof. Let Δ be a skew n-derivation on R with respect to the automorphism σ. Then for fixed $a_1, \dots, a_n \in R$ we obtain $\Delta(a_1, \dots, a_n) = \Delta_1(a_1, a_2, a_3)$ where $\Delta_1(x, y, z) = \Delta(x, y, z, a_4, \dots, a_n)$ is a skew 3-derivation with respect to σ. So it is sufficient to prove that every skew 3-derivation on R must map into the center of R. Let $\Delta : R \times R \times R \to R$ be a skew 3-derivation with respect to σ. For fixed $x_0, y_0, z_0 \in R$, we proceed to prove that $\Delta(x_0, y_0, z_0) \in Z(R)$. Obviously

$$\Delta(x_0, y, z) = \varphi_{x_0}(y, z),\ \Delta(x, y_0, z) = \varphi_{y_0}(x, z) \text{ and } \Delta(x, y, z_0) = \varphi_{z_0}(x, y)$$

are all σ-biderivations on R. Then by Remark 2 for every $t \in \{x_0, y_0, z_0\}$ there exist idempotents $\varepsilon_t, \varepsilon'_t, \varepsilon''_t \in C$ and invertible elements $p \in Q_s$, $\lambda_t \in C$ such that

- $\varepsilon_t + \varepsilon'_t + \varepsilon''_t = 1$, $\varepsilon_t\varepsilon'_t = \varepsilon_t\varepsilon''_t = \varepsilon'_t\varepsilon''_t = 0$,
- $\varepsilon_t\varphi_t(r, s) = \lambda_t\varepsilon_t p[r, s]$, $\varepsilon'_t\varphi_t(r, s) = 0$ and $\varepsilon''_t[r, s] = 0$ for all $r, s \in R$.

So for all $z \in R$, we obtain

$$\varepsilon_{x_0}\Delta(x_0, y_0, z) = \lambda_{x_0}\varepsilon_{x_0}p[y_0, z] \text{ and } \varepsilon_{y_0}\Delta(x_0, y_0, z) = \lambda_{y_0}\varepsilon_{y_0}p[x_0, z].$$

Then for all $z \in R$, we have

$$\lambda_{x_0}\varepsilon_{x_0}\varepsilon_{y_0}[y_0, z] = \varepsilon_{x_0}\varepsilon_{y_0}p^{-1}\Delta(x_0, y_0, z) = \lambda_{y_0}\varepsilon_{x_0}\varepsilon_{y_0}[x_0, z].$$

Hence for all $z \in R$, we get $\lambda_{x_0}\varepsilon_{x_0}\varepsilon_{y_0}\big[[y_0, z], [x_0, z]\big] = 0$. Then by Lemma 3 we obtain $\lambda_{x_0}\varepsilon_{x_0}\varepsilon_{y_0}[x_0, y_0] = 0$. Thus $\varepsilon_{x_0}\varepsilon_{y_0}[x_0, y_0] = 0$ since λ_{x_0} is invertible. Then

$$\varepsilon_{x_0}\varepsilon_{y_0}\varepsilon_{z_0}\Delta(x_0, y_0, z_0) = \varepsilon_{x_0}\varepsilon_{y_0}(\lambda_{z_0}\varepsilon_{z_0}p[x_0, y_0]) = \lambda_{z_0}\varepsilon_{z_0}p(\varepsilon_{x_0}\varepsilon_{y_0})[x_0, y_0] = 0.$$

Set

$$\begin{cases} \varepsilon_1 &= \varepsilon_{x_0}\varepsilon_{y_0}(1 - \varepsilon''_{z_0}) + \varepsilon'_{x_0}(1 - \varepsilon'_{y_0}) + \varepsilon'_{y_0}, \\ \varepsilon_2 &= \varepsilon_{x_0}(\varepsilon_{y_0}\varepsilon''_{z_0} + \varepsilon''_{y_0}) + \varepsilon''_{x_0}(1 - \varepsilon'_{y_0}). \end{cases}$$

It can be verified from direct computation that $\varepsilon_1, \varepsilon_2 \in C$ are idempotents such that

- $\varepsilon_1 + \varepsilon_2 = 1$,
- $\varepsilon_1\Delta(x_0, y_0, z_0) = 0$ and $\varepsilon_2[x, y] = 0$ for all $x, y \in R$.

So for all $w \in R$, we have

$$[\Delta(x_0, y_0, z_0), w] = \varepsilon_1[\Delta(x_0, y_0, z_0), w] + \varepsilon_2[\Delta(x_0, y_0, z_0), w] = 0.$$

Then $\Delta(x_0, y_0, z_0) \in Z(R)$ completes the proof. □

By Theorem 1 and [10, Theorem 3.2] we get the following result for prime rings.

Theorem 2. *A prime ring with a nonzero skew n-derivation $(n \geq 3)$ must be commutative.*

Proof. Let Δ be a nonzero skew n-derivation ($n \geq 3$) on a noncommutative prime ring R with respect to an automorphism σ. Then there exist $a_3, \ldots, a_n \in R$ such that $\Delta_1(x, y) = \Delta(x, y, a_3, \ldots, a_n)$ is a nonzero σ-biderivation on R. Then by Theorem 1 and [10, Theorem 3.2] there exists an invertible element $p \in Q_s$ such that $[p[x, y], z] = 0$ holds for all $x, y, z \in R$. Particularly for all $x, y, z \in R$

$$0 = [p[x, yx], z] = [p[x, y]x, z] = p[x, y][x, z].$$

Moreover for all $x, y, z \in R$ we have $[x, y]R[x, z] = 0$ since p is invertible. So R is commutative since R is prime. □

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