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ON THE WEAK ARTINIANNESS AND MINIMAX GENERALIZED LOCAL COHOMOLOGY MODULES

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Abstract. Let R be a commutative Noetherian ring, I an ideal of R, M and N two R-modules. We characterize the least integer i such that $H_I^i(M,N)$ is not weakly Artinian by using the notion of weakly filter regular sequences. Also, a local-global principle for minimax generalized local cohomology modules is shown and the result generalizes the corresponding result for local cohomology modules.

1. Introduction

Throughout this paper, let R be a commutative Noetherian ring, I a proper ideal of R, M and N two R-modules. The generalized local cohomology module

$$H_I^i(M,N) = \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}_R^i(M/I^nM, N)$$

was introduced by Herzog in [7]. It is clear that $H_I^i(R,N)$ is just the ordinary local cohomology module $H_I^i(N)$. For the details about local cohomology, we refer the reader to the book [3].

In [6, Definition 3.1], Hajikarimi gave the definition of weakly filter regular sequences in the case that R is a local ring. In Section 2, we generalize it into the general unnecessary local case. If M is a finitely generated R-module and N is a weakly Laskerian R-module, we prove that inf $\{i \mid H_I^i(M,N)$ is not weakly Artinian$\}$ is the length of any maximal weakly filter N-regular sequences in $I + \mathrm{Ann}M$.

In [2, Theorem 1.2] it is shown that, for a nonnegative integer t, if M and N are finitely generated R-modules such that $H_I^i(M,N)$ is finitely generated

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for all $i < t$, then $\operatorname{Hom}(R/I, H_I^t(M,N))$ is finitely generated. So it is natural to ask the following question.

Question. If M is a finitely generated R-module and N is a weakly Laskerian R-module, $H_I^i(M,N)$ is weakly Artinian for all $i < t$ (where t is a nonnegative integer). Is the module $\operatorname{Hom}(R/I, H_I^t(M,N))$ weakly Artinian?

We give a negative answer to this question (see Proposition 2.8).

Recall that an R-module M is called minimax if there is a finite submodule L such that M/L is Artinian (see [12]). The class of minimax modules includes all finite and all Artinian modules. Moreover, it is closed under taking submodules, quotients and extensions, i.e., it is a Serre subcategory of the category of R-modules. In [11, Theorem 1], Tehranian proved the local-global principle for the finiteness of generalized local cohomology modules which stated that, for a nonnegative integer t, if M and N are finitely generated R-modules, then $H_I^i(M,N)$ is finitely generated for all $i < t$ if and only if $H_{IR_p}^i(M_p, N_p)$ is finitely generated for all $i < t$ and all prime ideals p.

In Section 3, we obtain a similar principle for minimax generalized local cohomology modules, that is, for a nonnegative integer t, if M and N are finitely generated, then $H_I^i(M,N)$ is minimax for all $i < t$ if and only if $H_{IR_p}^i(M_p, N_p)$ is minimax for all $i < t$ and all prime ideals p, which generalizes the corresponding result in [1].

2. Weakly Artinian modules

Definition 2.1 ([6, Definition 2.1])**.** An R-module M is said to be weakly Artinian if $E_R(M)$, its injective envelope, can be written as

$$E_R(M) := \oplus_{i=1}^n \mu^0(\mathfrak{m}_i, M) E_R(R/\mathfrak{m}_i),$$

where $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ are maximal ideals of R.

Note that any Artinian module is weakly Artinian. The class of weakly Artinian modules is a Serre subcategory.

Definition 2.2. Let M be an R-module and $a_1, \ldots, a_n$ a sequence of elements of R. If the R-module $0 :_{M/(a_1,\ldots,a_{i-1})M} a_i$ is weakly Artinian for each $i = 1, \ldots, n$, then we say that $a_1, \ldots, a_n$ is a weakly filter M-regular sequence.

In [5, Definition 2.1], an R-module M is defined to be weakly Laskerian if the set of associated prime ideals of any quotient of M is finite. All finite and all Artinian modules are weakly Laskerian.

Next we will state the first main result of this paper.

Theorem 2.3. *Let M be a weakly Laskerian R-module and $t > 0$ an integer. Then the following are equivalent*:

(1) *I contains a weakly filter M-regular sequence of length t*;

(2) *$H_I^i(M)$ is weakly Artinian for all $i < t$.*

Proof. (1) $\Rightarrow$ (2). We use induction on t. When $t = 1$, there is $a \in I$ which is weakly filter M-regular, then $0 :_M a$ is weakly Artinian, so is $0 :_{H_I^0(M)} a$. It follows from [6, Lemma 2.8] that $H_I^0(M)$ is weakly Artinian. Now assume that $t > 1$ and the result is true for $t-1$. Suppose that there exist $a_1, \ldots, a_t \in I$ which form a weakly filter M-regular sequence. By the inductive hypothesis, $H_I^i(M)$ is weakly Artinian for all $i < t-1$. Now set $\overline{M} = M/a_1M$. As $a_2, \ldots, a_t$ is a weakly filter $\overline{M}$-regular sequence, $H_I^i(\overline{M})$ is weakly Artinian for all $i < t-1$ by the inductive hypothesis. From the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 :_M a_1 & \longrightarrow & M & \xrightarrow{a_1} & a_1M & \longrightarrow & 0 \\
 & & & & \swarrow a_1 \Big\downarrow a_1 & & & & \\
0 & \longrightarrow & a_1M & \longrightarrow & M & \longrightarrow & \overline{M} & \longrightarrow & 0
\end{array}$$

we get a commutative diagram with long exact rows,

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H_I^{i-1}(a_1M) & \longrightarrow & H_I^i(0 :_M a_1) & \longrightarrow & H_I^i(M) & \xrightarrow{a_1^{(i)}} & H_I^i(a_1M) & \longrightarrow & \cdots \\
 & & & & & & \swarrow a_1^{(i)} \Big\downarrow a_1 & & & & \\
\cdots & \longrightarrow & H_I^{i-1}(\overline{M}) & \longrightarrow & H_I^i(a_1M) & \longrightarrow & H_I^i(M) & \longrightarrow & H_I^i(\overline{M}) & \longrightarrow & \cdots
\end{array}$$

Since $0 :_M a_1$ is weakly Artinian, we get $H_I^i(0 :_M a_1) = 0$ for all $i > 0$. Thus, under the isomorphism $a_1^{(t-1)}$, we see that $0 :_{H_I^{t-1}(M)} a_1$ is a homomorphic image of $H_I^{t-2}(\overline{M})$ from the above diagram. Then $0 :_{H_I^{t-1}(M)} a_1$ is weakly Artinian. It follows from [6, Lemma 2.8] that $H_I^{t-1}(M)$ is weakly Artinian.

(2) $\Rightarrow$ (1). We use induction on t. When $t = 1$, $H_I^0(M)$ is weakly Artinian, so is $0 :_M I$. If one has $I \subseteq p$ for some $p \in \mathrm{Ass}M \backslash \mathrm{Max}R$, then $p \in \mathrm{Ass}(\mathrm{Hom}(R/I, M))$, which is a contradiction. So, there is $a_1 \in I$ such that $\mathrm{Hom}(R/(a_1), M) \cong 0 :_M a_1$ is weakly Artinian, that is, $a_1 \in I$ is weakly filter M-regular.

By the short exact sequence $0 \to 0 :_M a_1 \to M \to M/0 :_M a_1 \to 0$, we have the long exact sequence

$$\cdots \to H_I^i(0 :_M a_1) \to H_I^i(M) \to H_I^i(M/0 :_M a_1) \to H_I^{i+1}(0 :_M a_1) \to \cdots .$$

We see that $H_I^i(M/0 :_M a_1)$ is weakly Artinian for all $i < t$. Consider the exact sequence of local cohomology modules

$$\cdots \to H_I^i(M) \to H_I^i(M/a_1M) \to H_I^{i+1}(M/0 :_M a_1) \to \cdots$$

induced by the short exact sequence $0 \to M/0 :_M a_1 \xrightarrow{a_1} M \to M/a_1M \to 0$, we get $H_I^i(M/a_1M)$ is weakly Artinian for all $i < t-1$. By the inductive hypothesis, there exist $a_2, \ldots, a_t \in I$ which form a weakly filter M/a_1M-regular sequence. Thus, $a_1, \ldots, a_t$ is a weakly filter M-regular sequence. $\square$

Remark 2.4. In virtue of Theorem 2.3, any two maximal weakly filter M-regular sequences in I (if any exist) have the same length. Then we can give the following definition.

Definition 2.5. The weakly filter depth of I on M is defined as the length of any maximal weakly filter M-regular sequence in I, denoted by $w - f - \operatorname{depth}(I, M)$. Here, when the maximal weakly filter M-regular sequence in I does not exist, we understand that the length is ∞.

Next result extends [6, Corollary 3.6] from the local case to the general unnecessary local case, and it extends [4, Theorem 2.2] from the finitely generated case to the weakly Laskerian case, respectively.

Proposition 2.6. *Let R be a commutative Noetherian ring, M a finitely generated R-module and N a weakly Laskerian R-module. Then*

$$w - f - depth(I + AnnM, N) = \inf\{i \mid H_I^i(M, N) \text{ is not weakly Artinian}\},$$

where we understand the infimum of empty set is ∞.

Proof.

$$\begin{aligned} w - f - \operatorname{depth}(I + \operatorname{Ann}M, N) &= \inf\{i \mid H^i_{I+\operatorname{Ann}M}(N) \text{ is not weakly Artinian}\} \\ &= \inf\{i \mid H_I^i(M, N) \text{ is not weakly Artinian}\}. \end{aligned}$$

The second equality follows from [6, Theorem 2.9]. □

In [9, Corollary 2.3], it is shown that, for an integer t, if M and N are finitely generated and $H_I^i(N)$ is Artinian for all $i < t$, then $H_I^i(M, N)$ is also Artinian for $i < t$. Next we give the following corollary.

Corollary 2.7. *Let $t > 0$ be an integer, N a weakly Laskerian R-module and $H_I^i(N)$ is weakly Artinian for all $i < t$. Then for any finitely generated R-module M, $H_I^i(M, N)$ is also weakly Artinian for $i < t$.*

Proof. For any finitely generated R-module M, we have that

$$\begin{aligned} t \le w - f - \operatorname{depth}(I, N) &\le w - f - \operatorname{depth}(I + \operatorname{Ann}M, N) \\ &= \inf\{i \mid H_I^i(M, N) \text{ is not weakly Artinian}\} \end{aligned}$$

by Proposition 2.6. The result is clear. □

Next we will give a negative answer to the question mentioned in Section 1.

Proposition 2.8. *Let M be a finitely generated R-module, N a weakly Laskerian R-module. Then* $\operatorname{Hom}(R/I, H_I^t(M, N))$, *where $t = w - f - depth(I + AnnM, N)$, is not weakly Artinian but is weakly Laskerian.*

Proof. By Proposition 2.6, $H_I^t(M, N))$ is not weakly Artinian, so

$$\operatorname{Hom}(R/I, H_I^t(M, N))$$

is not weakly Artinian by [6, Lemma 2.8]. But $\operatorname{Hom}(R/I, H_I^t(M, N))$ is weakly Laskerian by [6, Theorem 2.4]. □

Proposition 2.9. *Let $(R, \mathfrak{m})$ be a local ring, M a weakly Laskerian R-module and $I \subseteq J$ be proper ideals of R. Then, for all $i < w - f - depth(I, M)$, we have $H_I^i(M) \cong H_J^i(M)$. In particular, for all $i < w - f - depth(I, M)$, we have $H_I^i(M) \cong H_{\mathfrak{m}}^i(M)$.*

Proof. We may assume that $I \neq J$, and so there exists an element $x \in J \backslash I$. Now by [10, Corollary 3.5], we get a short exact sequence

$$0 \to H_{Rx}^1(H_I^{i-1}(M)) \to H_{I+Rx}^i(M) \to H_{Rx}^0(H_I^i(M)) \to 0$$

for all $i \geq 0$. If $i < w - f - \mathrm{depth}(I, M)$, then $H_I^i(M)$ and $H_I^{i-1}(M)$ are weakly Artinian. Therefore, $H_{Rx}^1(H_I^{i-1}(M)) = 0$ and $H_{Rx}^0(H_I^i(M)) \cong H_I^i(M)$. It follows that $H_I^i(M) \cong H_{I+Rx}^i(M)$. Assume that $J = I + (x_1, \ldots, x_r)$, we can get the result by applying the above argument for finite steps. □

3. Minimax generalized local cohomology modules

In this section, we show the principle for minimax generalized local cohomology modules. Before this, we recall a known result.

Proposition 3.1 ([1, Proposition 2.2])**.** *Let M be an R-module. Then the following statements are equivalent*:

(1) *M is a minimax R-module*;

(2) *$M_{\mathfrak{m}}$ is a minimax $R_{\mathfrak{m}}$-module for all $\mathfrak{m} \in \mathrm{Max}R$ and M is a weakly Laskerian R-module.*

Next theorem generalizes [1, Theorem 2.8], which is our another main result of this article.

Theorem 3.2. *Let M and N be two finitely generated R-modules, and that $t > 0$ an integer. Then the following statements are equivalent*:

(1) *$H_I^i(M, N)$ is minimax for all $i < t$;*

(2) *$H_I^i(M, N)$ is I-cofinite and minimax for all $i < t$;*

(3) *$H_I^i(M, N)_{\mathfrak{m}}$ is a minimax $R_{\mathfrak{m}}$-module for all $i < t$ and all $\mathfrak{m} \in \mathrm{Max}R$;*

(4) *$H_I^i(M, N)_p$ is a minimax R_p-module for all $i < t$ and all prime ideals p of R.*

Proof. (1) ⇒ (2). We use induction on t. For $t = 1$, the assertion follows from the fact $H_I^0(M, N) = H_I^0(\mathrm{Hom}(M, N))$. Let $t > 1$ and suppose the case $t - 1$ is settled. Now we will prove that $H_I^{t-1}(M, N)$ is I-cofinite. We see that $\mathrm{Supp}(H_I^{t-1}(M, N)) \subseteq V(I)$. In virtue of [9, Theorem 2.4], we get $\mathrm{Hom}(R/I, H_I^{t-1}(M, N))$ is finitely generated. Thus $H_I^{t-1}(M, N)$ is I-cofinite by [8, Proposition 4.3].

(2) ⇒ (3) is clear by Proposition 3.1.

(3) ⇒ (2). By the short exact sequence

$$0 \to H_I^0(N) \to N \to N/H_I^0(N) \to 0,$$

we get the long exact sequence

$$\cdots \to H_I^i(M, H_I^0(N)) \to H_I^i(M,N) \to H_I^i(M, N/H_I^0(N)) \to H_I^{i+1}(M, H_I^0(N)) \to \cdots$$

for all $i \geq 0$. We can see that $H_I^i(M, N/H_I^0(N))$ is I-cofinite and minimax if and only if $H_I^i(M,N)$ is I-cofinite and minimax for all $i \geq 0$. Also, we get the exact sequence

$$H_I^i(M,N)_{\mathfrak{m}} \to H_I^i(M, N/H_I^0(N))_{\mathfrak{m}} \to H_I^{i+1}(M, H_I^0(N))_{\mathfrak{m}}$$

for all $i \geq 0$ and all $\mathfrak{m} \in \mathrm{Max}R$. We deduce that $H_I^i(M, N/H_I^0(N))_{\mathfrak{m}}$ is minimax for all $i < t$ and all $\mathfrak{m} \in \mathrm{Max}R$. So we can assume that $H_I^0(N) = 0$. Thus there exists $x \in I$ which is regular on N. From the short exact sequence $0 \to N \overset{x}{\to} N \to N/xN \to 0$, we get the exact sequences

$$H_I^{i-1}(M, N/xN) \to H_I^i(M,N) \overset{x}{\to} H_I^i(M,N)$$

and

$$H_I^i(M,N)_{\mathfrak{m}} \to H_I^i(M,N/xN)_{\mathfrak{m}} \to H_I^{i+1}(M,N)_{\mathfrak{m}}$$

for all i and all $\mathfrak{m} \in \mathrm{Max}R$. Now $H_I^i(M,N/xN)_{\mathfrak{m}}$ is minimax for all $i < t-1$ and all $\mathfrak{m} \in \mathrm{Max}R$. By the inductive hypothesis, we have $H_I^i(M,N/xN)$ is I-cofinite and minimax for all $i < t-1$. So $0 :_{H_I^i(M,N)} x$ is I-cofinite and minimax for all $i < t$. The result follows from [1, Theorem 2.6].

(1) $\Rightarrow$ (4) and (4) $\Rightarrow$ (3) are clear. □

Corollary 3.3. *Let M and N be two finitely generated R-modules and let $t > 0$ be an integer such that $H_I^i(M,N)$ is minimax for all $i < t$. Then $H_I^i(M,N)$ is I-cofinite for all $i < t$.*

The following corollary immediately follows by Corollary 3.3.

Corollary 3.4. *Let M and N be two finitely generated R-modules. Then* $\inf\{i \mid H_I^i(M,N)$ *is not minimax*$\} \leq \inf\{i \mid H_I^i(M,N)$ *is not I-cofinite*$\}$.

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