Bull. Korean Math. Soc. **50** (2013), No. 6, pp. 1841–1846
http://dx.doi.org/10.4134/BKMS.2013.50.6.1841

MORE ON SUMS OF HILBERT SPACE FRAMES

A. Najati, M. R. Abdollahpour, E. Osgooei, and M. M. Saem

Abstract. In this paper we establish some new results on sums of Hilbert space frames and Riesz bases. We also provide a correction to some recently established results in [2].

1. Introduction

Throughout this paper, $\mathcal{H}$ denotes a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. Recall that a sequence $\{f_i\}_{i\in I} \subseteq \mathcal{H}$ is a frame for $\mathcal{H}$ if there exist $0 < A \leqslant B < \infty$ such that

$$A\|f\|^2 \leqslant \sum_{i\in I} |\langle f, f_i\rangle|^2 \leqslant B\|f\|^2 \tag{1.1}$$

for all $f \in \mathcal{H}$. The constants A and B are called a lower and upper frame bounds, respectively.

We call a sequence $\{f_i\}_{i\in I} \subseteq \mathcal{H}$ a Bessel sequence for $\mathcal{H}$, if the right hand inequality in (1.1) holds for all $f \in \mathcal{H}$.

Let $\{f_i\}_{i\in I}$ be a Bessel sequence for $\mathcal{H}$. Then the bounded operator

$$T : \mathcal{H} \to l_2, \quad Tf = \{\langle f, f_i\rangle\}_{i\in I}$$

is called the *analysis operator* of $\{f_i\}_{i\in I}$ and its adjoint

$$T^* : l_2 \to \mathcal{H}, \quad T^*(\{c_i\}_{i\in I}) = \sum_{i\in I} c_i f_i,$$

is called the *synthesis operator* of $\{f_i\}_{i\in I}$. If $\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$, the *frame operator* for $\{f_i\}_{i\in I}$ is the operator $S : \mathcal{H} \to \mathcal{H}$ given by $Sf = \sum_{i\in I}\langle f, f_i\rangle f_i$. It is clear that $\langle Sf, f\rangle = \sum_{i\in I} |\langle f, f_i\rangle|^2$ for all $f \in \mathcal{H}$. Therefore, S is positive and invertible. This provides the frame decomposition

$$f = \sum_{i\in I}\langle f, S^{-1}f_i\rangle f_i = \sum_{i\in I}\langle f, f_i\rangle S^{-1}f_i$$

for all $f \in \mathcal{H}$.

Received February 18, 2012; Revised July 29, 2013.
2010 *Mathematics Subject Classification.* Primary 41A58, 42C15.
Key words and phrases. frame, Gabor frame, frame operator.

A sequence $\{f_i\}_{i\in I} \subseteq \mathcal{H}$ is called a Riesz basis for $\mathcal{H}$, if $\overline{\text{span}}\{f_i\}_{i\in I} = \mathcal{H}$ and there exist $0 < A \leqslant B < \infty$ such that

$$A \sum_i |c_i|^2 \leqslant \| \sum_i c_i f_i \|^2 \leqslant B \sum_i |c_i|^2 \tag{1.2}$$

holds for every finite scalar sequence $\{c_i\}$. The constants A and B are called the lower and upper Riesz basis bounds, respectively.

We will use the following lemma in the rest of paper.

Lemma 1.1 ([1], Lemma A.7.1). *If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and $T : \mathcal{H} \to \mathcal{K}$ is a bounded operator with closed range, then there exists a bounded operator $T^\dagger : \mathcal{K} \to \mathcal{H}$ such that*

$$TT^\dagger Tf = Tf, \quad f \in \mathcal{H}.$$

The operator $T^\dagger$ is called a pseudo-inverse of T.

2. Main results

The following assertion is stated in [2] as Proposition 2.1.

Assertion 2.1. *Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ with the frame operator S, frame bounds $A \leqslant B$ and let $L : \mathcal{H} \longrightarrow \mathcal{H}$ be a bounded operator. Then $\{Lf_i\}_{i\in I}$ is a frame for $\mathcal{H}$ if and only if L is invertible on $\mathcal{H}$. Moreover, in this case the frame operator for $\{Lf_i\}_{i\in I}$ is LSL^* and the new frame bounds are $A\|L^{-1}\|^{-2}, B\|L\|^2$.*

In this note, we show that Assertion 2.1 is not true in general. Indeed, if $\{f_i\}_{i\in I}$ is a frame for Hilbert space $\mathcal{H}$ and $L : \mathcal{H} \longrightarrow \mathcal{H}$ is a bounded invertible operator, then $\{Lf_i\}_{i\in I}$ is a frame for $\mathcal{H}$ but the converse is not true in general. In the proof of Proposition 2.1 of [2], the authors proved that LSL^* is invertible. But, this does not imply that L is invertible on $\mathcal{H}$. It should be noted that in [2], Proposition 2.1 has been used in Corollaries 2.2, 2.3 and in the proof of Proposition 4.1.

Example 2.2. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis for a Hilbert space $\mathcal{H}$. Define a shift operator L on $\mathcal{H}$ by $L(e_n) = e_{n-1}$ if $n > 1$ and $L(e_1) = 0$. It is clear that $\{L(e_n)\}_{n=1}^\infty$ is a frame for $\mathcal{H}$, but L is not invertible although $LL^* = I$. Moreover, $\{L^*(e_n)\}_{n=1}^\infty$ is not a frame for $\mathcal{H}$.

We can correct Assertion 2.1 as follows:

Proposition 2.3. *Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ with the frame operator S, frame bounds $A \leqslant B$ and let $L : \mathcal{H} \longrightarrow \mathcal{H}$ be a bounded operator. Then $\{Lf_i\}_{i\in I}$ is a frame for $\mathcal{H}$ if and only if L is surjective. Moreover, in this case the frame operator for $\{Lf_i\}_{i\in I}$ is LSL^* and the new frame bounds are $A\|L^\dagger\|^{-2}$ and $B\|L\|^2$, where $L^\dagger$ is the pseudo-inverse of L.*

Proof. If $\{Lf_i\}_{i\in I}$ is a frame for $\mathcal{H}$, then its frame operator LSL^* is invertible. So L is surjective. The converse follows from Corollary 5.3.2 of [1]. □

We also have:

Proposition 2.4. *Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ with the frame operator S and let $L : \mathcal{H} \longrightarrow \mathcal{H}$ be a bounded operator. Then $\{Lf_i\}_{i\in I}$ and $\{L^*f_i\}_{i\in I}$ are frames for $\mathcal{H}$ if and only if L is invertible. Moreover, in this case the frame operators for $\{Lf_i\}_{i\in I}$ and $\{L^*f_i\}_{i\in I}$ are LSL^* and L^*SL, respectively.*

Proof. If $\{Lf_i\}_{i\in I}$ and $\{L^*f_i\}_{i\in I}$ are frames for $\mathcal{H}$, then their frame operators LSL^* and L^*SL are invertible. So L and L^* are surjective and L is invertible. The converse is clear. □

In [2], Corollary 2.2 can be corrected as below.

Corollary 2.5. *Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ with the frame operator S, frame bounds $A \leqslant B$ and let $L : \mathcal{H} \longrightarrow \mathcal{H}$ be a bounded operator. Then $\{f_i + Lf_i\}_{i\in I}$ is a frame for $\mathcal{H}$ if and only if $I + L$ is surjective. Moreover, in this case the frame operator for the new frame is $(I + L)S(I + L^*)$ with the frame bounds $A\|(I+L)^\dagger\|^{-2}$ and $B\|I+L\|^2$, where $(I+L)^\dagger$ is a pseudo-inverse of $I+L$. In particular, if L is a positive operator (or just $L > -I$), then $\{f_i + Lf_i\}_{i\in I}$ is a frame for $\mathcal{H}$ with the frame operator $S + SL + SL^* + LSL^*$.*

Corollary 2.6. *Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ and $P : \mathcal{H} \longrightarrow \mathcal{H}$ be a bounded operator. If $P^2 = P$, then for all $a \neq -1$, $\{f_i + aPf_i\}_{i\in I}$ is a frame for $\mathcal{H}$.*

Proof. If $a \neq -1$, then we have $(I + aP)(I - \frac{a}{a+1}P) = I$. This implies that $I + aP$ is invertible and so $\{f_i + aPf_i\}_{i\in I}$ is a frame for $\mathcal{H}$. □

Proposition 2.7. *Let $\{f_i\}_{i\in I}$ be a sequence in $\mathcal{H}$ such that $\sum_{i\in I}\langle f, f_i\rangle f_i$ converges for all $f \in \mathcal{H}$. If $L : \mathcal{H} \longrightarrow \mathcal{H}$ is a bounded operator such that $\{Lf_i\}_{i\in I}$ and $\{L^*f_i\}_{i\in I}$ are frames for $\mathcal{H}$, then $\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$.*

Proof. Let us define

$$U : \mathcal{H} \longrightarrow \mathcal{H}, \quad U(f) := \sum_{i\in I}\langle f, f_i\rangle f_i.$$

Let S_L be the frame operator for $\{Lf_i\}_{i\in I}$. Then $S_L = LUL^*$ is invertible. So L is surjective. Similarly, we infer that L^* is surjective. Therefore L is invertible and so $\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$ with the frame operator $L^{-1}S_L(L^*)^{-1}$. □

Proposition 2.8. *Let $\{f_i\}_{i\in I}$ be a Riesz basis for $\mathcal{H}$ with analysis operator T, Riesz basis bounds $A \leq B$, and let $L : \mathcal{H} \longrightarrow \mathcal{H}$ be a bounded operator. Then $\{Lf_i\}_{i\in I}$ is a Riesz basis for $\mathcal{H}$ if and only if L is invertible on $\mathcal{H}$. Moreover in this case, the analysis operator for $\{Lf_i\}_{i\in I}$ is $T_L = TL^*$ and the new Riesz basis bounds are $\| L^{-1} \|^{-2} A$, $\| L \|^2 B$.*

Proof. Since the analysis operator for $\{Lf_i\}_{i\in I}$ is $T_L = TL^*$, L is invertible if and only if $\{Lf_i\}_{i\in I}$ is a Riesz basis for $\mathcal{H}$. □

Corollary 2.9. *If $\{f_i\}_{i\in I}$ is a Riesz basis for $\mathcal{H}$ and $L:\mathcal{H}\longrightarrow\mathcal{H}$ is a bounded operator, then $\{f_i+Lf_i\}_{i\in I}$ is a Riesz basis for $\mathcal{H}$ if and only if $I+L$ is invertible on $\mathcal{H}$. In this case the analysis operator for new frame is $T_{I+L}=T(I+L^*)$ and the new Riesz basis bounds are $\| (I+L)^{-1} \|^{-2} A$, $\| I+L \|^2 B$.*

We recall that if $\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$, then the frame $\{g_i\}_{i\in I}$ is called an alternate dual frame of $\{f_i\}_{i\in I}$, if

$$f=\sum_{i\in I}\langle f,g_i\rangle f_i, \quad f\in\mathcal{H}.$$

Corollary 2.10. *Let $\{f_i\}_{i\in I}$ be a Riesz basis for $\mathcal{H}$ with frame operator S and $\{g_i\}_{i\in I}$ be an alternate dual frame of $\{f_i\}_{i\in I}$. Suppose that a and b are real numbers such that $-1\notin\sigma(S^{-a+b-1})$. Then $\{S^a f_i+S^b g_i\}_{i\in I}$ is a Riesz basis for $\mathcal{H}$.*

Here, we also show that the equivalence of part (1) and (2) in Proposition 3.1 of [2], is not true in general. Indeed, if $T_1L_1^*+T_2L_2^*$ is an invertible operator, then $\{L_1f_i+L_2g_i\}_{i\in I}$ is a frame for $\mathcal{H}$ but the converse is not true.

Example 2.11. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$ and T be the analysis operator of $\{e_n\}_{n=1}^{\infty}$. Define a shift operator L on $\mathcal{H}$ as in Example 2.2. Letting $L_1=L_2=L$ and $f_n=g_n=e_n$ for each $n\in\mathbb{N}$, in Proposition 3.1 of [2], we see that $\{2L(e_n)\}_{n=1}^{\infty}$ is a frame for $\mathcal{H}$ but $2TL^*$ is not a surjective operator. Indeed, T is an invertible operator, but L^* is not surjective.

Proposition 2.12. *Let $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ be Bessel sequences in $\mathcal{H}$ with analysis operators T_1, T_2, respectively. Also, let $L_1,L_2:\mathcal{H}\longrightarrow\mathcal{H}$. Then the following are equivalent*:

(1) *$\{L_1f_i+L_2g_i\}_{i\in I}$ is a Riesz basis for $\mathcal{H}$.*

(2) *$T_1L_1^*+T_2L_2^*$ is an invertible operator on $\mathcal{H}$.*

Proof. (1) $\Leftrightarrow$ (2) $\{L_1f_i+L_2g_i\}_{i\in I}$ is a Riesz basis for $\mathcal{H}$ if and only if its analysis operator T is invertible on $\mathcal{H}$ where

$$\begin{aligned} Tf &= \{\langle f, L_1f_i+L_2g_i\rangle\}_{i\in I} \\ &= \{\langle L_1^*f,f_i\rangle+\langle L_2^*f,g_i\rangle\}_{i\in I} \\ &= T_1L_1^*f+T_2L_2^*f. \end{aligned}$$

□

3. Applications to Gabor frames

For $x,y\in\mathbb{R}$ we consider the operators E_x and T_y on $L^2(\mathbb{R})$ defined by $(E_xf)(t)=e^{2\pi ixt}f(t)$ and $(T_yf)(t)=f(t-y)$. It is easy to prove that E_x and T_y are unitary with $E_x^*=E_{-x}$ and $T_y^*=T_{-y}$. A Gabor frame is a frame for $L^2(\mathbb{R})$ of the form $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$, where $a,b>0$ and $g\in L^2(\mathbb{R})$ is a fixed function. We use (g,a,b) to denote $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$.

Proposition 3.1. *Let $S,T\in B(\mathcal{H})$. Then $I-TS$ is surjective if and only if $I-ST$ is surjective.*

Proof. Let $I - TS$ be surjective. Then, by using Lemma 1.1 we have

$$\begin{aligned}(I - ST)(I + S(I - TS)^{\dagger}T) &= (I - ST) + (S - STS)(I - TS)^{\dagger}T \\ &= I - ST + S(I - TS)(I - TS)^{\dagger}T \\ &= I - ST + ST = I,\end{aligned}$$

so $I - ST$ is surjective. □

Corollary 3.2. *Let $x, y \in \mathbb{R}$ and $c \in \mathbb{C}$ with $|c| = 1$. Then the following are equivalent*:

(i) $I + cT_xE_y$ *is a surjective operator on* $L^2(\mathbb{R})$.
(ii) $I + cE_yT_x$ *is a surjective operator on* $L^2(\mathbb{R})$.

Lemma 3.3. *Let T be a surjective and normal bounded linear operator on a Hilbert space. Then T is invertible.*

Proof. Assume that T is not injective, i.e., there exists non-zero $x \in \ker T$. By normality, $\|T^*x\| = \|Tx\| = 0$. Therefore x is in the orthogonal complement of the range of T. This implies that T is not surjective. □

In the following, we intend to correct Proposition 4.1 of [2].

Theorem 3.4. *Let $x, y \in \mathbb{R}$ such that $x^2 + y^2 \neq 0$ and let $c \in \mathbb{C}$ with $|c| = 1$. Then $I + cE_yT_x : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is not surjective.*

Proof. It is enough we take $x > 0$. Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a function defined by

$$f(t) := \sum_{k=1}^{n} (-1)^k c^k \rho_k e^{2\pi ikyt} \chi_{[kx,(k+1)x)}(t),$$

where $\rho_1 = 1$ and $\rho_{k+1} = \rho_k e^{-2\pi ikxy}$ for $k \geqslant 1$. By a simple computation, we get

$$\|f\|^2 = \int_{\mathbb{R}} |f(t)|^2\, dt = nx, \quad \|(I + cE_yT_x)f\|^2 = 2x.$$

Therefore $f \in L^2(\mathbb{R})$ and $I + cE_yT_x$ cannot be invertible. Since E_y and T_x are unitary, $I + cE_yT_x$ is normal. The previous lemma implies $I + cE_yT_x$ can not be surjective. □

Corollary 3.5. *Let $x, y \in \mathbb{R}$ such that $x^2 + y^2 \neq 0$ and let $c \in \mathbb{C}$ with $|c| = 1$. If (g, a, b) is a Gabor frame and $ay, bx \in \mathbb{Z}$, then $(g + cE_yT_xg, a, b)$ is not a Gabor frame.*

Proof. It is clear that $E_{mb}T_{na}(g + cE_yT_xg) = (I + dT_xE_y)(E_{mb}T_{na}g)$, where $|d| = 1$. If $(g + cE_yT_xg, a, b)$ is a Gabor frame, then $I + dT_xE_y$ is surjective on $L^2(\mathbb{R})$ by Proposition 2.1. So $I + dE_yT_x$ is surjective by Corollary 3.2. Using Theorem 3.4, we get a contradiction. □

Corollary 3.6. *Let $x, y \in \mathbb{R}$ such that $x^2 + y^2 \neq 0$ and let $c \in \mathbb{C}$ with $|c| = 1$. If (g, a, b) is a Gabor frame, then the following sequence is not a Gabor frame*

$$\{E_{mb}T_{na}(g + ce^{2\pi i(nay-mbx)}E_yT_xg)\}_{m,n\in\mathbb{Z}}.$$

Proof. Because

$$E_{mb}T_{na}(g + ce^{2\pi i(nay-mbx)}E_yT_xg) = (I + ce^{2\pi ixy}T_xE_y)(E_{mb}T_{na}g),$$

we get the result. □

Acknowledgment. The authors would like to thank the referees for their helpful comments to improve the quality of this manuscript.

References

[1] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2003.

[2] S. Obeidat, S. Samarah, Peter G. Casazza, and J. C. Tremain, *Sums of Hilbert space frames*, J. Math. Anal. Appl. **351** (2009), no. 2, 579–585.

A. Najati
Department of Mathematics
Faculty of Mathematical Sciences
University of Mohaghegh Ardabili
Ardabil 56199-11367, Iran
E-mail address: a.nejati@yahoo.com, a.najati@uma.ac.ir

M. R. Abdollahpour
Department of Mathematics
Faculty of Mathematical Sciences
University of Mohaghegh Ardabili
Ardabil 56199-11367, Iran
E-mail address: mrabdollahpour@yahoo.com, m.abdollah@uma.ac.ir

E. Osgooei
Department of Sciences
Urmia University of Technology
Band High Way, Urmia, Iran
E-mail address: e.osgooei@uut.ac.ir

M. M. Saem
Department of Mathematics
Faculty of Mathematical Sciences
University of Mohaghegh Ardabili
Ardabil 56199-11367, Iran
E-mail address: m.mohammadisaem@yahoo.com