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EXISTENCE OF THREE SOLUTIONS FOR A NAVIER BOUNDARY VALUE PROBLEM INVOLVING THE $p(x)$-BIHARMONIC

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Abstract. The existence of at least three weak solutions is established for a class of quasilinear elliptic equations involving the $p(x)$-biharmonic operators with Navier boundary value conditions. The technical approach is mainly based on a three critical points theorem due to Ricceri [11].

1. Introduction

In this paper, we consider the problem of the type

$$\begin{cases} \triangle^2_{p(x)} u = \lambda a(x) f(x,u) + \mu g(x,u), & x \in \Omega \\ u = \triangle u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with boundary of class C^1, $\lambda, \mu \geq 0$ are real numbers, $p(x) \in C^0(\overline{\Omega})$ with $\max\{2, \frac{N}{2}\} < p^- := \inf_{x\in\overline{\Omega}} p(x) \leq p^+ := \sup_{x\in\overline{\Omega}} p(x)$, $\triangle^2_{p(x)} := \triangle(|\triangle u|^{p(x)-2}\triangle u)$ is the operator of fourth order called the $p(x)$-biharmonic operator, which is a natural generalization of the p-biharmonic operator (where $p > 1$ is a constant).

In [10], the authors studied the following super-linear p-biharmonic elliptic problem with Navier boundary conditions:

$$\begin{cases} \triangle^2_p u = g(x,u), & x \in \Omega, \\ u = \triangle u = 0, & x \in \partial\Omega. \end{cases} \tag{2}$$

By means of Morse theory, the authors proved the existence of a nontrivial solution to (2) having a linking structure around the origin under the conditions: $\Omega \subseteq \mathbb{R}^N$ is bounded with smooth boundary, $N \geq 2p+1$, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for some $C > 0$, $|g(x,t)| \leq C(1+|t|^{q-1})$ for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$, $1 \leq q \leq p^* = \frac{Np}{N-2p}$. Moreover, in case of both

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resonance near zero and non-resonance at ∞, the existence of two nontrivial solutions was obtained.

In [9], the authors considered the following problem:

$$\begin{cases} \triangle_p^2 u = \lambda f(x,u) + \mu g(x,u), & x \in \Omega, \\ u = \triangle u = 0, & x \in \partial\Omega. \end{cases} \tag{3}$$

By the three critical points theorem obtained by Ricceri [11], they established the existence of three weak solutions to problem (3).

For more results for fourth-order elliptic equations with variable exponent, see [1, 2] and the reference therein.

To obtain the existence of at least three solutions of problem (1), the technical approach is mainly based on a three critical points theorem by B. Ricceri [11].

Theorem A. *Let X be a reflexive real Banach space; $I \subseteq \mathbb{R}$ an interval; $\Phi : X \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous C^1 functional, bounded on each bounded subset of X, whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbb{R}$ a C^1 functional with compact Gâteaux derivative. Assume that*

(i) $\lim_{\|u\|\to\infty}(\Phi(u) + \lambda\Psi(u)) = \infty$ *for all* $\lambda \in I$;

(ii) *There exists* $\rho \in \mathbb{R}$ *such that*:

$$\sup_{\lambda\in I}\inf_{t\in X}(\Phi(t) + \lambda(\Psi(t) + \rho)) < \inf_{t\in X}\sup_{\lambda\in I}(\Phi(t) + \lambda(\Psi(t) + \rho)).$$

Then there exists a non-empty open set $\Lambda \subseteq I$ and a positive real number σ with the following property: for each $\lambda \in \Lambda$ and every C^1 functional $J : X \to \mathbb{R}$ with compact Gâteaux derivative, there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$, the equation

$$\Phi'(u) + \lambda\Psi'(u) + \mu J'(u) = 0 \tag{4}$$

has at least three solutions in X whose norms are less than σ.

To obtain the existence of at least three solutions of (1), we assume the following conditions:

(A) $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $\sup_{|\zeta|\le s} |g(\cdot, \zeta)| \in L^1(\Omega)$ for all $s > 0$;

(B) $a(x) \in L^{r(x)}(\Omega)$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $|f(x,t)| \le b(x) + \alpha|t|^{q(x)-1}$ for $x \in \Omega$ and $t \in \mathbb{R}$, where $\alpha \ge 0$ is a constant, $b(x) \in L^{q^0(x)r^0(x)}(\Omega)$, $r(x), q(x) \in C(\Omega)$, $r^- > 1, p^- > q^+ \ge q^- \ge 1$, and

$$q(x) < \frac{r(x)-1}{r(x)} p^*(x), \ \ \forall x \in \Omega,$$

here

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & p(x) < N \\ \infty, & p(x) \ge N \end{cases}$$

and $r^0(x)$ is the conjugate function of $r(x)$, i.e., $\frac{1}{r(x)} + \frac{1}{r^0(x)} = 1$.

The paper is organized as follows. In Section 2, we recall some facts that will be needed in the paper. In Section 3, we establish our main results.

2. Notations and preliminaries

For the reader's convenience, we remind some background facts concerning the Lebesgue-Sobolev spaces with variable exponent and introduce some notations used below. For more details, we refer the reader to [5, 7, 12, 13].

Set

$$C_+(\Omega) = \{h : \ h \in C(\overline{\Omega}) \text{ and } h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For $p(x) \in C_+(\Omega)$, define the space

$$L^{p(x)}(\Omega) = \{u \mid u \text{ is a measurable real-valued funcion, } \int_\Omega |u(x)|^{p(x)}dx < \infty\}.$$

We can introduce a norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(x)} = \inf\{\lambda > 0 \mid \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx \leq 1\}.$$

and $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a Banach space, and we call it variable exponent Lebesgue space.

The space $W^{m,p(x)}(\Omega)$ is defined by

$$W^{m,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \mid D^\beta u \in L^{p(x)}(\Omega), \ |\beta| \leq m\},$$

where β is the multi-index and $|\beta|$ is the order, m is a positive integer.

$$W^{m,p(x)}(\Omega)$$

is a special class of so-called generalized Orlicz-Sobolev spaces. From [6], we know that $W^{m,p(x)}(\Omega)$ can be equipped with the norm $\|u\|_{W^{m,p(x)}(\Omega)}$ as Banach spaces, where

$$\|u\|_{W^{m,p(x)}(\Omega)} = \sum_{|\beta|\leq m} |D^\alpha u|_{p(x)}.$$

From [5], we know that spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$ are separable, reflexive and uniform convex Banach spaces.

Now we denote $X = W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$, where $W_0^{1,p(x)}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. For any $u \in X$, define

$$\|u\| = \inf\{\lambda > 0 \mid \int_\Omega |\frac{\triangle u(x)}{\lambda}|^{p(x)}dx \leq 1\}.$$

Then it is easy to see that X endowed with the above norm is also a separable, reflexive Banach space. We denote by X^* the dual space to X.

Remark 2.1. According to [14], $\|u\|_{W^{2,p(x)}(\Omega)}$ is equivalent to $|\triangle u|_{p(x)}$ in X. Consequently, the norms $\|u\|_{W^{2,p(x)}(\Omega)}$ and $\|u\|$ are equivalent.

From now on, we will use $\|\cdot\|$ instead of $\|\cdot\|_{W^{2,p(x)}(\Omega)}$ on X.

Proposition 2.1 (see [5, 12])**.** *The conjugate space of* $L^{p(x)}(\Omega)$ *is* $L^{p^0(x)}(\Omega)$. *For any* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{p^0(x)}(\Omega)$, *we have*

$$\int_\Omega |uv|dx \le (\frac{1}{p^-} + \frac{1}{(p^0)^-})|u|_{p(x)}|v|_{p^0(x)} \le 2|u|_{p(x)}|v|_{p^0(x)}.$$

Proposition 2.2 (see [5, 12])**.** *If we denote* $\rho(u) = \int_\Omega |u|^{p(x)}dx$, $\forall u \in L^{p(x)}(\Omega)$, *then*

(i) $|u|_{p(x)} < 1(= 1; > 1) \Leftrightarrow \rho(u) < 1(= 1; > 1)$;

(ii) $|u|_{p(x)} > 1 \Rightarrow |u|^{p^-}_{p(x)} \le \rho(u) \le |u|^{p^+}_{p(x)}$; $|u|_{p(x)} < 1 \Rightarrow |u|^{p^+}_{p(x)} \le \rho(u) \le |u|^{p^-}_{p(x)}$;

(iii) $|u|_{p(x)} \to 0(\infty) \Leftrightarrow \rho(u) \to 0(\infty)$.

From Proposition 2.2, the following inequalities hold:

$$\|u\|^{p^-} \le \int_\Omega |\triangle u(x)|^{p(x)}dx \le \|u\|^{p^+}, \text{ if } \|u\| \ge 1; \tag{5}$$

$$\|u\|^{p^+} \le \int_\Omega |\triangle u(x)|^{p(x)}dx \le \|u\|^{p^-}, \text{ if } \|u\| \le 1. \tag{6}$$

Proposition 2.3 (see [4])**.** *Suppose that the boundary of* Ω *possesses the cone property and* $a(x) \in L^{r(x)}(\Omega)$, $a(x) > 0$ *for a.e.* $x \in \Omega$, $r(x) \in C(\overline{\Omega})$ *and* $r^- > 1$. *If* $p(x), q(x) \in C(\overline{\Omega})$ *and*

$$1 \le q(x) < \frac{r(x) - 1}{r(x)} p^*(x), \quad \forall x \in \overline{\Omega},$$

then there is a compact embedding $X \hookrightarrow L^{q(x)}_{a(x)}(\Omega)$.

Proposition 2.4. *If* $\Omega \subset R^N$ *is a bounded domain, then the imbedding* $X \hookrightarrow C^0(\overline{\Omega})$ *is compact whenever* $\frac{N}{2} < p^-$.

Proof. It is well known that $X \hookrightarrow W^{2,p^-}(\Omega) \cap W^{1,p^-}_0(\Omega)$ is a continuous embedding, and the embedding $W^{2,p^-}(\Omega) \cap W^{1,p^-}_0(\Omega) \hookrightarrow C^0(\overline{\Omega})$ is compact when $\frac{N}{2} < p^-$ and Ω is bounded. So we obtain the embedding $X \hookrightarrow C^0(\overline{\Omega})$ which is compact whenever $\frac{N}{2} < p^-$. □

From Proposition 2.4, there exists a positive constant k depending on $p(x)$, N and Ω, such that

$$\|u\|_\infty = \sup_{x \in \overline{\Omega}} |u(x)| \le k\|u\|, \ \forall u \in X. \tag{7}$$

3. Existence of three solutions

Fix $x^0 \in \Omega$ and choose r_1, r_2 with $0 < r_1 < r_2$, such that $B(x^0, r_2) \subseteq \Omega$, where $B(x, r)$ stands for the open ball in $\mathbb{R}^N$ of radius r centered at x. Let

$$\sigma = \max\{[\frac{12(N+2)^2(r_1+r_2)}{(r_2-r_1)^3}]^{p^-} \frac{2\pi^{\frac{N}{2}}(r_2^N - r_1^N)}{N\Gamma(\frac{N}{2})}, [\frac{12(N+2)^2(r_1+r_2)}{(r_2-r_1)^3}]^{p^+} \frac{2\pi^{\frac{N}{2}}(r_2^N - r_1^N)}{N\Gamma(\frac{N}{2})}\},$$

$$\theta = \begin{cases} \min\{[\frac{3N}{(r_2-r_1)(r_2+r_1)}]^{p^-}\frac{2\pi^{\frac{N}{2}}[(r_2+r_1)^N-(2r_1)^N]}{2^N N\Gamma(\frac{N}{2})}, [\frac{3N}{(r_2-r_1)(r_2+r_1)}]^{p^+}\frac{2\pi^{\frac{N}{2}}[(r_2+r_1)^N-(2r_1)^N]}{2^N N\Gamma(\frac{N}{2})}\}, \\ \text{if } N < \frac{4r_1}{r_2-r_1}, \\ \min\{[\frac{12r_1}{(r_2-r_1)^2(r_2+r_1)}]^{p^-}\frac{2\pi^{\frac{N}{2}}[(r_2+r_1)^N-(2r_1)^N]}{2^N N\Gamma(\frac{N}{2})}, [\frac{12r_1}{(r_2-r_1)^2(r_2+r_1)}]^{p^+}\frac{2\pi^{\frac{N}{2}}[(r_2+r_1)^N-(2r_1)^N]}{2^N N\Gamma(\frac{N}{2})}\}, \\ \text{if } N \geq \frac{4r_1}{r_2-r_1}. \end{cases}$$

We define $\Phi : X \to \mathbb{R}$ as

$$\Phi(u) = \int_\Omega \frac{1}{p(x)}|\triangle u(x)|^{p(x)}dx. \tag{8}$$

Then

$$(\Phi'(u), v) = \int_\Omega |\triangle u|^{p(x)-2}\triangle u \triangle v dx, \quad \forall u, v \in X.$$

Denote

$$F(x,u) = \int_0^u a(x)f(x,t)dt, \ G(x,u) = \int_0^u g(x,t)dt,$$
$$\Psi(u) = -\int_\Omega F(x,u)dx, \ J(u) = -\int_\Omega G(x,u)dx.$$

Then for $\forall u, v \in X$,

$$(\Psi'(u), v) = -\int_\Omega a(x)f(x,u)vdx,$$
$$(J'(u), v) = -\int_\Omega g(x,u)vdx.$$

We say that $u \in X$ is a weak solution of problem (1) if

$$\int_\Omega |\triangle u|^{p(x)-2}\triangle u\triangle v dx = \lambda \int_\Omega a(x)f(x,u)vdx + \mu \int_\Omega g(x,u)vdx, \ \forall v \in X,$$

i.e.,

$$(\Psi'(u), v) + \lambda(\Psi'(u), v) + \mu(J'(u), v) = 0.$$

It follows that we can find the weak solutions of (1) applying Theorem A.

We first obtain the following results.

Lemma 3.1. *If Φ is defined in (8), then $(\Phi')^{-1} : X^* \to X$ exists and it is continuous.*

Proof. First, we show that Φ' is uniformly monotone. In fact, for any $\zeta, \eta \in R^N$, we have the following inequality (see [8]):

$$(|\zeta|^{p-2}\zeta - |\eta|^{p-2}\eta)(\zeta - \eta) \geq \frac{1}{2^p}|\zeta - \eta|^p, \ p \geq 2.$$

Thus, we deduce that

$$(\Phi'(u) - \Phi'(v), u - v) \geq \frac{1}{2^{p^+}}\int_\Omega |\triangle u - \triangle v|^{p(x)}dx, \ \forall u, v \in X,$$

i.e., Φ' is uniformly monotone.

From (5), we can see that for any $u \in X$ with $\|u\| \geq 1$, and hence we have that

$$\frac{(\Phi'(u), u)}{\|u\|} \geq \|u\|^{p^- - 1}.$$

This means that Φ' is coercive on X.

By a standard argument, we know that Φ' is hemicontinuous. Therefore, the conclusion follows immediately by applying Theorem 26.A [15]. $\square$

Lemma 3.2. *If condition* (B) *holds, then for any* $\lambda \in R$, $\Phi(u) + \lambda\Psi(u)$ *is coercive on* X.

Proof. For $|f(x,t)| \leq b(x) + \alpha|t|^{q(x)-1}$ and the Young's inequality, we have that

$$|F(x,t)| \leq |a(x)|(b(x)|t| + \frac{\alpha}{q(x)}|t|^{q(x)}) \leq |a(x)|((b(x))^{q^0(x)} + (1+\alpha)|t|^{q(x)}).$$

Then from condition (B) and Proposition 2.3 we know that $F(x,u)$ is integrable on Ω for any $u \in X$, $\Psi(u)$ is well defined.

Combining it with Proposition 2.3, we have

$$\begin{aligned}
&\Phi(u) + \lambda\Psi(u) \\
= &\int_\Omega \frac{1}{p(x)}|\nabla u(x)|^{p(x)}dx - \lambda\int_\Omega F(x,u)dx \\
\geq &\frac{\|u\|^{p^-}}{p^+} - |\lambda|\int_\Omega |a(x)|[|b(x)|^{q^0(x)} + (1+\alpha)|u|^{q(x)}]dx \\
\geq &\frac{\|u\|^{p^-}}{p^+} - |\lambda|C_1 - |\lambda|(1+\alpha)|u|^{q^\dagger}_{(q(x),|a(x)|)} \\
\geq &\frac{\|u\|^{p^-}}{p^+} - |\lambda|C_1 - |\lambda|C_2\|u\|^{q^\dagger},
\end{aligned}$$

where

$$\dagger = \begin{cases} +, & \text{if } |u|_{q(x),|a(x)|} \leq 1, \\ -, & \text{if } |u|_{q(x),|a(x)|} \geq 1 \end{cases}$$

and C_1, C_2 are positive constants. Since $q^\dagger < p^-$, we can see that $\Phi(u) + \lambda\Psi(u)$ is coercive. $\square$

Furthermore, we suppose

(C) There exist two positive constants c, d with $k > c$ and $k\theta\min\{d^{p^+}, d^{p^-}\} > c$ such that $F(x,t) \geq 0$ for each $(x,t) \in \{\Omega\backslash B(x^0, r_1)\} \times [0,d]$, and

$$m(\Omega)\max_{(x,t)\in\overline{\Omega}\times[-c,c]} F(x,t) \leq \frac{p^-}{p^+\sigma\max\{d^{p^+}, d^{p^-}\}}(\frac{c}{k})^{p^+}\int_{B(x^0,r_1)} F(x,d)dx,$$

where $m(\Omega)$ is the Lebesgue measure of Ω.

Then we have the following result.

Lemma 3.3. *If condition* (C) *holds, then there exist* $r > 0$ *and* $u^* \in X$ *such that*

$$\Phi(u^*) > r \tag{9}$$

and

$$m(\Omega) \max_{(x,t)\in\overline{\Omega}\times[-c,c]} F(x,t) \le rp^- \frac{\int_\Omega F(x,u^*)dx}{\int_\Omega \mid \triangle u(x)|^{p(x)}dx}. \tag{10}$$

Proof. Let us define

$$u^*(x) = \begin{cases} 0, \quad x \in \overline{\Omega}\backslash B(x^0, r_2), \\ \frac{d[3(l^4 - r_2^4) - 4(r_1+r_2)(l^3-r_2^3)+6r_1r_2(l^2-r_2^2)]}{(r_2-r_1)^3(r_1+r_2)}, \ x \in B(x^0,r_2)\backslash B(x^0,r_1), \\ d, \quad x \in B(x^0, r_1), \end{cases}$$

where $l = dist(x, x^0) = \sqrt{\sum_{i=1}^N (x_i - x_i^0)^2}$. Then, we have

$$\frac{\partial u^*(x)}{\partial x_i} = \begin{cases} 0, \quad x \in \overline{\Omega}\backslash B(x^0, r_2) \cup B(x^0,r_1), \\ \frac{12d[l^2(x_i-x_i^0)-(r_1+r_2)l(x_i-x_i^0)+r_1r_2(x_i-x_i^0)]}{(r_2-r_1)^3(r_1+r_2)}, \ x\in B(x^0,r_2)\backslash B(x^0,r_1), \end{cases}$$

$$\frac{\partial^2 u^*(x)}{\partial x_i^2} = \begin{cases} 0, \quad x \in \overline{\Omega}\backslash B(x^0, r_2) \cup B(x^0,r_1), \\ \frac{12d[r_1r_2+(2l-r_1-r_2)(x_i-x_i^0)^2/l-(r_1+r_2-l)l]}{(r_2-r_1)^3(r_1+r_2)}, \ x\in B(x^0,r_2)\backslash B(x^0,r_1), \end{cases}$$

and

$$\sum_{i=1}^N \frac{\partial^2 u^*(x)}{\partial x_i^2} = \begin{cases} 0, \quad x \in \overline{\Omega}\backslash B(x^0, r_2) \cup B(x^0,r_1), \\ \frac{12d[(N+2)l^2-(N+1)(r_1+r_2)l+Nr_1r_2]}{(r_2-r_1)^3(r_1+r_2)}, \ x \in B(x^0,r_2)\backslash B(x^0,r_1). \end{cases}$$

It is easy to verify that $u^* \in X$ and, in particular, we have

$$\theta \min\{d^{p^+}, d^{p^-}\} \le \int_\Omega |\triangle u^*|^{p(x)}dx \le \sigma \max\{d^{p^+}, d^{p^-}\}. \tag{11}$$

If we let

$$r = \frac{1}{p^+}(\frac{c}{k})^{p^+},$$

from (11) and the assumption that $k\theta \min\{d^{p^+}, d^{p^-}\} > c$, we have

$$\begin{aligned} \Phi(u^*) &= \int_\Omega \frac{1}{p(x)}|\triangle u^*(x)|^{p(x)}dx \\ &\ge \frac{1}{p^+}\int_\Omega |\triangle u^*(x)|^{p(x)}dx \\ &\ge \frac{1}{p^+}\theta \min\{d^{p^+}, d^{p^-}\} \\ &> \frac{1}{p^+}(\frac{c}{k})^{p^+} = r. \end{aligned} \tag{12}$$

Therefore, (9) follows.

Since $0 \le u^* \le d$ for any $x \in \Omega$, the condition (C) ensures that

$$\int_{\Omega \setminus B(x^0,r_2)} F(x,u^*)dx + \int_{B(x^0,r_2)\setminus B(x^0,r_1)} F(x,u^*)dx \ge 0.$$

Therefore, we have

$$\begin{aligned} m(\Omega) \max_{(x,t)\in\overline{\Omega}\times[-c,c]} F(x,t) &\le \frac{p^-}{p^+\sigma \max\{d^{p^+}, d^{p^-}\}}(\frac{c}{k})^{p^+} \int_{B(x^0,r_1)} F(x,d)dx \\ &\le rp^- \frac{\int_{B(x^0,r_1)} F(x,d)dx}{\int_\Omega |\triangle u(x)|^{p(x)}dx} \\ &\le rp^- \frac{\int_\Omega F(x,u^*)dx}{\int_\Omega |\triangle u(x)|^{p(x)}dx}. \end{aligned}$$

This implies (10). □

Finally, we have the following main theorem.

Theorem 3.1. *Assume conditions* (A)*,* (B) *and* (C) *hold. Then there exist a non-empty open set $\Lambda \subseteq R$ and a positive real number σ with the following property: for each $\lambda \in \Lambda$, there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$, problem* (1) *has at least three weak solutions whose norms are less than σ.*

Proof. By the definitions of Φ, Ψ, and J, we know that Ψ' is compact, and Φ is weakly lower semi-continuous and bounded on each bounded subset of X. From Lemma 3.1 we can see that $(\Phi')^{-1}$ is well defined, from condition (A), and J is well defined and continuously Gâteaux differentiable on X, with compact derivative. Then we can use Theorem A to obtain the result. Now we show that the hypotheses of Theorem A are fulfilled.

From Lemma 3.2, we can see (i) is satisfied.

From (7) we know that

$$\sup_{x\in\overline{\Omega}} |u(x)| \le k\|u\|, \ \forall u \in X.$$

Hence, if we let $r = \frac{1}{p^+}(\frac{c}{k})^{p^+}$, for each $u \in X$ such that

$$\Phi(u) \le r, \tag{13}$$

by (5), (6) and the assumption $k > c$, we have

$$\begin{aligned} \sup_{x\in\overline{\Omega}} |u(x)| &\le k\|u\| \\ &\le k \max\{(\int_\Omega |\triangle u(x)|^{p(x)}dx)^{\frac{1}{p^+}}, (\int_\Omega |\triangle u(x)|^{p(x)}dx)^{\frac{1}{p^-}}\} \\ &\le k(rp^+)^{\frac{1}{p^+}} \\ &\le c. \end{aligned} \tag{14}$$

Due to Lemma 3.3, there exists $u^* \in X$ such that

$$\Phi(u^*) > r > 0$$

and

$$m(\Omega) \max_{(x,t)\in\overline{\Omega}\times[-c,c]} F(x,t) \le rp^- \frac{\int_\Omega F(x,u^*)dx}{\int_\Omega |\bigtriangleup u(x)|^{p(x)}dx}.$$

Let $u_1(x) = u^*(x)$ on Ω, and then by (14) we have

$$\begin{aligned}
\sup_{u\in\Phi^{-1}((-\infty,r])} (-\Psi(u)) &\le \int_\Omega \sup_{\{u|\Phi(u)\le r\}} F(x,u)dx \\
&\le \int_\Omega \sup_{|t|\le c} F(x,t)dx \\
&< m(\Omega) \max_{(x,t)\in\overline{\Omega}\times[-c,c]} F(x,t)dx \\
&\le rp^- \frac{\int_\Omega F(x,u_1)dx}{\int_\Omega |\bigtriangleup u_1(x)|^{p(x)}dx} \\
&\le r\frac{-\Psi(u_1)}{\Phi(u_1)}.
\end{aligned}$$

Fixing any $h > 1$, it is easy to see that

$$\sup_{u\in\Phi^{-1}((-\infty,r])} (-\Psi(u)) + \frac{r\frac{-\Psi(u_1)}{\Phi(u_1)} - \sup_{u\in\Phi^{-1}((-\infty,r])}(-\Psi(u))}{h} < r\frac{-\Psi(u_1)}{\Phi(u_1)}.$$

By Proposition 1.3 of [3], when ρ satisfies

$$\sup_{u\in\Phi^{-1}((-\infty,r])} (-\Psi(u)) + \frac{r\frac{-\Psi(u_1)}{\Phi(u_1)} - \sup_{u\in\Phi^{-1}((-\infty,r])}(-\Psi(u))}{h} < \rho < r\frac{-\Psi(u_1)}{\Phi(u_1)},$$

we have

$$\sup_{\lambda\in R} \inf_{u\in X} (\Phi(u) + \lambda(\rho + \Psi(u))) < \inf_{u\in X} \sup_{\lambda\in[0,\alpha_1]} (\Phi(u) + \lambda(\rho + \Psi(u))),$$

where $\alpha_1 = \frac{hr}{r\frac{-\Psi(u_1)}{\Phi(u_1)} - \sup_{u\in\Phi^{-1}((-\infty,r])}(-\Psi(u))} > 0$.

Then (ii) of Theorem A holds with $I = [0, \alpha_1]$. Then all the hypotheses of Theorem A are fulfilled. By Theorem A, we know that there exist an open interval $\Lambda \subseteq I$ and a positive constant σ such that for any $\lambda \in \Lambda$, there exists $\delta > 0$ and for each $\mu \in [0, \delta]$, problem (1) has at least three weak solutions whose norms are less than σ. $\square$

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