Bull. Korean Math. Soc. **50** (2013), No. 6, pp. 1781–1797
http://dx.doi.org/10.4134/BKMS.2013.50.6.1781

LAGUERRE CHARACTERIZATIONS OF HYPERSURFACES IN $\mathbb{R}^n$

Shichang Shu and Yanyan Li

Abstract. Let $x : M \to \mathbb{R}^n$ be an $n-1$-dimensional hypersurface in $\mathbb{R}^n$, $\mathbf{L}$ be the Laguerre Blaschke tensor, $\mathbf{B}$ be the Laguerre second fundamental form and $\mathbf{D} = \mathbf{L} + \lambda\mathbf{B}$ be the Laguerre para-Blaschke tensor of the immersion x, where λ is a constant. The aim of this article is to study Laguerre Blaschke isoparametric hypersurfaces and Laguerre para-Blaschke isoparametric hypersurfaces in $\mathbb{R}^n$ with three distinct Laguerre principal curvatures one of which is simple. We obtain some classification results of such isoparametric hypersurfaces.

1. Introduction

In Laguerre differential geometry, T. Li and C. Wang [5] studied invariants of hypersurfaces in Euclidean space $\mathbb{R}^n$ under the Laguerre transformation group. The Laguerre transformations are the Lie sphere transformations which take oriented hyperplanes in $\mathbb{R}^n$ to oriented hyperplanes and preserve the tangential distance.

Let $U\mathbb{R}^n$ be the unit tangent bundle over $\mathbb{R}^n$. An oriented sphere in $\mathbb{R}^n$ centered at p with radius r can be regarded as the oriented sphere $\{(x,\xi) \mid x-p = r\xi\}$ in $U\mathbb{R}^n$, where x is the position vector and ξ the unit normal vector of the sphere. An oriented hyperplane in $\mathbb{R}^n$ with constant unit normal vector ξ and constant real number c can be regarded as the oriented hyperplane $\{(x,\xi) \mid x \cdot \xi = c\}$ in $U\mathbb{R}^n$. A diffeomorphism $\phi : U\mathbb{R}^n \to U\mathbb{R}^n$ which takes oriented spheres to oriented spheres, oriented hyperplanes to oriented hyperplanes, preserving the tangential distance of any two spheres, is called a Laguerre transformation. All Laguerre transformations in $U\mathbb{R}^n$ form a group of dimension $\frac{(n+1)(n+2)}{2}$, called Laguerre transformation group. An oriented hypersurface $x : M \to \mathbb{R}^n$ can be identified as the submanifold $(x,\xi) : M \to U\mathbb{R}^n$, where ξ is the unit normal of x. Two hypersurfaces $x, x^* : M \to \mathbb{R}^n$ are called

---

Received May 22, 2011; Revised November 28, 2011.

2010 *Mathematics Subject Classification.* 53C42, 53C20.

Key words and phrases. Laguerre characterization, Laguerre form, Laguerre Blaschke tensor, Laguerre second fundamental form.

Project supported by NSF of Shaanxi Province (SJ08A31) and NSF of Shaanxi Educational Committee(11JK0479, 2010JK893).

Laguerre equivalent, if there is a Laguerre transformation $\phi : U\mathbb{R}^n \to U\mathbb{R}^n$ such that $(x^*, \xi^*) = \phi \circ (x, \xi)$ (see [4]).

In [5], T. Li and C. Wang gave a complete Laguerre invariant system for hypersurfaces in $\mathbb{R}^n$. They proved that two umbilical free oriented hypersurfaces in $\mathbb{R}^n$ with non-zero principal curvatures are Laguerre equivalent if and only if they have the same Laguerre metric g and Laguerre second fundamental form $\mathbf{B}$. We should notices that the Laguerre geometry of surfaces in $\mathbb{R}^3$ has been studied by Blaschke in [1] and other authors in [3, 6, 7].

Let $\mathbb{R}^{n+3}_2$ be the space $\mathbb{R}^{n+3}$ equipped with the inner product $\langle X, Y\rangle = -X_1Y_1 + X_2Y_2 + \cdots + X_{n+2}Y_{n+2} - X_{n+3}Y_{n+3}$. Let C^{n+2} be the light-cone in $\mathbb{R}^{n+3}$ given by $C^{n+2} = \{X \in \mathbb{R}^{n+3}_2 \mid \langle X, X\rangle = 0\}$. Let $L\mathbb{G}$ be the subgroup of orthogonal group $O(n+1, 2)$ on $\mathbb{R}^{n+3}_2$ given by $L\mathbb{G} = \{T \in O(n+1, 2) \mid \zeta T = \zeta\}$, where $\zeta = (1, -1, \mathbf{0}, 0)$ and $\mathbf{0} \in \mathbb{R}^n$ is a light-like vector in $\mathbb{R}^{n+3}_2$.

Let $x : M \to \mathbb{R}^n$ be an umbilic free hypersurface with non-zero principal curvatures, $\xi : M \to S^{n-1}$ be its unit normal vector. Let $\{e_1, e_2, \ldots, e_{n-1}\}$ be the orthonormal basis for TM with respect to $dx \cdot dx$, consisting of unit principal vectors. The structure equations of $x : M \to \mathbb{R}^n$ are (see [4])

$$e_j(e_i(x)) = \sum_k \Gamma^k_{ij} e_k(x) + k_i \delta_{ij} \xi, \;\; e_i(\xi) = -k_i e_i(x), \;\; i, j, k = 1, \ldots, n-1, \tag{1.1}$$

where $k_i \neq 0$ is the principal curvature corresponding to e_i. Let

$$r_i = \frac{1}{k_i}, \;\; r = \frac{r_1 + r_2 + \cdots + r_{n-1}}{n-1}, \tag{1.2}$$

be the curvature radius and mean curvature radius of x respectively. We define $Y = \rho(x \cdot \xi, -x \cdot \xi, \xi, 1) : M \to C^{n+2} \subset \mathbb{R}^{n+3}_2$, where $\rho = \sqrt{\sum_i (r_i - r)^2} > 0$. From [5], we know that the *Laguerre metric* g of the immersion x can be defined by $g = \langle dY, dY\rangle$. Let $\{E_1, E_2, \ldots, E_{n-1}\}$ be an orthonormal basis for g with dual basis $\{\omega_1, \omega_2, \ldots, \omega_{n-1}\}$. The *Laguerre form* $\mathbf{C}$, *Laguerre Blaschke tensor* $\mathbf{L}$ and *Laguerre second fundamental form* $\mathbf{B}$ of the immersion x are defined by

$$\mathbf{C} = \sum_{i=1}^{n-1} C_i \omega_i, \;\; \mathbf{L} = \sum_{i,j=1}^{n-1} L_{ij} \omega_i \otimes \omega_j, \;\; \mathbf{B} = \sum_{i,j=1}^{n-1} B_{ij} \omega_i \otimes \omega_j, \tag{1.3}$$

respectively, where C_i, L_{ij} and B_{ij} are defined by formulas (2.10)–(2.12) in Section 2. We should notices that g, $\mathbf{C}$, $\mathbf{L}$ and $\mathbf{B}$ are Laguerre invariants (see [5]).

By making use of the two important Laguerre invariants, the Laguerre Blaschke tensor $\mathbf{L}$ and the Laguerre second fundamental form $\mathbf{B}$ of the immersion x, we define a symmetric $(0, 2)$ tensor $\mathbf{D} = \mathbf{L} + \lambda\mathbf{B}$ which is so called the *Laguerre para-Blaschke tensor* of x, where λ is a constant. An eigenvalue of the Laguerre Blaschke tensor is called a *Laguerre Blaschke eigenvalue* of x, an eigenvalue of the Laguerre second fundamental form is called a *Laguerre principal curvature* of x and an eigenvalue of the Laguerre para-Blaschke tensor is called a *Laguerre para-Blaschke eigenvalue* of x. An umbilic free hypersurface

$x : M \to \mathbb{R}^n$ is called a *Laguerre isoparametric hypersurface* if $\mathbf{C} \equiv 0$ and the Laguerre principal curvatures of the immersion x are constant, an umbilic free hypersurface $x : M \to \mathbb{R}^n$ is called a *Laguerre Blaschke isoparametric hypersurface* if $\mathbf{C} \equiv 0$ and the Laguerre Blaschke eigenvalues of the immersion x are constant, and an umbilic free hypersurface $x : M \to \mathbb{R}^n$ is called a *Laguerre para-Blaschke isoparametric hypersurface* if $\mathbf{C} \equiv 0$ and the Laguerre para-Blaschke eigenvalues of the immersion x are constant. An umbilic free hypersurface $x : M \to \mathbb{R}^n$ is called a *Laguerre para-isotropic hypersurface*, if there are two functions λ and μ on x such that $\mathbf{L} + \lambda\mathbf{B} + \mu g = 0$ and $\mathbf{C} \equiv 0$. If $\lambda = 0$, we call x a *Laguerre isotropic hypersurface*. It should be noted that if x is a Laguerre para-isotropic hypersurface, or a Laguerre isotropic hypersurface, then the Laguerre para-Blaschke eigenvalues, or the Laguerre Blaschke eigenvalues of x are all equal.

We define the Laguerre embedding $\tau : U\mathbb{R}^n_0 \to U\mathbb{R}^n$ (see [5]). Let $\mathbb{R}^{n+1}_1$ be the Minkowski space with the inner product $\langle X, Y\rangle = X_1Y_1 + \cdots + X_nY_n - X_{n+1}Y_{n+1}$. Let $\nu = (1, \mathbf{0}, 1)$ be the light-like vector in $\mathbb{R}^{n+1}_1$, $\mathbf{0} \in \mathbb{R}^{n-1}$. Let $\mathbb{R}^n_0$ be the degenerate hyperplane in $\mathbb{R}^{n+1}_1$ defined by $\mathbb{R}^n_0 = \{X \in \mathbb{R}^{n+1}_1 \mid \langle X, \nu\rangle = 0\}$. We define

$$U\mathbb{R}^n_0 = \{(x, \xi) \in \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1 \mid \langle x, \nu\rangle = 0, \langle \xi, \xi\rangle = 0, \langle \xi, \nu\rangle = 1\}. \tag{1.4}$$

The Laguerre embedding $\tau : U\mathbb{R}^n_0 \to U\mathbb{R}^n$ is defined by

$$\tau(x, \xi) = (x', \xi') \in U\mathbb{R}^n, \tag{1.5}$$

where $x = (x_1, x_0, x_1) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$, $\xi = (\xi_1 + 1, \xi_0, \xi_1) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$ and

$$x' = \left(-\frac{x_1}{\xi_1}, x_0 - \frac{x_1}{\xi_1}\xi_0\right), \quad \xi' = \left(1 + \frac{1}{\xi_1}, \frac{\xi_0}{\xi_1}\right). \tag{1.6}$$

Let $x : M \to \mathbb{R}^n_0$ be a space-like oriented hypersurface in the degenerate hyperplane $\mathbb{R}^n_0$. Let ξ be the unique vector in $\mathbb{R}^{n+1}_1$ satisfying $\langle \xi, dx\rangle = 0$, $\langle \xi, \xi\rangle = 0$, $\langle \xi, \nu\rangle = 1$. From $\tau(x, \xi) = (x', \xi') \in U\mathbb{R}^n$, we may obtain a hypersurface $x' : M \to \mathbb{R}^n$.

We should notice that it is one of the important aims to characterize hypersurfaces in terms of Laguerre invariants. Concerning this topic, recently, T. Li, H. Li and C. Wang [4] studied the Laguerre geometry of hypersurfaces with parallel Laguerre second fundamental form in $\mathbb{R}^n$ and obtained the following result:

Theorem 1.1 ([4]). *Let $x : M \to \mathbb{R}^n$ be an umbilic free hypersurface with non-zero principal curvatures. If the Laguerre second fundamental form of x is parallel, then x is Laguerre equivalent to an open part of one of the following hypersurfaces*:

(1) *the oriented hypersurface $x : S^{k-1} \times H^{n-k} \to \mathbb{R}^n$ given by Example 2.1*; *or*

(2) *the image of τ of the oriented hypersurface $x : \mathbb{R}^{n-1} \to \mathbb{R}^n_0$ given by Example 2.2.*

The aim of this article is to continue this topic, we shall study Laguerre Blaschke isoparametric hypersurfaces and Laguerre para-Blaschke isoparametric hypersurfaces in $\mathbb{R}^n$ with three distinct Laguerre principal curvatures one of which is simple. We obtain the following results:

Theorem 1.2. *Let* $x : M \to \mathbb{R}^n$ *be an* $n-1$*-dimensional Laguerre Blaschke isoparametric hypersurface in* $\mathbb{R}^n (n \geq 5)$ *with three distinct Laguerre principal curvatures one of which is simple. Then* x *is a Laguerre isoparametric hypersurface with non-parallel Laguerre second fundament form or a Laguerre isotropic hypersurface.*

Theorem 1.3. *Let* $x : M \to \mathbb{R}^n$ *be an* $n-1$*-dimensional Laguerre para-Blaschke isoparametric hypersurface in* $\mathbb{R}^n (n \geq 5)$ *and* $\mathbf{D} = \mathbf{L} + \lambda \mathbf{B}$, $\lambda \neq 0$, *be the Laguerre para-Blaschke tensor of* x*. If* x *has three distinct Laguerre principal curvatures one of which is simple, then*

(i) x *is a Laguerre isoparametric hypersurface with non-parallel Laguerre second fundament form, or*

(ii) x *is a Laguerre para-isotropic hypersurface, or*

(iii) x *is Laguerre equivalent to an open part of the image of* τ *of the oriented hypersurface* $x : \mathbb{R}^{n-1} \to \mathbb{R}^n_0$ *given by Example 2.2.*

From Theorem 1.2 and Theorem 1.3, we easily see that:

Corollary 1.4. *Let* $x : M \to \mathbb{R}^n$ *be an* $n-1$*-dimensional Laguerre para-Blaschke isoparametric hypersurface in* $\mathbb{R}^n (n \geq 5)$ *and* $\mathbf{D} = \mathbf{L} + \lambda \mathbf{B}$ *be the Laguerre para-Blaschke tensor of* x*. If* x *has three distinct Laguerre principal curvatures one of which is simple, then*

(i) x *is a Laguerre isoparametric hypersurface with non-parallel Laguerre second fundament form, or*

(ii) x *is a Laguerre isotropic hypersurface for* $\lambda = 0$*, or*

(iii) x *is a Laguerre para-isotropic hypersurface or* x *is Laguerre equivalent to an open part of the image of* τ *of the oriented hypersurface* $x : \mathbb{R}^{n-1} \to \mathbb{R}^n_0$ *given by Example 2.2 for* $\lambda \neq 0$.

2. Laguerre invariants and fundamental formulas

In this section, we review the Laguerre invariants and fundamental formulas on Laguerre geometry of hypersurfaces in $\mathbb{R}^n$, for more details, see [5].

Let $x : M \to \mathbb{R}^n$ be an $n-1$-dimensional umbilical free hypersurface with vanishing Laguerre form in $\mathbb{R}^n$. Let $\{E_1, \ldots, E_{n-1}\}$ denote a local orthonormal frame for Laguerre metric $g = \langle dY, dY \rangle$ with dual frame $\{\omega_1, \ldots, \omega_{n-1}\}$. Putting $Y_i = E_i(Y)$, then we have

$$N = \frac{1}{n-1} \Delta Y + \frac{1}{2(n-1)^2} \langle \Delta Y, \Delta Y \rangle Y, \tag{2.1}$$

$$\langle Y, Y \rangle = \langle N, N \rangle = 0, \quad \langle Y, N \rangle = -1, \tag{2.2}$$

and the following orthogonal decomposition:

$$\mathbb{R}^{n+3}_2 = \mathrm{Span}\{Y, N\} \oplus \mathrm{Span}\{Y_1, \dots, Y_{n-1}\} \oplus \mathbb{V}, \tag{2.3}$$

where $\{Y, N, Y_1, \dots, Y_{n-1}, \eta, \wp\}$ forms a moving frame in $\mathbb{R}^{n+3}_2$ and $\mathbb{V} = \{\eta, \wp\}$ is called *Laguerre normal bundle* of x. We use the following range of indices throughout this paper:

$$1 \le i, j, k, l, m \le n-1.$$

The structure equations on x with respect to the Laguerre metric g can be written as

$$dY = \sum_i \omega_i Y_i, \tag{2.4}$$

$$dN = \sum_i \psi_i Y_i + \varphi\eta, \tag{2.5}$$

$$dY_i = -\psi_i Y - \omega_i N + \sum_j \omega_{ij} Y_j + \omega_{in+1}\eta, \tag{2.6}$$

$$d\wp = -\varphi Y - \sum_i \omega_{in+1} Y_i, \tag{2.7}$$

where $\{\psi_i, \omega_{ij}, \omega_{in+1}, \varphi\}$ are 1-forms on x with

$$\omega_{ij} + \omega_{ji} = 0, \quad d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \tag{2.8}$$

and

$$\psi_i = \sum_j L_{ij}\omega_j, \ L_{ij} = L_{ji}, \ \omega_{in+1} = \sum_j B_{ij}\omega_j, \ B_{ij} = B_{ji}, \ \varphi = \sum_i C_i\omega_i. \tag{2.9}$$

We define $\tilde{E}_i = r_i e_i$, $1 \le i \le n-1$, then $\{\tilde{E}_1, \dots, \tilde{E}_{n-1}\}$ is an orthonormal basis for $III = d\xi \cdot d\xi$ and $\{E_i = \rho^{-1}\tilde{E}_i\}$ is an orthonormal basis for the Laguerre metric g with dual frame $\{\omega_1, \dots, \omega_{n-1}\}$. L_{ij}, B_{ij} and C_i are locally defined functions and satisfy

$$L_{ij} = \rho^{-2}\left\{\mathrm{Hess}_{ij}(\log\rho) - \tilde{E}_i(\log\rho)\tilde{E}_j(\log\rho) + \frac{1}{2}\left(|\nabla \log\rho|^2 - 1\right)\delta_{ij}\right\}, \tag{2.10}$$

$$B_{ij} = \rho^{-1}(r_i - r)\delta_{ij}, \tag{2.11}$$

$$C_i = -\rho^{-2}\left\{\tilde{E}_i(r) - \tilde{E}_i(\log\rho)(r_i - r)\right\}, \tag{2.12}$$

where $g = \sum_i (r_i - r)^2 III = \rho^2 III$, r_i and r are defined by (1.2), Hess_{ij} and ∇ are the Hessian matrix and the gradient with respect to the third fundamental form $III = d\xi \cdot d\xi$ of x (see [5]).

Defining the covariant derivative of C_i, L_{ij}, B_{ij} by

$$\sum_j C_{i,j}\omega_j = dC_i + \sum_j C_j\omega_{ji}, \tag{2.13}$$

$$\sum_k L_{ij,k}\omega_k = dL_{ij} + \sum_k L_{ik}\omega_{kj} + \sum_k L_{kj}\omega_{ki}, \tag{2.14}$$

$$\sum_k B_{ij,k}\omega_k = dB_{ij} + \sum_k B_{ik}\omega_{kj} + \sum_k B_{kj}\omega_{ki}. \tag{2.15}$$

We have from [5] that

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \quad R_{ijkl} = -R_{jikl}, \tag{2.16}$$

$$\sum_i B_{ii} = 0, \quad \sum_{i,j} B_{ij}^2 = 1, \quad \sum_i B_{ij,i} = (n-2)C_j, \quad \mathrm{tr}\mathbf{L} = -\frac{R}{2(n-2)}. \tag{2.17}$$

$$L_{ij,k} = L_{ik,j}, \tag{2.18}$$

$$C_{i,j} - C_{j,i} = \sum_k (B_{ik}L_{kj} - B_{jk}L_{ki}), \tag{2.19}$$

$$B_{ij,k} - B_{ik,j} = C_j\delta_{ik} - C_k\delta_{ij}, \tag{2.20}$$

$$R_{ijkl} = L_{jk}\delta_{il} + L_{il}\delta_{jk} - L_{ik}\delta_{jl} - L_{jl}\delta_{ik}, \tag{2.21}$$

where R_{ijkl} and R denote the curvature tensor and the scalar curvature with respect to the Laguerre metric g on x. Since the Laguerre form $\mathbf{C} \equiv 0$, we have for all indices i, j, k

$$B_{ij,k} = B_{ik,j}, \quad \sum_k B_{ik}L_{kj} = \sum_k B_{kj}L_{ki}. \tag{2.22}$$

Denote by $\mathbf{D} = \sum_{i,j} D_{ij}\omega_i \otimes \omega_j$ the $(0,2)$ Laguerre para-Blaschke tensor, then

$$D_{ij} = L_{ij} + \lambda B_{ij}, \quad 1 \le i, j \le n, \tag{2.23}$$

where λ is a constant. The covariant derivative of D_{ij} is defined by

$$\sum_k D_{ij,k}\omega_k = dD_{ij} + \sum_k D_{ik}\omega_{kj} + \sum_k D_{kj}\omega_{ki}. \tag{2.24}$$

From (2.18) and (2.22), we have for all indices i, j, k that

$$D_{ij,k} = D_{ik,j}. \tag{2.25}$$

We recall the following examples of hypersurfaces in $\mathbb{R}^n$ with parallel Laguerre second fundamental form (see [4]):

Example 2.1 ([4]). For any integer k, $1 \le k \le n-1$, we define a hypersurface $x : S^{k-1} \times H^{n-k} \to \mathbb{R}^n$ by

$$x(u,v,w) = \left(\frac{u}{w}(1+w), \frac{v}{w} \right),$$

where $H^{n-k} = \{(v,w) \in \mathbb{R}_1^{n-k+1} \mid v \cdot v - w^2 = -1, w > 0\}$ denotes the hyperbolic space embedded in the Minkowski space $\mathbb{R}_1^{n-k+1}$. From [4], we know that x has two distinct Laguerre principal curvatures

$$B_1 = -\sqrt{\frac{n-k}{(k-1)(n-1)}}, \quad B_2 = \sqrt{\frac{k-1}{(n-k)(n-1)}},$$

the Laguerre form is zero and the Laguerre second fundamental form of x is parallel.

Example 2.2 ([4]). For any positive integers $m_1, \ldots, m_s$ with $m_1 + \cdots + m_s = n-1$ and any non-zero constants $\lambda_1, \ldots, \lambda_s$, we define $x : \mathbb{R}^{n-1} \to \mathbb{R}_0^n$ to be a spacelike oriented hypersurface in $\mathbb{R}_0^n$ given by

$$x = \left\{ \frac{\lambda_1 |u_1|^2 + \cdots + \lambda_s |u_s|^2}{2}, u_1, u_2, \ldots, u_s, \frac{\lambda_1 |u_1|^2 + \cdots + \lambda_s |u_s|^2}{2} \right\},$$

where $(u_1, \ldots, u_s) \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_s} = \mathbb{R}^{n-1}$ and $|u_i|^2 = u_i \cdot u_i, i = 1, \ldots, s$. Then $\tau \circ (x, \xi) = (x', \xi') : \mathbb{R}^{n-1} \to U\mathbb{R}^n$, and we obtain the hypersurfaces $x' : \mathbb{R}^{n-1} \to \mathbb{R}^n$. From [4], we know that x has $s(s \ge 3)$ distinct Laguerre principal curvatures:

$$B_i = \frac{r_i - r}{\sqrt{\sum_i (r_i - r)^2}}, \quad 1 \le i \le s,$$

where

$$r_i = \frac{1}{k_i}, \quad r = \frac{k_1 r_1 + k_2 r_2 + \cdots + k_s r_s}{n-1},$$

and $k_i \ne 0$ is the principal curvature corresponding to e_i. We also know that the Laguerre form is zero, $L_{ij} = 0$ for $1 \le i, j \le n-1$ and the Laguerre second fundamental form of x is parallel.

3. Propositions and lemmas

Throughout this section, we shall make the following convention on the ranges of indices:

$$1 \le a, b \le m_1, \quad m_1 + 1 \le p, q \le m_1 + m_2,$$
$$m_1 + m_2 + 1 \le \alpha, \beta \le m_1 + m_2 + m_3 = n-1, \quad 1 \le i, j, k \le n-1.$$

Let L, B and D denote the $n \times n$-symmetric matrices (L_{ij}), (B_{ij}) and (D_{ij}), respectively, where L_{ij}, B_{ij} and D_{ij} are defined by (2.10), (2.11) and (2.23). From (2.22) and (2.23), we know that $BL = LB$, $DL = LD$ and $BD = DB$. Thus, we may choose a local orthonormal basis $\{E_1, E_2, \ldots, E_n\}$ such that

$$L_{ij} = L_i \delta_{ij}, \quad B_{ij} = B_i \delta_{ij}, \quad D_{ij} = D_i \delta_{ij},$$

where L_i, B_i and D_i are the Laguerre Blaschke eigenvalues, the Laguerre principal curvatures and the Laguerre para-Blaschke eigenvalues of the immersion x.

In the proof of the following propositions and theorems, we agree on the fact that a local orthonormal basis $\{E_1, E_2, \ldots, E_n\}$ may be always chosen such that $L_{ij} = L_i\delta_{ij}, B_{ij} = B_i\delta_{ij}, D_{ij} = D_i\delta_{ij}$.

Proposition 3.1. *Let $x : M \to \mathbb{R}^n$ be an $n-1$-dimensional hypersurface with vanishing Laguerre form in $\mathbb{R}^n$. If the multiplicity of a Laguerre principal curvature is constant and greater than 1, then this Laguerre principal curvature is constant along its leaf.*

Proof. Let B_i, $i = 1, \ldots, n-1$, be the Laguerre principal curvatures of x with constant multiplicities. We choose a local orthonormal frame $\{E_1, \ldots, E_{n-1}\}$ such that E_i is a unit principal vector with respect to B_i. From (2.15), we have

$$B_{ij,k} = E_k(B_i)\delta_{ij} + \Gamma^j_{ik}(B_i - B_j), \tag{3.1}$$

where Γ^j_{ik} is the Levi-Civita connection for the Laguerre metric g given by

$$\omega_{ij} = \sum_k \Gamma^j_{ik}\omega_k, \quad \Gamma^j_{ik} = -\Gamma^i_{jk}. \tag{3.2}$$

From (2.22), we know that $B_{ii,j} = B_{ij,i}$. Thus from (3.1), we get

$$E_j(B_i) = \Gamma^j_{ii}(B_i - B_j) \quad \text{for} \quad i \neq j. \tag{3.3}$$

Without loss of generality, we may assume that B_1 is the Laguerre principal curvature of x with constant multiplicity m_1 and $m_1 \geq 2$, that is, for $1 \leq a \leq m_1$ we have $B_a = B_1$. From (3.3), we have

$$E_a(B_1) = \Gamma^a_{11}(B_1 - B_a) = 0 \quad \text{for} \quad a \neq 1,$$

and

$$E_1(B_1) = E_1(B_a) = \Gamma^1_{aa}(B_a - B_1) = 0 \quad \text{for} \quad a \neq 1.$$

Thus

$$E_a(B_1) = 0 \quad \text{for any} \quad a.$$

This implies that B_1 is constant along its leaf. We complete the proof of Proposition 3.1. □

We may prove the following proposition by reasoning as in [2].

Proposition 3.2. *Let $x : M \to \mathbb{R}^n$ be an $n-1$-dimensional hypersurface in $\mathbb{R}^n (n \geq 5)$ with vanishing Laguerre form and three distinct Laguerre principal curvatures B_1, B_2, B_3 one of which is simple. Then either B_1, B_2, B_3 are constants or $B_{ap,n-1} = 0$ for every a, p.*

Proof. From (2.16), (2.8) and (3.2), the curvature tensor of x may be given by (see [2])

$$R_{ijkl} = E_k(\Gamma^j_{il}) - E_l(\Gamma^j_{ik}) + \sum_m \Gamma^j_{im}\Gamma^m_{kl} \tag{3.4}$$

$$-\sum_m \Gamma^j_{im}\Gamma^m_{lk} + \sum_m \Gamma^m_{il}\Gamma^j_{mk} - \sum_m \Gamma^m_{ik}\Gamma^j_{ml}.$$

Since x has three distinct Laguerre principal curvatures B_1, B_2, B_3 one of which is simple and $n \geq 5$, without loss of generality, we may assume that $m_3 = 1$, $m_1 \geq 1$ and $m_2 \geq 2$. From (2.17), we have

$$\begin{aligned} &m_1 dB_1 + m_2 dB_2 + m_3 dB_3 = 0,\\ &m_1 B_1 dB_1 + m_2 B_2 dB_2 + m_3 B_3 dB_3 = 0. \end{aligned}$$

Thus

$$\frac{m_1 dB_1}{B_3 - B_2} = \frac{m_2 dB_2}{B_1 - B_3} = \frac{m_3 dB_3}{B_2 - B_1}. \tag{3.5}$$

From Proposition 3.1 and (3.5), we have

$$E_p(B_2) = E_p(B_1) = E_p(B_3) = 0, \tag{3.6}$$

and from (3.1), we have

$$\Gamma^p_{ab} = \Gamma^\alpha_{ab} = 0, a \neq b, \quad \Gamma^\alpha_{pq} = 0, p \neq q, \quad \Gamma^p_{aa} = \Gamma^p_{bb}, \quad \Gamma^\alpha_{aa} = \Gamma^\alpha_{bb}, \tag{3.7}$$

$$\Gamma^p_{a\alpha} = \frac{B_{ap,\alpha}}{B_1 - B_2}, \quad \Gamma^a_{\alpha p} = \frac{B_{\alpha a,p}}{B_3 - B_1}, \quad \Gamma^\alpha_{pa} = \frac{B_{p\alpha,a}}{B_2 - B_3}. \tag{3.8}$$

(i) If $m_1 \geq 2$, from Proposition 3.1 and (3.5), we have

$$E_a(B_1) = E_a(B_2) = E_a(B_3) = 0. \tag{3.9}$$

From (3.1), (3.3), (3.6) and (3.9), we have

$$\Gamma^p_{ab} = \Gamma^a_{pq} = 0, \quad \Gamma^a_{n-1n-1} = \Gamma^p_{n-1n-1} = 0, \tag{3.10}$$

$$\Gamma^{n-1}_{aa} = \frac{E_{n-1}(B_1)}{B_1 - B_3}, \quad \Gamma^{n-1}_{pp} = \frac{E_{n-1}(B_2)}{B_2 - B_3}. \tag{3.11}$$

From (3.8), we have

$$\begin{aligned} &\Gamma^p_{an-1} = \frac{B_{ap,n-1}}{B_1 - B_2}, \quad \Gamma^p_{n-1b} = \frac{B_{bp,n-1}}{B_3 - B_2}, \quad \Gamma^{n-1}_{bq} = \frac{B_{bq,n-1}}{B_1 - B_3},\\ &\Gamma^{n-1}_{qb} = \frac{B_{bq,n-1}}{B_2 - B_3}. \end{aligned} \tag{3.12}$$

Thus, from (3.4), (3.7) and (3.10)–(3.12), we have

$$\begin{aligned} R_{apbq} &= E_b(\Gamma^p_{aq}) - E_q(\Gamma^p_{ab}) + \sum_m \Gamma^p_{am}\Gamma^m_{bq}\\ &\quad - \sum_m \Gamma^p_{am}\Gamma^m_{qb} + \sum_m \Gamma^m_{aq}\Gamma^p_{mb} - \sum_m \Gamma^m_{ab}\Gamma^p_{mq}\\ &= \Gamma^p_{an-1}\Gamma^{n-1}_{bq} - \Gamma^p_{an-1}\Gamma^{n-1}_{qb} + \Gamma^{n-1}_{aq}\Gamma^p_{n-1b} - \Gamma^{n-1}_{ab}\Gamma^{n-1}_{pq}\\ &= \frac{B_{ap,n-1}B_{bq,n-1} + B_{aq,n-1}B_{bp,n-1} + E_{n-1}(B_1)E_{n-1}(B_2)\delta_{ab}\delta_{pq}}{(B_1 - B_3)(B_3 - B_2)}. \end{aligned} \tag{3.13}$$

On the other hand, from (2.21), we have

$$R_{apbq} = -(L_a + L_p)\delta_{ab}\delta_{pq}. \tag{3.14}$$

(3.13) and (3.14) imply that

$$\begin{aligned} &B_{ap,n-1}B_{bq,n-1} + B_{aq,n-1}B_{bp,n-1} \\ &\quad = \{-(L_a + L_p)(B_1 - B_3)(B_3 - B_2) - E_{n-1}(B_1)E_{n-1}(B_2)\}\delta_{ab}\delta_{pq}. \end{aligned}$$

Putting

$$\varrho_{a,p} = -(L_a + L_p)(B_1 - B_3)(B_3 - B_2) - E_{n-1}(B_1)E_{n-1}(B_2), \tag{3.15}$$

we get

$$B_{ap,n-1}B_{bq,n-1} + B_{aq,n-1}B_{bp,n-1} = \varrho_{a,p}\delta_{ab}\delta_{pq}.$$

If $a = b$, we have

$$2B_{ap,n-1}B_{aq,n-1} = \varrho_{a,p}\delta_{pq}. \tag{3.16}$$

From (3.15), we know that $\varrho_{a,p}$ depends on a, p. If $L_1 = L_2 = \cdots = L_{n-1}$, from (3.15), we see that for any a, p, all $\varrho_{a,p}$ are equal. If there is p_0, such that $B_{ap_0,n-1} \neq 0$, $1 \leq a \leq m_1$. By (3.16), we have $B_{ap,n-1} = 0$ for other $p(p \neq p_0)$. By (3.16) again, if $p = q$, then $B^2_{ap,n-1} = \frac{\varrho_{a,p}}{2}$ for any p. Since for any a, p, all $\varrho_{a,p}$ are equal, we have $B^2_{ap_0,n-1} = \frac{\varrho_{a,p_0}}{2} = \frac{\varrho_{a,p}}{2} = B^2_{ap,n-1} = 0$ for $p_0, p(p \neq p_0)$. Thus $B_{ap_0,n-1} = 0$, this is a contradiction. Therefore we know that $B_{ap,n-1} = 0$ for any a, p.

If at least two of $L_1, L_2, \ldots, L_{n-1}$ are not equal, since $m_2 \geq 2$ and $m_1 \geq 2$, we may prove that there exists at most one p, such that $\varrho_{a,p} \neq 0$ for any a, $1 \leq a \leq m_1$ and there exists at most one a, such that $\varrho_{a,p} \neq 0$ for any p, $m_1 + 1 \leq p \leq m_1 + m_2$. In fact, if there exists more than one p, for example p_1, p_2, $(p_1 \neq p_2)$ such that $\varrho_{a,p_1} \neq 0$, $\varrho_{a,p_2} \neq 0$. By (3.16), we have $B^2_{ap,n-1} = \frac{\varrho_{a,p}}{2}$ for any p. Thus $B^2_{ap_1,n-1} = \frac{\varrho_{a,p_1}}{2} \neq 0$, $B^2_{ap_2,n-1} = \frac{\varrho_{a,p_2}}{2} \neq 0$. By (3.16) again, we see that $B_{ap_1,n-1}B_{ap_2,n-1} = 0$, this is a contradiction. Thus, we know that there exists at most one p, such that $\varrho_{a,p} \neq 0$ for any a, $1 \leq a \leq m_1$. By the same proof as above, we also know that there exists at most one a, such that $\varrho_{a,p} \neq 0$ for any p, $m_1 + 1 \leq p \leq m_1 + m_2$.

If there exists at most one p, such that $\varrho_{a,p} \neq 0$ for any a, possibly, say $\varrho_{a,p_0} \neq 0$ for any a. From (3.15), we have

$$L_a + L_p = -\frac{E_{n-1}(B_1)E_{n-1}(B_2)}{(B_1 - B_3)(B_3 - B_2)}, \quad p \neq p_0. \tag{3.17}$$

By (3.17), we know that $L_a = L_b$ for any a, b. On the other hand, since we know that there exists at most one a, such that $\varrho_{a,p} \neq 0$ for any p, possibly, say $\varrho_{a_0,p} \neq 0$ for any p. By (3.15), we also have

$$L_a + L_p = -\frac{E_{n-1}(B_1)E_{n-1}(B_2)}{(B_1 - B_3)(B_3 - B_2)}, \quad a \neq a_0,$$

and we know that $L_p = L_q$ for any p, q. Thus, from (3.15) we see that only $\varrho_{a,p} = 0$ for any a, p holds exactly. Thus, by (3.16), we have $B_{ap,n-1} = 0$ for any p and a.

(ii) If $m_1 = 1$, from (3.3) and (3.7), we have

$$\Gamma^1_{pq} = \Gamma^{n-1}_{pq} = 0, \quad \Gamma^p_{n-1n-1} = \Gamma^p_{11} = 0, \tag{3.18}$$

$$\begin{aligned} \Gamma^1_{n-1n-1} &= \frac{E_1(B_3)}{B_3 - B_1}, \quad \Gamma^1_{pp} = \frac{E_1(B_2)}{B_2 - B_1}, \\ \Gamma^{n-1}_{11} &= \frac{E_{n-1}(B_1)}{B_1 - B_3}, \quad \Gamma^{n-1}_{pp} = \frac{E_{n-1}(B_2)}{B_2 - B_3}. \end{aligned} \tag{3.19}$$

From (3.4), (3.7), (3.8), (3.18) and (3.19), by a similar calculation as in (i), we have

$$2B_{1p,n-1}B_{1q,n-1} = \upsilon_p \delta_{pq} \tag{3.20}$$

for any p, q, where

(3.21)

$$\begin{aligned} \upsilon_p = (B_1 - B_2)(B_1 - B_3)\Bigg\{&-(L_p + L_{n-1}) + \frac{E_1(B_2)E_1(B_3)}{(B_1 - B_2)(B_1 - B_3)} \\ &+ \frac{[E_{n-1}(B_2) - E_{n-1}(B_3)]E_{n-1}(B_2)}{(B_2 - B_3)^2} - \frac{E_{n-1}(E_{n-1}(B_2))}{B_2 - B_3} + \frac{E_{n-1}(B_2)}{(B_2 - B_3)^2}\Bigg\}. \end{aligned}$$

From (3.21), we know that υ_p depends on p. If $L_1 = L_2 = \cdots = L_{n-1}$, from (3.21), we see that for any p, all υ_p are equal. By the same proof as in (i), we know that $B_{1p,n-1} = 0$ for any p.

If at least two of $L_1, L_2, \ldots, L_{n-1}$ are not equal, since $m_2 \geq 2$, by the same proof as in (i), we easily know that there exists at most one p, such that $\upsilon_p \neq 0$.

If for any p, $\upsilon_p = 0$, by (3.20), we have $B_{1p,n-1} = 0$.

If there is p_0, such that $\upsilon_{p_0} \neq 0$ and $\upsilon_p = 0$, for other $p (p \neq p_0)$, we have

$$\upsilon_{p_0} = \upsilon_{p_0} - \upsilon_p = (B_1 - B_2)(B_1 - B_3)(L_{p_0} - L_p). \tag{3.22}$$

On the other hand, since $m_1 = 1$, $m_3 = 1$ and $B_{ij,k}$ is symmetric for all indices i, j, k, interchanging 1 and n in the above equations, we also have,

$$2B_{n-1p,1}B_{n-1q,1} = \omega_p \delta_{pq}, \tag{3.23}$$

where

$$\begin{aligned} \omega_p = (B_3 - B_2)(B_3 - B_1)\Bigg\{&-(L_p + L_1) + \frac{E_{n-1}(B_2)E_{n-1}(B_1)}{(B_3 - B_2)(B_3 - B_1)} \\ &+ \frac{[E_1(B_2) - E_1(B_1)]E_1(B_2)}{(B_2 - B_1)^2} - \frac{E_1(E_1(B_2))}{B_2 - B_1} + \frac{E_1(B_2)}{(B_2 - B_1)^2}\Bigg\}. \end{aligned} \tag{3.24}$$

From (3.24), we know that ω_p depends on p. By the same assertion as above, we know that there exists at most one p, such that $\omega_p \neq 0$.

If for any p, $\omega_p = 0$, by (3.23), we have $B_{1p,n-1} = 0$. Otherwise, we may prove that $\omega_{p_0} \neq 0$ for the above p_0 in (3.22). In fact, by (3.20), we have

$B^2_{1p_0,n-1} = \frac{\upsilon_{p_0}}{2} \neq 0$. On the other hand, by (3.23), we have $B^2_{n-1p_0,1} = \frac{\omega_{p_0}}{2}$. Since $B_{1p_0,n-1} = B_{n-1p_0,1}$, we have $\omega_{p_0} = \upsilon_{p_0} \neq 0$. Since there exists at most one p, such that $\omega_p \neq 0$, we know that for other $p (p \neq p_0)$, $\omega_p = 0$. By (3.24), we also have

$$\upsilon_{p_0} = \omega_{p_0} = \omega_{p_0} - \omega_p = (B_3 - B_2)(B_3 - B_1)(L_{p_0} - L_p). \tag{3.25}$$

Thus, from (3.22) and (3.25), we have

$$(B_1 - B_3)(B_1 - 2B_2 + B_3)(L_{p_0} - L_p) = 0.$$

If $L_{p_0} = L_p$, by (3.22), we have $\upsilon_{p_0} = 0$, this contradicts with $\upsilon_{p_0} \neq 0$. Thus

$$B_1 - 2B_2 + B_3 = 0,$$

and

$$dB_1 - 2dB_2 + dB_3 = 0. \tag{3.26}$$

From (3.5) and (3.26), we easily know that $dB_1 = dB_2 = dB_3 = 0$, that is, B_1, B_2, B_3 are constants. We complete the proof of Proposition 3.2. ☐

4. Proof of theorems

Proof of Theorem 1.2. Let B_1, B_2, B_3 be the three distinct Laguerre principal curvatures of multiplicities m_1, m_2, m_3 and one of which is simple. Since $n \geq 5$, without loss of generality, we may assume that $m_3 = 1$, $m_1 \geq 1$ and $m_2 \geq 2$. By Proposition 3.2, we know that either B_1, B_2, B_3 are constants or $B_{ap,n-1} = 0$ for any a, p. In the first case, we know that x is a Laguerre isoparametric hypersurface with non-parallel Laguerre second fundament form. In the second case, if $B_{ap,n-1} = 0$ for any a, p, we may consider two cases:

(i) If $m_1 \geq 2$, since $B_{ap,n-1} = 0$ for every a, p, setting $p = q$ in (3.16), we have $\varrho_{a,p} = 0$ for any a, p. From (3.15), we get for any $1 \leq a \leq m_1$ and $m_1 + 1 \leq p \leq m_1 + m_2$,

$$L_a + L_p = -\frac{E_{n-1}(B_1)E_{n-1}(B_2)}{(B_1 - B_3)(B_3 - B_2)}. \tag{4.1}$$

Thus, we know that $L_a = L_b$ for any a, b and $L_p = L_q$ for any p, q. This implies that x has at most three distinct Laguerre Blaschke eigenvalues L_a, L_p, L_{n-1} with multiplicities m_1, m_2, 1 and $m_1 \geq 2, m_2 \geq 2$.

(ii) If $m_1 = 1$, since $B_{1p,n-1} = 0$ for any p, setting $p = q$ in (3.20), we have $\upsilon_p = 0$ for any p, $m_1 + 1 \leq p \leq m_1 + m_2$. By (3.21),

$$\begin{aligned} L_p = & -L_{n-1} + \frac{E_1(B_2)E_1(B_3)}{(B_1 - B_2)(B_1 - B_3)} \\ & + \frac{[E_{n-1}(B_2) - E_{n-1}(B_3)]E_{n-1}(B_2)}{(B_2 - B_3)^2} - \frac{E_{n-1}(E_{n-1}(B_2))}{B_2 - B_3} + \frac{E_{n-1}(B_2)}{(B_2 - B_3)^2}. \end{aligned} \tag{4.2}$$

Thus, we know that x has at most three distinct Laguerre Blaschke eigenvalues L_1, L_p, L_{n-1} with multiplicities 1, m_2, 1 and $m_2 \geq 2$.

Next, we may prove that $x : M \to \mathbb{R}^n$ is only a Laguerre isotropic hypersurface, that is, the number of distinct Laguerre Blaschke eigenvalues is only 1. In fact, if L_a, L_p, L_{n-1} are the three constant Laguerre Blaschke eigenvalues with multiplicities m_1, m_2, 1 and the number of distinct Laguerre Blaschke eigenvalues is 2 or 3. From (2.14), we have

$$L_{ij,k} = E_k(L_i)\delta_{ij} + \Gamma^j_{ik}(L_i - L_j), \tag{4.3}$$

where Γ^j_{ik} is the Levi-Civita connection for the Laguerre metric g.

From (2.18), we know that $L_{ii,j} = L_{ij,i}$. Thus

$$E_j(L_i) = \Gamma^j_{ii}(L_i - L_j) \quad \text{for} \quad i \neq j. \tag{4.4}$$

If the number of distinct Laguerre Blaschke eigenvalues is 3, that is, $L_a \neq L_p \neq L_{n-1}$, $1 \leq a \leq m_1$, $m_1 + 1 \leq p \leq m_1 + m_2$ and $m_2 \geq 2$, from (4.4), we have

$$0 = E_a(L_p) = \Gamma^a_{pp}(L_p - L_a), \quad 0 = E_{n-1}(L_p) = \Gamma^{n-1}_{pp}(L_p - L_{n-1}). \tag{4.5}$$

Thus $\Gamma^a_{pp} = \Gamma^{n-1}_{pp} = 0$.

On the other hand, from (2.15), we have

$$B_{ij,k} = E_k(B_i)\delta_{ij} + \Gamma^j_{ik}(B_i - B_j), \tag{4.6}$$

where Γ^j_{ik} is the Levi-Civita connection for the Laguerre metric g.

From (2.22), we know that $B_{ii,j} = B_{ij,i}$. Thus

$$E_j(B_i) = \Gamma^j_{ii}(B_i - B_j) \quad \text{for} \quad i \neq j. \tag{4.7}$$

We get

$$E_a(B_p) = \Gamma^a_{pp}(B_p - B_a) = 0, \quad E_{n-1}(B_p) = \Gamma^{n-1}_{pp}(B_p - B_{n-1}) = 0.$$

That is, $E_a(B_2) = 0$, $E_{n-1}(B_2) = 0$. Since we assume that $m_2 \geq 2$, from Proposition 3.1, we have $E_p(B_2) = 0$. Thus, B_2 is constant. Therefore, from (3.5), we know that B_1 and B_3 are constants.

If the number of distinct Laguerre Blaschke eigenvalues is 2, when $m_1 \geq 2$, without loss of the generality, we may assume that $L_a = L_p \neq L_{n-1}$. By (4.4), we have

$$0 = E_{n-1}(L_a) = \Gamma^{n-1}_{aa}(L_a - L_{n-1}).$$

Thus, $\Gamma^{n-1}_{aa} = 0$. On the other hand, by (4.7)

$$E_{n-1}(B_1) = \Gamma^{n-1}_{aa}(B_1 - B_3) = 0.$$

From (3.5), we have

$$\frac{m_1 E_i(B_1)}{B_3 - B_2} = \frac{m_2 E_i(B_2)}{B_1 - B_3} = \frac{m_3 E_i(B_3)}{B_2 - B_1}. \tag{4.8}$$

By Proposition 3.1, we have $E_a(B_1) = 0$, $1 \leq a \leq m_1$ and $E_p(B_2) = 0$, $m_1 + 1 \leq p \leq m_1 + m_2$. By (4.8), we have $E_a(B_3) = E_p(B_3) = E_{n-1}(B_3) = 0$, that is, B_3 is constant. By (3.5) again, we know that B_1 and B_2 are constants.

When $m_1 = 1$, without loss of the generality, we may assume that $L_1 = L_{n-1} \neq L_p$. By (4.4), we have

$$0 = E_1(L_p) = \Gamma^1_{pp}(L_p - L_1), \quad 0 = E_{n-1}(L_p) = \Gamma^{n-1}_{pp}(L_p - L_{n-1}).$$

Thus $\Gamma^1_{pp} = \Gamma^{n-1}_{pp} = 0$. On the other hand, by (4.7)

$$E_1(B_2) = \Gamma^1_{pp}(B_2 - B_1) = 0, \quad E_{n-1}(B_2) = \Gamma^{n-1}_{pp}(B_2 - B_3) = 0.$$

From Proposition 3.1, we have $E_p(B_2) = 0$, $2 \leq p \leq 1 + m_2$. Thus B_2 is constant. By (3.5) again, we know that B_1 and B_3 are constants.

To sum up, we know that if the number of distinct Laguerre Blaschke eigenvalues is 2 or 3, then B_1, B_2 and B_3 are constants.

From (4.6), we have $B_{ab,k} = B_{pq,k} = B_{\alpha\beta,k} = 0$ for any $a, b, p, q, \alpha, \beta, k$. Since we know that $B_{ap,n-1} = 0$ for every a, p, we get $B_{ij,k} = 0$ for any i, j, k, that is, the Laguerre second fundamental form of x is parallel. From (2.15), it follows that

$$0 = dB_i\delta_{ij} + (B_i - B_j)\omega_{ij}, \quad 1 \leq i, j \leq n-1. \tag{4.9}$$

If $B_i \neq B_j$, we have $\omega_{ij} = 0$. If for some k such that $\omega_{ik} \neq 0$ and $\omega_{kj} \neq 0$, by (4.9) we have $B_i = B_k = B_j$, this is in contradiction with $B_i \neq B_j$. Thus, from (2.16), we have $R_{ijij} = 0$ for $B_i \neq B_j$. From (2.21), it follows that

$$L_i + L_j = 0 \ \text{ for } \ B_i \neq B_j. \tag{4.10}$$

Let $B_1 = B_a$, $B_2 = B_p$, $B_3 = B_\alpha$ be the three distinct Laguerre principal curvatures with multiplicities m_1, m_2, m_3 one of which is simple, where $1 \leq a \leq m_1$, $m_1 + 1 \leq p \leq m_1 + m_2$, $m_1 + m_2 + 1 \leq \alpha \leq n - 1$. Since $B_a \neq B_p$, $B_a \neq B_\alpha$ and $B_p \neq B_\alpha$, from (4.10), we have $L_a + L_p = 0$, $L_a + L_\alpha = 0$ and $L_p + L_\alpha = 0$. This implies that $L_a = 0$, $L_p = 0$ and $L_\alpha = 0$ for any a, p, α. That is $L_i = 0$ for any i. This is a contradiction with the assumption that the number of distinct Laguerre Blaschke eigenvalues is 2 or 3. Therefore, we know that the number of distinct Laguerre Blaschke eigenvalues is only 1, that is, x is only a Laguerre isotropic hypersurface. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. By the same assertion as in the proof of Theorem 1.2, we know that either B_1, B_2, B_3 are constants and x is a Laguerre isoparametric hypersurface with non-parallel Laguerre second fundament form, or $B_{ap,n-1} = 0$ for any a, p.

If $B_{ap,n-1} = 0$ for any a, p, we may consider two cases:

(i) If $m_1 \geq 2$, since $B_{ap,n-1} = 0$ for every a, p, setting $p = q$ in (3.16), we have $\varrho_{a,p} = 0$ for any a, p. From (3.15) and (2.23), we get for any $1 \leq a \leq m_1$ and $m_1 + 1 \leq p \leq m_1 + m_2$,

$$D_a + D_p = \lambda(B_1 + B_2) - \frac{E_{n-1}(B_1)E_{n-1}(B_2)}{(B_1 - B_3)(B_3 - B_2)}. \tag{4.11}$$

From (4.11), we know that for any a, all D_a are the same and for any p, all D_p are the same. Thus, we know that x has at most three distinct Laguerre para-Blaschke eigenvalues D_a, D_p, D_{n-1} with multiplicities m_1, m_2, 1 and $m_1 \geq 2, m_2 \geq 2$.

(ii) If $m_1 = 1$, since $B_{1p,n-1} = 0$ for any p, setting $p = q$ in (3.20), we have $\upsilon_p = 0$ for any p, $m_1 + 1 \leq p \leq m_1 + m_2$. By (3.21) and (2.23),

(4.12)
$$\begin{aligned} D_p = &-D_{n-1} + \lambda(B_2 + B_3) + \frac{E_1(B_2)E_1(B_3)}{(B_1 - B_2)(B_1 - B_3)} \\ &+ \frac{[E_{n-1}(B_2) - E_{n-1}(B_3)]E_{n-1}(B_2)}{(B_2 - B_3)^2} - \frac{E_{n-1}(E_{n-1}(B_2))}{B_2 - B_3} + \frac{E_{n-1}(B_2)}{(B_2 - B_3)^2}. \end{aligned}$$

From (4.12), we know that for any p, all D_p are the same. Thus, x has at most three distinct para-Blaschke eigenvalues D_1, D_p, D_{n-1} with multiplicities 1, m_2, 1 and $m_2 \geq 2$.

If the number of the distinct Laguerre para-Blaschke eigenvalues of D_a, D_p, D_{n-1} is 1, then x is a Laguerre para-isotropic hypersurface.

If the number of the distinct Laguerre para-Blaschke eigenvalues of D_a, D_p, D_{n-1} is 2, we may prove that this case does not occur. In fact, if $m_1 \geq 2$, without loss of the generality, we may assume that $D_a = D_p \neq D_{n-1}$. From (2.24), we have

$$D_{ij,k} = E_k(D_i)\delta_{ij} + \Gamma_{ik}^j(D_i - D_j),$$

where Γ_{ik}^j is the Levi-Civita connection for the Laguerre metric g.

From (2.25), we know that $D_{ii,j} = D_{ij,i}$. Thus

$$E_j(D_i) = \Gamma_{ii}^j(D_i - D_j) \quad \text{for} \quad i \neq j. \tag{4.13}$$

By (4.13), we have

$$0 = E_{n-1}(D_a) = \Gamma_{aa}^{n-1}(D_a - D_{n-1}).$$

Thus, $\Gamma_{aa}^{n-1} = 0$.

From (4.7), we get

$$E_{n-1}(B_1) = \Gamma_{aa}^{n-1}(B_1 - B_3) = 0.$$

Combining with $E_a(B_1) = 0$, $1 \leq a \leq m_1$, $E_p(B_2) = 0$, $m_1 + 1 \leq p \leq m_1 + m_2$, (4.8) and (3.5), we easily see that B_1, B_2 and B_3 are constants.

If $m_1 = 1$, without loss of the generality, we may assume that $D_1 = D_{n-1} \neq D_p$. By (4.13), we have

$$0 = E_1(D_p) = \Gamma_{pp}^1(D_p - D_1), \quad 0 = E_{n-1}(D_p) = \Gamma_{pp}^{n-1}(D_p - D_{n-1}).$$

Thus $\Gamma_{pp}^1 = \Gamma_{pp}^{n-1} = 0$. On the other hand, by (4.7)

$$E_1(B_2) = \Gamma_{pp}^1(B_2 - B_1) = 0, \quad E_{n-1}(B_2) = \Gamma_{pp}^{n-1}(B_2 - B_3) = 0.$$

Combining with $E_p(B_2) = 0$, $2 \leq p \leq 1 + m_2$ and (3.5), we easily see that B_1, B_2 and B_3 are constants.

By the same assertion as in the proof of Theorem 1.2, we know that the Laguerre second fundamental form of x is parallel and $L_i = 0$ for any i. Since $\lambda \neq 0$, it follows that x has three distinct Laguerre para-Blaschke eigenvalues $\lambda B_1, \lambda B_2, \lambda B_3$. This is a contradiction.

If the number of the distinct Laguerre para-Blaschke eigenvalues of D_a, D_p, D_{n-1} is 3, that is, $D_a \neq D_p \neq D_{n-1}$, $1 \leq a \leq m_1$, $m_1 + 1 \leq p \leq m_1 + m_2$ and $m_2 \geq 2$, from (4.13), we have

$$0 = E_a(D_p) = \Gamma^a_{pp}(D_p - D_a), \quad 0 = E_{n-1}(D_p) = \Gamma^{n-1}_{pp}(D_p - D_{n-1}). \tag{4.14}$$

Thus $\Gamma^a_{pp} = \Gamma^{n-1}_{pp} = 0$.

By (4.7), we get

$$E_a(B_p) = \Gamma^a_{pp}(B_p - B_a) = 0, \quad E_{n-1}(B_p) = \Gamma^{n-1}_{pp}(B_p - B_{n-1}) = 0.$$

That is, $E_a(B_2) = 0, \quad E_{n-1}(B_2) = 0$. Combining with $E_p(B_2) = 0$ and (3.5), we know that B_1, B_2 and B_3 are constants.

By the same assertion as in the proof of Theorem 1.2, we know that the Laguerre second fundamental form of x is parallel and $L_i = 0$ for any i. From the result of Theorem 1.1 and Example 2.1–Example 2.2, we know that x is Laguerre equivalent to an open part of the image of τ of the oriented hypersurface $x : \mathbb{R}^{n-1} \to \mathbb{R}^n_0$ given by Example 2.2. This completes the proof of Theorem 1.3. □

Acknowledgment. The authors would like to thank the referee for his/her many valuable remarks and suggestions that really improve the paper.

References

[1] W. Blaschke, *Vorlesungen über Differential geometrie*, Springer, Berlin, Heidelberg, New York, Vol. 3, 1929.

[2] G. H. Li, *Möbius hypersurfaces in S^{n+1} with three distinct principal curvatures*, J. Geom. **80** (2004), no. 1-2, 154–165.

[3] T. Z. Li, *Laguerre geometry of surfaces in $\mathbb{R}^3$*, Acta Math. Sin. (Engl. Ser.) **21** (2005), no. 6, 1525–1534.

[4] T. Z. Li, H. Z. Li, and C. P. Wang, *Classification of hypersurfaces with parallel Laguerre second fundamental form in $\mathbb{R}^n$*, Differential Geom. Appl. **28** (2010), no. 2, 148–157.

[5] T. Z. Li and C. P. Wang, *Laguerre geometry of hypersurfaces in $\mathbb{R}^n$*, Manuscripta Math. **122** (2007), no. 1, 73–95.

[6] E. Musso and L. Nicolodi, *A variational problem for surfaces in Laguerre geometry*, Trans. Amer. Math. Soc. **348** (1996), no. 11, 4321–4337.

[7] ______, *Laguerre geometry of surfaces with plane lines of curvature*, Abh. Math. Sem. Univ. Hamburg **69** (1999), 123–138.

Shichang Shu
School of Mathematics and Information Science
Xianyang Normal University
Xianyang 712000 Shaanxi, P. R. China
E-mail address: `shushichang@126.com`

Yanyan Li
School of Mathematics and Information Science
Xianyang Normal University
Xianyang 712000 Shaanxi, P. R. China
E-mail address: `muzilyy@163.com`