Journal of the Korean Data & Information Science Society
2013, **24**(6), 1455–1464

http://dx.doi.org/10.7465/jkdi.2013.24.6.1455
한국데이터정보과학회지

Bayesian analysis of an exponentiated half-logistic distribution under progressively type-II censoring

Suk Bok Kang[1] · Jung In Seo[2] · Yongku Kim[3]

[12]Department of Statistics, Yeungnam University
[3]Department of Statistics, Kyungpook National University

Received 30 June 2013, revised 24 July 2013, accepted 16 August 2013

Abstract

This paper develops maximum likelihood estimators (MLEs) of unknown parameters in an exponentiated half-logistic distribution based on a progressively type-II censored sample. We obtain approximate confidence intervals for the MLEs by using asymptotic variance and covariance matrices. Using importance sampling, we obtain Bayes estimators and corresponding credible intervals with the highest posterior density and Bayes predictive intervals for unknown parameters based on progressively type-II censored data from an exponentiated half logistic distribution. For illustration purposes, we examine the validity of the proposed estimation method by using real and simulated data.

Keywords: Bayes predictive interval, exponentiated half-logistic distribution, HPD credible interval, importance sampling, progressively type-II censored sample.

1. Introduction

Many reliability and survival analysis studies have used half-logistic distributions, particularly for censored data. Several studies have drawn inferences about half logistic distributions. Balakrishnan and Puthenpura (1986) introduce the best linear unbiased estimator of location and scale parameters of half-logistic distributions by considering linear functions of order statistics. Balakrishnan and Wong (1991) obtain approximate maximum likelihood estimators (AMLEs) of location and scale parameters of half-logistic distributions by using a type-II right-censored sample. Kang *et al.* (2008) derive AMLEs and MLEs of scale parameters of half-logistic distributions by using progressively type-II censored samples. Kang *et al.* (2009) propose AMLEs of scale parameters of half-logistic distributions by considering double-hybrid censored samples. Kang and Seo (2011) recently examine exponentiated half-logistic distributions and propose two types of AMLEs of scale parameters and reliability functions by using progressively type-II right-censored samples. Kim *et al.* (2011a) derive Bayesian estimators of shape parameters, reliability functions, and failure rate functions

[1] Professor, Department of Statistics, Yeungnam University, Kyungsan 712-749, Korea.
[2] Ph.D student, Department of Statistics, Yeungnam University, Kyungsan 712-749, Korea.
[3] Corresponding author: Assistant professor, Department of Statistics, Kyungpook National University, Daegu 702-701, Korea. E-mail: kim.1252@knu.ac.kr

of exponentiated half-logistic and half-triangle distributions by using type-II right-censored data (Kim *et al.*, 2011b, 2011c).

The probability density function (PDF) and cumulative distribution function (CDF) of a random variable X with an exponentiated half-logistic distribution is given by

$$f(x) = \frac{\lambda}{\sigma}\left(\frac{1-e^{-x/\sigma}}{1+e^{-x/\sigma}}\right)^{\lambda-1}\frac{2e^{-x/\sigma}}{\left(1+e^{-x/\sigma}\right)^2} \tag{1.1}$$

and

$$F(x) = \left(\frac{1-e^{-x/\sigma}}{1+e^{-x/\sigma}}\right)^{\lambda}, \quad x>0, \quad \lambda,\, \sigma>0, \tag{1.2}$$

where σ is the scale parameter and λ is the shape parameter. As a special case, if $\lambda = 1$, then this distribution is a half-logistic distribution.

The rest of this paper is organized as follows: Section 2 introduces a maximum likelihood estimation method, and Section 3 details the Bayesian estimation for a given setting and its computation. Section 4 examines the performance of the proposed approach through simulation studies.

2. Maximum likelihood estimation

In this section, we first develop MLEs of unknown parameters of exponentiated half-logistic distributions by using progressively type II censored samples and then construct asymptotic confidence intervals by using an asymptotic variance-covariance matrix.

Let us consider the following progressively type-II censoring scheme. Suppose n randomly selected units with exponentiated half logistic distribution in (1.1) were placed on a life test, only m are completely observed until failure. At the time of the first failure, r_1 of the $n-1$ surviving units are randomly withdrawn (or censored) from the life testing experiment. At the time of the next failures, r_2 of the $n-2-r_1$ surviving units are randomly censored, and so on. Finally, at the time of the mth failures, all the remaining $r_m = n-m-r_1-\cdots-r_{m-1}$ surviving units are censored.

Let $X_{1:m:n} \le X_{2:m:n} \le \cdots \le X_{m:m:n}$ denote such a progressively type-II censored sample and $(r_1, \ldots, r_m)$ be the progressively censoring scheme. Note that the case $m = n$ with $r_1 = \cdots = r_m = 0$ corresponds to the complete sample situation, whereas the case $r_1 = \cdots = r_{m-1} = 0$, $r_m = n-m$, the usual type-II censored sample.

We can express the likelihood function based on the progressively type-II censored sample as

$$\begin{aligned} L &= C\prod_{i=1}^{m} f(x_{i:m:n};\sigma,\,\lambda)\left[1-F(x_{i:m:n};\sigma,\,\lambda)\right]^{r_i} \\ &\propto \sigma^{-m}\lambda^{m}\exp\left[-\lambda\sum_{i=1}^{m}\log\left(\frac{1+e^{-x_i/\sigma}}{1-e^{-x_i/\sigma}}\right)-\frac{1}{\sigma}\sum_{i=1}^{m}x_i\right]\prod_{i=1}^{m}\frac{1}{1-e^{-2x_i/\sigma}}\left[1-\left(\frac{1-e^{-x_i/\sigma}}{1+e^{-x_i/\sigma}}\right)^{\lambda}\right]^{r_i} \end{aligned} \tag{2.1}$$

where $C = n(n-1-r_1)(n-2-r_1-r_2)\cdots(n-m+1-r_1-\cdots-r_{m-1})$.

Differentiating the natural logarithm of the likelihood function (2.1) for σ and λ, we have the likelihood equation for σ and λ:

$$\begin{aligned}\frac{\partial}{\partial\sigma}\log L(\sigma,\lambda) =& -\frac{1}{\sigma}\Bigg[m+\left(1-\frac{1}{\lambda}\right)\sum_{i=i}^{m}\frac{f(x_{i:m:n};\sigma,\lambda)}{F(x_{i:m:n};\sigma,\lambda)}\frac{x_{i:m:n}}{\sigma}-\sum_{i=1}^{m}[F(x_{i:m:n};\sigma,\lambda)]^{\frac{1}{\lambda}}\frac{x_{i:m:n}}{\sigma}\\ &-\sum_{i=1}^{m}r_i\frac{f(x_{i:m:n};\sigma,\lambda)}{1-F(x_{i:m:n};\sigma,\lambda)}\frac{x_{i:m:n}}{\sigma}\Bigg]\\ =&0 \end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\frac{\partial}{\partial\lambda}\log L(\sigma,\lambda) &= \frac{1}{\lambda}\left[m+\sum_{i=1}^{m}\left(1-r_i\frac{F(x_{i:m:n};\sigma,\lambda)}{1-F(x_{i:m:n};\sigma,\lambda)}\right)\log F(x_{i:m:n};\sigma,\lambda)\right]\\ &=0.\end{aligned} \tag{2.3}$$

We can find MLEs of σ and λ by solving equations (2.2) and (2.3), respectively. Unfortunately, these equations cannot be solved explicitly, and therefore we solve them by using numerical methods such as the Newton-Raphson method. MLEs of σ and λ are denoted by $\hat{\sigma}$ and $\hat{\lambda}$, respectively.

The asymptotic variance-covariance matrix of the MLEs $\hat{\sigma}$ and $\hat{\lambda}$ is given by

$$\hat{\Sigma} = \begin{pmatrix} -\frac{\partial^2\log L}{\partial\sigma^2} & -\frac{\partial^2\log L}{\partial\sigma\partial\lambda} \\ -\frac{\partial^2\log L}{\partial\lambda\partial\sigma} & -\frac{\partial\log L}{\partial\lambda^2}\end{pmatrix}^{-1}_{(\sigma,\lambda)=(\hat{\sigma},\hat{\lambda})} = \begin{pmatrix}\widehat{Var}(\hat{\sigma}) & \widehat{Cov}(\hat{\sigma},\hat{\lambda}) \\ \widehat{Cov}(\hat{\sigma},\hat{\lambda}) & \widehat{Var}(\hat{\lambda})\end{pmatrix} \tag{2.4}$$

By the asymptotic normality of the MLE, we can obtain the approximate confidence intervals of the scale parameter σ and the shape parameter λ as

$$\hat{\sigma}\pm Z_{\alpha/2}\sqrt{\widehat{Var}(\hat{\sigma})} \quad\text{and}\quad \hat{\lambda}\pm Z_{\alpha/2}\sqrt{\widehat{Var}(\hat{\lambda})} \tag{2.5}$$

where $Z_{\alpha/2}$ is a standard normal variate.

3. Bayesian inference

3.1. Bayesian estimation

In this section, we consider the squared error loss function (SELF), which is a symmetric loss function that assigns equal losses to overestimation and underestimation. Therefore, under SELF, we can define a Bayes estimator by the posterior expectation. Based on SELF, after deriving Bayes estimators, we can construct corresponding credible intervals with the highest posterior density (HPD) and Bayes predictive intervals.

Suppose that σ and λ have an inverted gamma prior and a gamma prior, respectively, and that these two priors are independent:

$$\pi(\sigma) = \frac{\delta^\gamma}{\Gamma(\gamma)}\sigma^{-\gamma-1}e^{-\delta/\sigma}, \tag{3.1}$$

$$\pi(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}. \tag{3.2}$$

Then the joint prior of σ and λ is

$$\pi(\sigma,\,\lambda) = \frac{\beta^{\alpha}\delta^{\gamma}}{\Gamma(\alpha)\Gamma(\gamma)}\sigma^{-\gamma-1}\lambda^{\alpha-1}e^{-(\beta\lambda+\delta/\sigma)}. \tag{3.3}$$

Then it follows from (2.1) and (3.3) that the joint posterior distribution of λ and σ as

$$\begin{aligned}\pi(\sigma,\,\lambda|\underline{x}) &= \frac{L(\sigma,\,\lambda)\,\pi(\sigma,\,\lambda)}{\int_{\sigma}\int_{\lambda}L(\sigma,\,\lambda)\,\pi(\sigma,\,\lambda)\mathrm{d}\lambda\mathrm{d}\sigma} \\ &= \frac{h_1(\sigma)h_2(\lambda|\sigma)h_3(\sigma,\,\lambda)}{\int_{\sigma}\int_{\lambda}h_1(\sigma)h_2(\lambda|\sigma)h_3(\sigma,\,\lambda)\mathrm{d}\lambda\mathrm{d}\sigma}\end{aligned} \tag{3.4}$$

where

$$h_1(\sigma) = \frac{(\delta+\sum_{i=1}^m x_i)^{\gamma+m}}{\Gamma(\gamma+m)}\sigma^{-(\gamma+m+1)}\exp\left(-\frac{\delta+\sum_{i=1}^m x_i}{\sigma}\right),$$

$$h_2(\lambda|\sigma) = \frac{\left[\beta+\sum_{i=1}^m \log\left(\frac{1-e^{-x_i/\sigma}}{1+e^{-x_i/\sigma}}\right)\right]^{\alpha+m}}{\Gamma(\alpha+m)}\lambda^{\alpha+m-1}\exp\left[-\lambda\left(\beta+\sum_{i=1}^m\log\left(\frac{1-e^{-x_i/\sigma}}{1+e^{-x_i/\sigma}}\right)\right)\right],$$

$$h_3(\sigma,\,\lambda) = \frac{1}{\left[\beta+\sum_{i=1}^m \log\left(\frac{1-e^{-x_i/\sigma}}{1+e^{-x_i/\sigma}}\right)\right]^{\alpha+m}}\prod_{i=1}^m\frac{1}{1-e^{-2x_i/\sigma}}\left[1-\left(\frac{1-e^{-x/\sigma}}{1+e^{-x/\sigma}}\right)^{\lambda}\right]^{r_i}.$$

Note that $h_1(\sigma)$ is an inverted gamma distribution and $h_2(\lambda|\sigma)$ is a gamma distribution.

Let $\theta(\sigma,\lambda)$ be any function of σ and λ. Then, from (3.4), the Bayes estimator of $\theta(\sigma,\lambda)$ under SELF is

$$\hat{\theta}_S(\sigma,\lambda) = \frac{\int_{\sigma}\int_{\lambda}\theta(\sigma,\lambda)h_1(\sigma)h_2(\lambda|\sigma)h_3(\sigma,\,\lambda)\mathrm{d}\sigma\mathrm{d}\lambda}{\int_{\sigma}\int_{\lambda}h_1(\sigma)h_2(\lambda|\sigma)h_3(\sigma,\,\lambda)\mathrm{d}\sigma\mathrm{d}\lambda}. \tag{3.5}$$

If $\theta(\sigma,\,\lambda)=\sigma$, then we have the Bayes estimator $\hat{\theta}_S(\sigma,\lambda)=\hat{\sigma}_S$, whereas if $\theta(\sigma,\,\lambda)=\lambda$, then we have $\hat{\theta}_S(\sigma,\lambda)=\hat{\lambda}_S$. Unfortunately, we cannot obtain the Bayes estimator $\hat{\theta}_S(\sigma,\lambda)$ in this case because the ratio of integrals in (3.5) does not to take a closed form. Alternatively, we can use Lindley's (1980) approximation method for an approximate ratio of integral in (3.5). However, because it is not possible to construct HPD credible intervals, we consider importance sampling to obtain Bayes estimators and construct corresponding HPD credible intervals and Bayes predictive intervals. With this technique, there is no need to compute a normalizing constant.

3.2. Importance sampling

Applying importance sampling to (3.5), we can approximate the Bayes estimators of σ and λ as

$$\hat{\sigma}_S \simeq \frac{\sum_{j=1}^N \sigma_j h_3(\sigma_j,\,\lambda_j)}{\sum_{j=1}^N h_3(\sigma_j,\,\lambda_j)} \tag{3.6}$$

and

$$\hat{\lambda}_S \simeq \frac{\sum_{j=1}^{N} \lambda_j h_3(\sigma_j,\, \lambda_j)}{\sum_{j=1}^{N} h_3(\sigma_j,\, \lambda_j)}. \tag{3.7}$$

Here we generate σ_j and λ_j (for $j = 1, 2, \cdots, N$) from $h_1(\sigma)$ and $h_2(\lambda|\sigma)$, respectively.

To construct HPD credible intervals of the scale parameter σ, we follow Chen and Shao (1999). Let

$$\sigma_p = \inf\{\sigma : \Pi(\sigma,\, \lambda|\underline{x}) \geq p\}, \quad 0 < p < 1, \tag{3.8}$$

where $\Pi(\sigma,\, \lambda|\underline{x})$ is a posterior cumulative distribution function.

Observe that for given σ^*,

$$\Pi(\sigma^*,\, \lambda|\underline{x}) = E\left(\mathrm{I}_{\sigma \leq \sigma^*}|\underline{x}\right), \tag{3.9}$$

where $\mathrm{I}_{\sigma \leq \sigma^*}$ is an indicator function. Then we can obtain a simulation consistent estimator of $\Pi(\sigma^*,\, \lambda|\underline{x})$ as

$$\hat{\Pi}(\sigma^*,\, \lambda|\underline{x}) = \frac{\sum_{j=1}^{N} \mathrm{I}_{\sigma \leq \sigma^*} h_3(\sigma_j,\, \lambda_j)}{\sum_{j=1}^{N} h_3(\sigma_j,\, \lambda_j)}. \tag{3.10}$$

Let

$$w_{(j)} = \frac{h_3(\sigma_{(j)},\, \lambda_{(j)})}{\sum_{j=1}^{N} h_3(\sigma_{(j)},\, \lambda_{(j)})}, \tag{3.11}$$

where $\sigma_{(j)}$ is the ordered value of σ_j for $j = 1, \ldots, N$. Then we have

$$\hat{\Pi}(\sigma^*,\, \lambda|\underline{x}) = \begin{cases} 0 & \text{if } \sigma^* < \sigma_{(1)} \\ \sum_{k=1}^{j} w_{(k)} & \text{if } \sigma_{(j)} \leq \sigma^* < \sigma_{(j+1)}. \\ 1 & \text{if } \sigma^* \geq \sigma_{(n)} \end{cases} \tag{3.12}$$

Using (3.12), we can approximate σ_p as follows:

$$\sigma_p = \begin{cases} \sigma_{(1)} & \text{if } p = 0 \\ \sigma_{(j)} & \text{if } \sum_{k=i}^{j-1} w_{(k)} < p \leq \sum_{k=i}^{j} w_{(k)}. \end{cases} \tag{3.13}$$

Let $R_k = \left(\hat{\sigma}^{(k/N)},\, \hat{\sigma}^{(k/N+(1-p))}\right)$ for $k = 1, \ldots, [pN]$. Here $[z]$ denotes the largest integer less than or equal to z. Then choosing R_k with the smallest width, we obtain the HPD credible interval of the scale parameter σ.

Applying the same weight $w_{(k)}$ and procedure to the shape parameter λ, we can obtain its HPD credible interval of the shape parameter λ.

3.3. Bayesian prediction

To construct a two-sided Bayes predictive interval, we first compute the Bayes predictive density of $X_{(m+1)}$.

Given $x_{(1)} < \cdots < x_{(m)}$, the Bayes predictive density of $X_{(m+1)}$ is

$$f^*_{X_{(m+1)}|\underline{x}}(y) = \int_\sigma \int_\lambda f_{X_{(m+1)}|\underline{x}}(y|\sigma, \lambda)\pi(\sigma, \lambda|\underline{x})d\lambda d\sigma, \qquad y > x_{(m)}, \tag{3.14}$$

where $f_{X_{(m+1)}|\underline{x}}(\cdot|\sigma, \lambda)$ is the conditional density of $X_{(m+1)}$ given $x_{(1)} < \cdots < x_{(m)}$. By the Markov property of conditional order statistics, we can write (3.14) as

$$f^*_{X_{(m+1)}|\underline{x}}(y) = \int_\sigma \int_\lambda f_{X_{(m+1)}|X_{(m)}=x_{(m)}}(y|\sigma, \lambda)\pi(\sigma, \lambda|\underline{x})d\lambda d\sigma, \tag{3.15}$$

where

$$f_{X_{(m+1)}|X_{(m)}=x_{(m)}}(y|\sigma, \lambda) = \frac{(n-m)f(y|\sigma, \lambda)(1-F(y|\sigma, \lambda))^{n-m-1}}{(1-F(x_{(m)}|\sigma, \lambda))^{n-m}}, \qquad y > x_{(m)}. \tag{3.16}$$

In addition, letting $1 - F(y|\sigma, \lambda) = u$ in (3.16), we have

$$\int_y^\infty f(y|\sigma, \lambda)(1-F(y|\sigma, \lambda))^{n-m-1}dy = \int_0^{1-F(y|\sigma, \lambda)} u^{n-m-1}du.$$

Therefore, the conditional survival density of $X_{(m+1)}$ given $x_{(m)}$ is

$$T_{X_{(m+1)}|X_{(m)}=x_{(m)}}(y|\sigma, \lambda) = \frac{(1-F(y|\sigma, \lambda))^{n-m}}{(1-F(x_{(m)}|\sigma, \lambda))^{n-m}}, \qquad y > x_{(m)}. \tag{3.17}$$

Here, the Bayes predictive survival density of $X_{(m+1)}$ is

$$T^*_{X_{(m+1)}|\underline{x}}(y) = \int_\sigma \int_\lambda T_{X_{(m+1)}|X_{(m)}=x_{(m)}}(y|\sigma, \lambda)\pi(\sigma, \lambda|\underline{x})d\lambda d\sigma \tag{3.18}$$

Using the technique illustrated in the Section 3.1, we can approximate the Bayes predictive density (3.15) and the Bayes predictive survival density (3.17) as

$$\hat{f}^*_{X_{(m+1)}|\underline{x}}(y) \simeq \sum_{j=1}^N f_{X_{(m+1)}|X_{(m)}=x_{(m)}}(y|\sigma_j, \lambda_j)w_j \tag{3.19}$$

and,

$$\hat{T}^*_{X_{(m+1)}|\underline{x}}(y) \simeq \sum_{j=1}^N T_{X_{(m+1)}|X_{(m)}=x_{(m)}}(y|\sigma_j, \lambda_j)w_j, \text{ respectively,} \tag{3.20}$$

where

$$w_j = \frac{h_3(\sigma_j, \lambda_j)}{\sum_{j=1}^N h_3(\sigma_j, \lambda_j)}, \qquad j = 1, 2, \ldots, N. \tag{3.21}$$

Using the approximate Bayes predictive survival density (3.20), we can construct a two-sided Bayes predictive interval for X_{m+1}. Here a $100\nu\%$ Bayes prediction interval for $X_{(m+1)}$ is

$P(L < X_{(m+1)} < U|\underline{x}) = \nu$. Therefore, we can obtain a symmetric $100\nu\%$ Bayes predictive interval $(L,\ U)$ for X_{m+1} by solving the following two non-linear equations.

$$P(X_{(m+1)} > L|\underline{x}) = T^{*}_{X_{(m+1)}|\underline{x}}(L) = \frac{1+\nu}{2}, \tag{3.22}$$

$$P(X_{(m+1)} > U|\underline{x}) = T^{*}_{X_{(m+1)}|\underline{x}}(U) = \frac{1-\nu}{2}. \tag{3.23}$$

Because the above equations (3.22) and (3.23) cannot be solved explicitly, we can solve them by using numerical methods such as the Newton-Raphson method.

4. Illustrative example and a simulation study

In this section, we present two examples to verify the performance of the proposed estimation method.

4.1. Real data

Many authors (e.g., Balakrishnan and Kannan, 2001; Balakrishnan *et al.*, 2004; Alaboud, 2009) used the data in Nelson (1982), which represents log failure times to breakdown of an insulating fluid testing experiment (see Table 4.1). Kang and Seo (2011) use the data for the Kolmogorov test to examine whether the data follow an exponentiated half-logistic distribution when the shape parameter $\lambda = 1$. To check the goodness of fit for an exponentiated half-logistic distribution with unknown scale and shape parameters, we first calculate MLEs of unknown parameters σ and λ for uncensored data. Then we use the results to obtain values of the Kolmogorov test statistic D_n and associated p-values. Table 4.2 shows these values. The test statistic D_n is less than the test statistic computed by Kang and Seo (2011). Therefore, we conclude that the data follow an exponentiated half-logistic distribution with unknown scale and shape parameters. In this example, we have $n = 16$ and $m = 11$. Table 4.3 shows the censoring scheme and observations

Table 4.1 Log faliure times to breakdown of an insulating fluid testing experiment

0.270027	1.02245	1.15057	1.42311	1.54116	1.57898	1.8718	1.9947
2.08069	2.11263	2.48989	3.45789	3.48186	3.52371	3.60305	4.28895

Table 4.2 The Kolmogorov test statistic and the associated p-value for uncensored data

$\hat{\sigma}$	$\hat{\lambda}$	D_n	p-value
1.03738	2.43119	0.15318	0.7945

We calculate the MLEs $\hat{\sigma}$ and $\hat{\lambda}$ from equations (2.2) and (2.3) and obtain the corresponding 95% approximate confidence intervals by using equation (2.5). In addition, we obtained the Bayes estimators, 95% HPD credible intervals, and 95% Bayes predictive intervals by using importance sampling method as illustrated in Section 4.3. To examine the effect the hyperparameter, we compute Bayes estimators based on vague priors with very small shape and rate parameters (here $\alpha = 0.01$, $\beta = 0.01$, $\gamma = 0.01$, and $\delta = 0.01$). In addition, we obtain the corresponding 95% HPD credible intervals. Tables 4.4 and 4.5 show these values.

Table 4.3 Progressively type-II censored data

i	1	2	3	4	5	6
x	0.270027	1.02245	1.15057	1.42311	1.54116	1.57898
r_i	0	0	1	0	0	0

i	7	8	9	10	11
x	1.8718	1.9947	2.11263	2.48989	3.45789
r_i	2	0	0	2	0

As shown in Tables 4.4 and 4.5, all the estimators have nearly the same values for all unknown parameters. The length of the asymptotic confidence interval is shorter than that of the HPD credible interval obtained under the vague prior for the scale parameter σ, whereas that of the HPD credible interval under the vague prior is shorter than that of the asymptotic confidence interval for the shape parameter λ. In addition, the HPD credible intervals obtained using informative priors have the shortest length for both of the unknown parameters σ and λ. Based on vague and informative priors, the 95% Bayes predictive intervals for X_9 are (3.4635 4.4195) and (3.4633 4.3453), respectively.

Table 4.4 MLEs/Bayes estimators and confidence intervals/95% HPD credible intervals of σ

	$\hat{\sigma}$	$\hat{\sigma}_s$ ($\gamma = 0.01$, $\delta = 0.01$)	$\hat{\sigma}_s$ ($\gamma = 5.8$, $\delta = 7.3$)
Estimates	0.9791	1.1110	1.0563
c.i	(0.4923, 1.4658)	(0.6298, 1.7970)	(0.6301, 1.5912)
Length of c.i	0.9735	1.1672	0.9611

Table 4.5 MLEs/Bayes estimators and confidence intervals/95% HPD credible intervals of λ

	$\hat{\lambda}$	$\hat{\lambda}_s$ ($\alpha = 0.01$, $\beta = 0.01$)	$\hat{\lambda}_s$ ($\alpha = 5.7$, $\beta = 3.0$)
Estimates	2.5337	2.4426	2.4761
c.i	(0.4926, 4.5749)	(0.7921, 4.4291)	(1.0481, 4.0989)
Length of c.i	4.0823	3.6370	3.0508

4.2. Simulation assessment

We also consider progressively type-II censored data generated from exponentiated half-logistic distributions with $\sigma = 1.0$ and $\lambda = 0.5$ by using the algorithm in Balakrishnan and Sandhu (1995). Table 4.6 shows the censoring scheme and samples.

Table 4.6 The progressively type-ll censored data ($n = 20$, $m = 15$)

i	1	2	3	4	5	6	7	8
x	0.00183	0.01846	0.02421	0.03247	0.04391	0.12350	0.12460	0.25958
r_i	0	0	0	0	0	0	0	0

i	9	10	11	12	13	14	15
x	0.42189	0.64591	1.27032	1.56578	1.73546	1.86236	4.28857
r_i	5	0	0	0	0	0	0

As in the case of real data, we obtain MLEs and Bayes estimators by using the formulas presented in Sections 2 and 3. By setting $E(\sigma) = 1.0$ and $Var(\sigma) = 0.1$ from the prior distribution (3.2), we obtain the hyperparameters α and β of the gamma prior (3.2) by solving the prior information $E(\sigma)$ and $V(\sigma)$, respectively. Similarly, we obtain the hyperparameters γ and δ of the inverted gamma prior (3.1). By using these values, we obtain the Bayes estimators for unkonwn parameters λ and σ with informative priors. We also compute Bayes estimators for vague priors. Tables 4.7 and 4.8 show these values.

Table 4.7 The MLEs/Bayes estimators and the confidence interval/95% HPD credible interval of σ

	$\hat{\sigma}$	$\hat{\sigma}_s$ ($\gamma = 0.01$, $\delta = 0.01$)	$\hat{\sigma}_s$ ($\gamma = 12$, $\delta = 11$)
Estimates	1.2832	1.2466	1.1203
c.i	(0.3274, 2.2390)	(0.5380, 2.1889)	(0.6236, 1.6471)
Length of c.i	1.9116	1.6509	1.0235

Table 4.8 The MLEs/Bayes estimators and the confidence interval/95% HPD credible interval of λ

	$\hat{\lambda}$	$\hat{\lambda}_s$ ($\alpha = 0.01$, $\beta = 0.01$)	$\hat{\lambda}_s$ ($\alpha = 2.5$, $\beta = 5.0$)
Estimates	0.4627	0.5102	0.4926
c.i	(0.2249, 0.7005)	(0.2903, 0.7339)	(0.2760, 0.6971)
Length of c.i	0.4756	0.4436	0.4211

As shown in Tables 4.7 and 4.8, the estimator closest to the actual value is the Bayes estimator under the assumption of informative priors. For confidence and HPD credible intervals, we obtain results that are consistent with those for real data. Based on vague and informative priors, the 95% Bayes predictive intervals for X_{16} are (4.2942, 5.2870) and (4.2941, 5.2646), respectively. Because we obtain the hyperparameters of informative priors from prior means and very small variances, the Bayes estimators and corresponding intervals have extreme priors, and these Bayes estimators are approximately the same as those estimators under vague priors. However, it is noticed that Bayes estimators and Bayes HPD credible intervals are a little bit sensitive to the choice of hyperparameters.

5. Concluding remarks

In this papers we present the MLEs and Bayesian estimators of unknown parameters in an exponentiated half-logistic distribution based on a progressively type-II censored sample. We also obtain corresponding credible intervals with the highest posterior density and Bayes predictive intervals using the importance sampling technique. Finally, we compared the performance of Bayesian estimations based on informative and vague priors. Results shows that Bayesian inference is sensitive to the choice of hyperparameters. Thus, it is highly suggested to use vague priors (less informative) or other objective priors.

References

Alaboud, F. M. (2009). Bayesian estimations for the extreme value distribution using progressive censored data and asymmetric loss. *International Mathematical Forum*, **8**, 1603–1622.

Balakrishnan, N. and Kannan, N. (2001). Point and interval estimation for the logistic distribution based on progressively type-ll censored samples. In *Handbook of Statistics*, **20**, edited by Balakrishnan, N. and Rao, C. R., Elsevier, Oxford, 431–456.

Balakrishnan, N., Kannan, N., Lin, C. T., and Wu, S. J. S. (2004). Inference for the extreme value distribution under progressively type-II censoring. *Journal of Statistical Computation and Simulation*, **74**, 25–45.

Balakrishnan, N. and Puthenpura, N. (1986). Best linear unbiased estimators of location and scale parameters of the half logistic distribution. *Journal of Statistics and Computer Simulation*, **25**, 193–204.

Balakrishnan, N. and Sandhu, R. A. (1995). A simple simulational algorithm for generating progressively type-ll censored samples. *The American Statistician*, **49**, 229–230.

Balakrishnan, N. and Wong, K. H. T. (1991). Approximate MLEs for the location and scale parameters of the half-logistic distribution with type-ll right censoring. *IEEE Transactions on Reliability*, **40**, 140–145.

Chen, M. H. and Shao, Q. M. (1999). Monte Carlo estimation of Bayesian credible and HPD intervals. *Journal of Computational and Graphical Statistics*. **8**, 69–92.

Kang, S. B., Cho, Y. S., and Han, J. T. (2008). Estimation for the half logistic distribution under progressively type-ll censoring. *Communications of the Korean Statistical Society*, **15**, 815–823.

Kang, S. B., Cho, Y. S., and Han, J. T. (2009). Estimation for the half logistic distribution based on double hybrid censored samples. *Communications of the Korean Statistical Society*, **16**, 1055–1066.

Kang, S. B. and Seo, J. I (2011). Estimation in an exponentiated half logistic distribution under progressively type-II censoring. *Communications of the Korean Statistical Society*, **18**, 657–666.

Kim, Y., Kang, S. B., and Seo, J. I. (2011). Bayesian estimations on the exponentiated distribution family with type-II right censoring. *Communications of the Korean Statistical Society*, **18**, 603–613.

Kim, Y., Kang, S. B., and Seo, J. I. (2011). Bayesian estimations on the exponentiated half triangle distribution under type-I hybrid censoring. *Journal of the Korean Data & Information Science Society*, **22**, 565–574.

Kim, Y., Kang, S. B., and Seo, J. I. (2011). Bayesian estimation in the generalized half logistic distribution under progressively type-II censoring. *Journal of the Korean Data & Information Science Society*, **22**, 977–987.

Lindley, D. V. (1980). Approximate Bayesian methods estimations. In *Bayesian Statistics*, edited by Bernardo, J. M., De Groot, M. H., Lindley, D. V. and Smith, A. F. M., Valencia Press, Spain.

Nelson, W. B. (1982). *Applied life data analysis*, John Willey & Sons, New York.