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A DEFINITE INTEGRAL FORMULA

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Abstract. A remarkably large number of integral formulas have been investigated and developed. Certain large number of integral formulas are expressed as derivatives of some known functions. Here we choose to recall such a formula to present explicit expressions in terms of Gamma function, Psi function and Polygamma functions. Some simple interesting special cases of our main formulas are also considered. It is also pointed out that the same argument can establish explicit integral formulas for other those expressed in terms of derivatives of some known functions.

1. Introduction and preliminaries

A remarkably large number of integral formulas have been investigated and developed (see, *e.g.*, [1–5]). Certain large number of integral formulas are expressed as derivatives of some known functions (see [1, pp. 152-155]). For example, we choose to recall such a formula (see [1, p. 153, Entry 4.1.5-163]):

$$\begin{aligned}\int_0^1 x^{-\frac{1}{2}}\,(1-x)^{2a+1}\,(4-x)^a \left(\ln\frac{(1-x)^2}{4-x}\right)^n dx \\ = \frac{2\sqrt{\pi}}{3}\,\mathrm{D}_a^n \left[\frac{2^{2a}\,\Gamma(a+1)}{\Gamma\left(a+\frac{3}{2}\right)}\right] \quad (\Re(a) > -1;\, n \in \mathbb{N}),\end{aligned} \tag{1.1}$$

where $\Gamma(a)$ is the familiar Gamma function (see, *e.g.*, [6, Section 1.1]) and $\mathbb{N}$ denotes the set of positive integers.

Here we aim at presenting explicit expressions of the integral formula in (1.1) only for $n = 1, 2, 3$. Some special cases of our main formulas are also considered. To do this, we recall some related known functions and their properties.

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The Psi (or Digamma) function ψ is defined by

$$\psi(z) := \frac{d}{dz}\{\log\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \qquad \text{or} \qquad \log\Gamma(z) = \int_1^z \psi(t)\,dt. \tag{1.2}$$

The Polygamma functions $\psi^{(n)}(z)$ $(n \in \mathbb{N})$ are defined by

$$\psi^{(n)}(z) := \frac{d^{n+1}}{dz^{n+1}}\log\Gamma(z) = \frac{d^n}{dz^n}\psi(z) \quad (n \in \mathbb{N}_0;\ z \in \mathbb{C}\setminus\mathbb{Z}_0^-), \tag{1.3}$$

where $\mathbb{C}$ and $\mathbb{Z}_0^-$ denote the sets of complex numbers and nonpositive integers, respectively. In terms of the generalized (or Hurwitz) Zeta function $\zeta(s,a)$ defined by

$$\zeta(s,a) := \sum_{k=0}^{\infty}(k+a)^{-s} \quad (\Re(s) > 1;\ a \in \mathbb{C}\setminus\mathbb{Z}_0^-), \tag{1.4}$$

we can write

$$\psi^{(n)}(z) = (-1)^{n+1}\,n!\,\zeta(n+1,z) \quad (n \in \mathbb{N};\ z \in \mathbb{C}\setminus\mathbb{Z}_0^-), \tag{1.5}$$

which may be used to deduce the properties of $\psi^{(n)}(z)$ $(n \in \mathbb{N})$ from those of $\zeta(s,z)$ $(s = n+1;\ n \in \mathbb{N})$ and vice versa. The Riemann Zeta function $\zeta(s)$ is defined by

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty}\frac{1}{n^s} = \frac{1}{1-2^{-s}}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^s} & (\Re(s) > 1) \\ \displaystyle\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0;\ s \neq 1). \end{cases} \tag{1.6}$$

It is easy to see from the definitions (1.4) and (1.6) that

$$\zeta(s) = \zeta(s,1) = (2^s-1)^{-1}\,\zeta\left(s,\frac{1}{2}\right) = 1 + \zeta(s,2). \tag{1.7}$$

From the definition (1.4) of $\zeta(s,a)$, it easily follows that

$$\zeta(s,a) = \zeta(s,n+a) + \sum_{k=0}^{n-1}(k+a)^{-s} \quad (n \in \mathbb{N}). \tag{1.8}$$

The Gamma function Γ satisfies the following fundamental functional relation:

$$\Gamma(z+1) = z\,\Gamma(z), \tag{1.9}$$

so that, obviously,

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)} \quad (n \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}). \tag{1.10}$$

By making use of the relation (1.10) together with $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, we obtain

$$\Gamma(n+1) = n! \quad \text{and} \quad \Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!\,\sqrt{\pi}}{2^{2n}\,n!} \quad (n \in \mathbb{N}_0). \tag{1.11}$$

The Psi function yields the following special values:

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k} \quad (n \in \mathbb{N}) \tag{1.12}$$

and

$$\psi\left(n+\frac{1}{2}\right) = -\gamma - 2\,\log\,2 + 2\sum_{k=0}^{n-1} \frac{1}{2k+1} \quad (n \in \mathbb{N}_0), \tag{1.13}$$

where γ denotes the Euler-Mascheroni constant defined by

$$\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{k} - \log\,n\right) \cong 0.57721\,56649\,\cdots \tag{1.14}$$

and an empty sum is understood (throughout this paper) to be nil.

The following well-known identity is also recalled (see, *e.g.*, [6, p. 166, Eq. (18)]

$$\zeta(2n) = (-1)^{n+1}\,\frac{(2\pi)^{2n}}{2\,(2n)!}\,B_{2n} \quad (n \in \mathbb{N}_0)\,, \tag{1.15}$$

where B_n denotes the Bernoulli numbers (see, *e.g.*, [6, Section 1.7]).

2. Integral formulas

We begin by presenting three main integral formulas as in the following theorem.

Theorem. *Each of the following integral formulas holds true*:

$$\begin{aligned}\int_0^1 x^{-\frac{1}{2}}\,(1-x)^{2a+1}\,(4-x)^{a}\,\ln\frac{(1-x)^2}{4-x}\,dx &= \frac{\sqrt{\pi}}{3}\,\frac{2^{2a+1}\,\Gamma(a+1)}{\Gamma\left(a+\frac{3}{2}\right)} \\ &\cdot\left[2\,\ln\,2 + \psi(a+1) - \psi\left(a+\frac{3}{2}\right)\right] \quad (\Re(a) > -1).\end{aligned} \tag{2.1}$$

$$\int_0^1 x^{-\frac{1}{2}}\,(1-x)^{2a+1}\,(4-x)^a\,\left(\ln\frac{(1-x)^2}{4-x}\right)^2\,dx$$
$$= \frac{\sqrt{\pi}}{3}\,\frac{2^{2a+1}\,\Gamma(a+1)}{\Gamma\left(a+\frac{3}{2}\right)}\left[\left\{2\,\ln\,2+\psi(a+1)-\psi\left(a+\frac{3}{2}\right)\right\}^2 \right. \tag{2.2}$$
$$\left. +\,\psi'(a+1)-\psi'\left(a+\frac{3}{2}\right)\right] \quad (\Re(a)>-1).$$

$$\int_0^1 x^{-\frac{1}{2}}\,(1-x)^{2a+1}\,(4-x)^a\,\left(\ln\frac{(1-x)^2}{4-x}\right)^3\,dx$$
$$\begin{aligned} &= \frac{\sqrt{\pi}}{3}\,\frac{2^{2a+1}\,\Gamma(a+1)}{\Gamma\left(a+\frac{3}{2}\right)}\left[\left\{2\,\ln\,2+\psi(a+1)-\psi\left(a+\frac{3}{2}\right)\right\}^3 \right. \\ &\quad +3\left\{2\,\ln\,2+\psi(a+1)-\psi\left(a+\frac{3}{2}\right)\right\}\left(\psi'(a+1)-\psi'\left(a+\frac{3}{2}\right)\right) \\ &\quad \left. +\,\psi^{(2)}(a+1)-\psi^{(2)}\left(a+\frac{3}{2}\right)\right] \quad (\Re(a)>-1). \end{aligned} \tag{2.3}$$

Proof. For convenience and simplicity, let the function $f(a)$ be defined by

$$f(a) := \frac{2^{2a}\,\Gamma(a+1)}{\Gamma\left(a+\frac{3}{2}\right)}.$$

Then the logarithmic derivative of $f(a)$ gives

$$f'(a) = f(a)\,g(a), \tag{2.4}$$

where

$$g(a) := 2\,\ln\,2+\psi(a+1)-\psi\left(a+\frac{3}{2}\right).$$

Applying the Leibniz's rule for the differentiation of the product of two functions to (2.4) $n-1$ times, we obtain

$$\begin{aligned} \mathrm{D}_a^n\,f(a) =& g(a)\,f^{(n-1)}(a) \\ &+\sum_{k=1}^{n-1}\left\{\psi^{(k)}(a+1)-\psi^{(k)}\left(a+\frac{3}{2}\right)\right\}\,f^{(n-1-k)}(a). \end{aligned} \tag{2.5}$$

Setting $n=1,\,2,\,3$ in the formula (2.5) and using the resulting identities, we get the desired integral formulas in Theorem.

□

Furthermore we can evaluate various interesting special cases of our main results in Theorem. Here, by using suitably chosen formulas in Section 1, we can give the corresponding formulas of (2.1), (2.2) and (2.3) only when $a=0$, $a=\frac{1}{2}$ and $a=1$ as in the following corollary.

Corollary. *Each of the following integral formulas holds true*:

$$\int_0^1 x^{-\frac{1}{2}}\,(1-x)\,\ln\frac{(1-x)^2}{4-x}\,dx=\frac{8}{3}\,(2\,\ln 2-1). \tag{2.6}$$

$$\int_0^1 x^{-\frac{1}{2}}\,(1-x)^2\,(4-x)^{\frac{1}{2}}\,\ln\frac{(1-x)^2}{4-x}\,dx=\frac{2\,\pi}{3}. \tag{2.7}$$

$$\int_0^1 x^{-\frac{1}{2}}\,(1-x)^3\,(4-x)\,\ln\frac{(1-x)^2}{4-x}\,dx=\frac{32}{9}\left(4\,\ln 2-\frac{5}{3}\right). \tag{2.8}$$

$$\int_0^1 x^{-\frac{1}{2}}\,(1-x)\left(\ln\frac{(1-x)^2}{4-x}\right)^2dx=\frac{4}{3}\left[4(2\,\ln 2-1)^2+4-\frac{\pi^2}{3}\right]. \tag{2.9}$$

$$\int_0^1 x^{-\frac{1}{2}}\,(1-x)^2\,(4-x)^{\frac{1}{2}}\left(\ln\frac{(1-x)^2}{4-x}\right)^2dx=\frac{2\pi}{3}\left(\frac{\pi^2}{3}-2\right). \tag{2.10}$$

$$\begin{aligned}&\int_0^1 x^{-\frac{1}{2}}\,(1-x)^3\,(4-x)\left(\ln\frac{(1-x)^2}{4-x}\right)^2dx\\ &\qquad=\frac{32}{9}\left[\left(4\,\ln 2-\frac{5}{3}\right)^2+\frac{31}{9}-\frac{\pi^2}{3}\right].\end{aligned} \tag{2.11}$$

$$\begin{aligned}&\int_0^1 x^{-\frac{1}{2}}\,(1-x)\left(\ln\frac{(1-x)^2}{4-x}\right)^3dx\\ &\quad=\frac{8}{3}\left[4\,(2\,\ln 2-1)^3+\left(4-\frac{\pi^2}{3}\right)(2\,\ln 2-1)+6\,\zeta(3)-8\right].\end{aligned} \tag{2.12}$$

$$\begin{aligned}&\int_0^1 x^{-\frac{1}{2}}\,(1-x)^2\,(4-x)^{\frac{1}{2}}\left(\ln\frac{(1-x)^2}{4-x}\right)^3dx\\ &\qquad=\frac{2\pi}{3}\left[6+\pi^2-12\,\zeta(3)\right].\end{aligned} \tag{2.13}$$

$$\int_0^1 x^{-\frac{1}{2}}\,(1-x)^3\,(4-x)\,\left(\ln\frac{(1-x)^2}{4-x}\right)^3\,dx=\frac{32}{9}\left[\left(4\,\ln 2-\frac{5}{3}\right)^3\right.$$
$$\left.+\left(\frac{31}{3}-\pi^2\right)\left(4\,\ln 2-\frac{5}{3}\right)+12\,\zeta(3)-\frac{394}{27}\right]. \tag{2.14}$$

Concluding Remarks. By using the formula (2.5), we can present explicit integral formulas for the involved integrand $n\in\mathbb{N}$ with $n\geq 4$ even though they become more and more complicated. A similar argument here can establish explicit integral formulas for those integral formulas recorded in [1, pp. 152-155] expressed in terms of derivatives of certain known functions.

References

[1] Y. A. Brychkov, *Handbook of Special Functions, Derivatives, Integrals, Series and Other Formulas*, CRC Press, 2008.

[2] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products* (Corrected and Enlarged edition prepared by A. Jeffrey), Academic Press, New York, 1980; Sixth edition, 2000.

[3] A. Jeffrey, *Handbook of Mathematical Formulas and Integrals*, Second edition, Academic Press, 2000.

[4] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and Series*, Vol. **1**: *Elementary Functions*, Gordon and Breach Science Publishers, New York, 1986.

[5] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and Series*, Vol. **3**: *More Special Functions*, Gordon and Breach Science Publishers, New York, 1990.

[6] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London, and New York, 2012.

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