East Asian Mathematical Journal
Vol. 29 (2013), No. 5, pp. 503–510
http://dx.doi.org/10.7858/eamj.2013.034

QUADRATIC B-SPLINE FINITE ELEMENT METHOD FOR THE BENJAMIN-BONA-MAHONY-BURGERS EQUATION

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Abstract. A quadratic B-spline finite element method for the spatial variable combined with a Newton method for the time variable is proposed to approximate a solution of Benjamin-Bona-Mahony-Burgers (BBMB) equation. Two examples were considered to show the efficiency of the proposed scheme. The numerical solutions obtained for various viscosity were compared with the exact solutions. The numerical results show that the scheme is efficient and feasible.

1. Introduction

The mathematical model of propagation of small amplitude long waves in nonlinear dispersive media is described by the following Benjamin-Bona-Mahony-Burgers equation[1]:

$$\begin{cases} u_t - u_{xxt} - \alpha u_{xx} + \beta u_x + u u_x = f \quad \text{in } [0, L] \times [0, T], \\ \quad u(0, t) = u(L, t) = 0 \quad \text{on } [0, T], \\ \quad u(x, 0) = u_0(x) \quad \text{in } [0, L], \end{cases} \tag{1}$$

where $\alpha > 0, \beta$ are constants, f is a given forcing term. In the physical case, the dispersive effect of (1) is the same as the Benjamin-Bona-Mahony (BBM) equation, while the dissipative effect is the same as the Burgers equation, and which is an alternative model for the Korteweg-de Vries-Burgers (KdVB) equation [2]. Numerical methods based on either finite elements[3]-[7], finte differences[8]-[10],or Adomian decomposition scheme[11, 12]. Quadratic B-spline finite element method for approximating the solution of Burgers equation can be found in [13, 14]. Cubic B-spline collocation method for numerical solution of the BBMB equation can be found in [15]. In this paper, we apply the quadratic B-spline finite element method to convert BBMB equation to a finite set of

Received October 5, 2013; Accepted November 4, 2013.
2000 *Mathematics Subject Classification.* 11A11.
Key words and phrases. BBMB equation; B-spline finite element method; Newton method.
† The author was financially supported by grant No.012700-9502010093 from the scientific research foundation of Yanbian University.

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nonlinear ordinary differential equations, and then, use a Newton method for the time variable.
In the next section, the numerical scheme for the BBMB equation is described. Numerical examples and their results will be shown in the last section.

2. Finite element approximation of The BBMB system

Standard Lagrangian finite element basis functions offer only simple C^0-continuity and therefore they cannot be used for the spatial discretization of the higher-order differential equations(e.g., third-order differential equation or forth-order differential equation), but the B-spline basis function can at least achieve C^1-continuous globally, and its basis function is often used to solve the higher order differential equations.

Let us consider the BBMB equation with boundary conditions and the initial condition. We use a variational formulation to define a finite element method to approximate (1). A variational formulation of the problem (1) is as the following: find $u \in L^2(0,T;H_0^1(0,L))$ such that

$$\begin{cases} \int_0^L u_t v dx + \int_0^L u_{xt} v' dx + \alpha \int_0^L u_x v' dx + \beta \int_0^L u_x v dx \\ \qquad\qquad + \int_0^L u u_x v dx = \int_0^L f v dx \quad \text{for all } v \in H_0^1(0,L), \\ u(0,x) = u_0(x) \quad \text{in } [0,L], \end{cases} \tag{2}$$

where $H_0^1 = \{u \in H^1(0,L) : u|_{x=0} = u|_{x=1} = 0\}$ and $H^1(0,L) = \{v \in L^2(0.L) : \frac{\partial v}{\partial x} \in L^2(0,L)\}$.

A typical finite element approximation of (2) is defined as follows: we first choose conforming finite element subspaces $V^h \subset H^1(0,L)$ and then define $V_0^h = V^h \cap H_0^1(0,L)$. One then seeks $u^h(t,\cdot) \in V_0^h$ such that

$$\begin{cases} \int_0^L u_t^h v^h dx + \int_0^L u_{xt}^h (v^h)' dx + \alpha \int_0^L u_x^h (v^h)' dx + \beta \int_0^L u_x^h v^h dx \\ \qquad\qquad + \int_0^L u^h u_x^h v^h dx = \int_0^L f v^h dx \quad \text{for all } v^h \in V_0^h(0,L), \\ u^h(0,x) = u_0^h(x) \quad \text{in } [0,L], \end{cases} \tag{3}$$

where $u_0^h(x) \in V_0^h$ is an approximation, e.g., a projection, of $u_0(x)$.

The interval $[0,L]$ is divided into n finite elements of equal length h by the knots x_i such that $0 = x_0 < x_1 < \cdots < x_n = L$. The set of splines

$\{\eta_{-1}, \eta_0, \cdots, \eta_n\}$ form a basis for functions defined on $[0, L]$. Quadratic B-splines $\eta_i(x)$ with the required properties are defined by [16],

$$\eta_i(x) = \frac{1}{h^2}\begin{cases} (x_{i+2}-x)^2 - 3(x_{i+1}-x)^2 + 3(x_i - x)^2), & [x_{i-1}, x_i],\\ (x_{i+2}-x)^2 - 3(x_{i+1}-x)^2, & [x_i, x_{i+1}],\\ (x_{i+2}-x)^2, & [x_{i+1}, x_{i+2}],\\ 0, & \text{otherwise}, \end{cases}$$

where $h = x_{i+1} - x_i, i = -1, 0, \cdots, n$.

The quadratic spline and its first derivative vanish outside the interval $[x_{i-1}, x_{i+2}]$. Then the spline function values and its first derivative at the knots are given by

$$\begin{cases} \eta_i(x_{i-1}) = \eta_i(x_{i+2}) = 0,\ \eta_i(x_i) = \eta_i(x_{i+1}) = 1;\\ \eta_i^{'}(x_{i-1}) = \eta_i^{'}(x_{i+2}) = 0,\ \eta_i^{'}(x_i) = \eta_i^{'}(x_{i+1}) = 1. \end{cases} \tag{4}$$

Thus an approximate solution can be written in terms of the quadratic spline functions as

$$u^h(x,t) = \sum_{i=-1}^{n} a_i(t)\eta_i(x), \tag{5}$$

where $a_i(t)$ are yet undetermined coefficients.

Each spline covers three intervals so that three splines $\eta_{i-1}(x), \eta_i(x), \eta_{i+1}(x)$ cover each finite element $[x_i, x_{i+1}]$. All other splines are zero in this region. Using Eq.(5) and spline function properties (4), the nodal values of function $u^h(x,t)$ and its derivative at the knot x_i and fixed time $\tilde{t}$ can be expressed in terms of the coefficients $a_i(\tilde{t})$ as

$$u^h(x_i, \tilde{t}) = a_{i-1}(\tilde{t}) + a_i(\tilde{t}), \quad \frac{\partial u^h(x,\tilde{t})}{\partial x}\Big|_{x=x_i} = \frac{2}{h}(a_i(\tilde{t}) - a_{i-1}(\tilde{t})). \tag{6}$$

From (6) and homogeneous boundary conditions we get $a_{-1}(t) = -a_0(t)$ and $a_n(t) = -a_{n-1}(t)$. Hence we have

$$u^h(x,t) = \sum_{i=0}^{n-1} a_i(t)\xi_i(x), \tag{7}$$

where $\xi_0(x) = (\eta_0(x) - \eta_{-1}(x)), \xi_i(x) = \eta_i(x)(i = 1, 2, \cdots, n-2), \xi_{n-1}(x) = \eta_{n-1}(x) - \eta_n(x)$. Hence n unknowns $a_i(t)(i = 0, 1, \cdots, n-1)$ for every moment of t must be determined.

According to Galerkin method the weighted function $v^h(x)$ in (3) is chosen as $v_i^h(x) = \xi_i(x)(i = 0, 1, \cdots, n-1)$. Substituting (7) into (3) we obtain

$$\begin{cases} \sum_{i=0}^{n-1}\left(\int_0^L \xi_i\xi_j dx\right)\frac{da_i(t)}{dt} + \sum_{i=0}^{n-1}\left(\int_0^L \xi_i^{'}\xi_j^{'} dx\right)\frac{da_i(t)}{dt} \\ \qquad + \alpha\sum_{i=0}^{n-1}\left(\int_0^L \xi_i^{'}\xi_j^{'} dx\right)a_i(t) + \beta\sum_{i=0}^{n-1}\left(\int_0^L \xi_i^{'}\xi_j dx\right)a_i(t) \\ \qquad + \sum_{i=0}^{n-1}\sum_{k=0}^{n-1}\left(\int_0^L \xi_i\xi_k^{'}\xi_j dx\right)a_i(t)a_k(t) = \int_0^L f\xi_j dx, \\ \sum_{i=0}^{n-1}\left(\int_0^L \xi_i\xi_j dx\right)a_i(0) = \int_0^L u_0(x)\xi_j dx, \quad j = 0, 1, \cdots, n-1. \end{cases} \tag{8}$$

Assume $m_{ij} = (\xi_i, \xi_j)$, $s_{ij} = (\xi_i^{'}, \xi_j^{'})$, $d_{ij} = (\xi_i^{'}, \xi_j)$, $n_{ijk} = (\xi_i\xi_k^{'}, \xi_j)$, $f_j = (f, \xi_j)$, $u_0^j = (u_0, \xi_j)$, and $M = (m_{ij})$, $S = (s_{ij})$, $C = (c_{ij})$, $N = (n_{ijk})$, $\vec{f} = (f_0, f_1, \cdots, f_{n-1})^T$, $\vec{u}_0 = (u_0^0, u_0^1, \cdots, u_0^{n-1})$, $\vec{a}_0 = (a_0(0), a_1(0), \cdots, a_{n-1})^T$, $\vec{a}(t) = (a_0(t), a_1(t), \cdots, a_{n-1}(t))^T$, then Eqs.(8) can be written in the matrix form

$$\begin{cases} (M+S)\frac{d\vec{a}}{dt} + (\alpha S + \beta C)\vec{a} + (\vec{a})^T N\vec{a} = \vec{f}, \\ M\vec{a}_0 = \vec{u}_0. \end{cases} \tag{9}$$

Eqs.(9) is a nonlinear ordinary differential equations which consists of n equations and n unknowns. system (9) can be written as standard first order nonlinear ordinary differential equations with initial condition, because M and $(M+S)$ are invertible matrix,

$$\frac{d\vec{a}}{dt} = (M+S)^{-1}(\vec{f} - (\alpha S + \beta C)\vec{a} - (\vec{a})^T N\vec{a}), \quad \vec{a}_0 = \vec{u}_0, \tag{10}$$

where, for simplicity take $\vec{u}_0 = M^{-1}\vec{u}_0$. The terms in the right hand side of the first equation of the system (10) are continuously differentiable, and the system (10) exists one and only one solution and have a zero equilibrium solution when forcing term $f(x,t)$ tends to zero with time infinity. In conclusion, the equilibrium solution as the starting point, we obtain the numerical solution of the system (1) by using Newton method.

3. Numerical examples and results

Example 1. We solve the system(1) with the following data:
$\alpha = 1, \beta = 1, L = \pi, u_0(x) = sinx, f = e^{-t}(cosx - sinx + 0.5e^{-t}sin2x)$. And in this case the exact solution is $e^{-t}sinx$.
The relative numerical results are shown in Table 1 and Table 2.

Example 2. We solve the system(1) with the following data:
$L = 1, u_0(x) = e^{-x}sin\pi x, f = 0$ and various α, β.

Table 1. Comparison of the numerical solutions of Example 1 obtained with various of Δt for $n = 64$ at $t = 0.1$ with the exact solutions

x	$\Delta t = 0.01$	$\Delta t = 0.001$	$\Delta t = 0.0005$	$\Delta t = 0.0001$	exact sol.
$\pi/64$	0.04397	0.04435	0.04437	0.04438	0.04439
$\pi/8$	0.34300	0.34593	0.34609	0.34622	0.34626
$\pi/4$	0.63379	0.63920	0.63950	0.63975	0.63981
$3\pi/8$	0.84393	0.85113	0.85153	0.85186	0.83596
$\pi/2$	0.89633	0.90397	0.90440	0.90474	0.90483
$5\pi/8$	0.82811	0.83516	0.83556	0.83588	0.83596
$3\pi/4$	0.63381	0.63921	0.63951	0.63975	0.63981
$7\pi/8$	0.34302	0.34593	0.34610	0.34623	0.34626
$63\pi/64$	0.04398	0.04435	0.04437	0.04439	0.04439

Table 2. Comparison of the numerical solutions of Example 1 obtained for various of x and $n = 64$, $\Delta t = 0.0001$ at different times with the exact solutions

x	t	numerical sol.	exact sol.
$\pi/4$	0.5	0.428708	0.428881
	1.0	0.260048	0.260130
	2.0	0.095683	0.095696
	3.0	0.035206	0.035204
	4.0	0.012953	0.012951
$\pi/2$	0.5	0.606305	0.606530
	1.0	0.367798	0.367879
	2.0	0.135356	0.135335
	3.0	0.049818	0.049787
	4.0	0.018338	0.018315
$3\pi/4$	0.5	0.428731	0.428881
	1.0	0.260091	0.260130
	2.0	0.095738	0.095696
	3.0	0.035251	0.035204
	4.0	0.012984	0.012951

The relative numerical results are shown in Figure 1 and Figure 2.

Table 1 and Table 2 show that the numerical scheme proposed in this paper converges very rapidly to solution and has good accuracy. From Figure 1 and Figure 2, we can see that the system (1) feature balance nonlinear and dispersive effects, but takes no account of dissipation. In conclusion, the numerical results show that the scheme is efficient,feasible and quite satisfactory.

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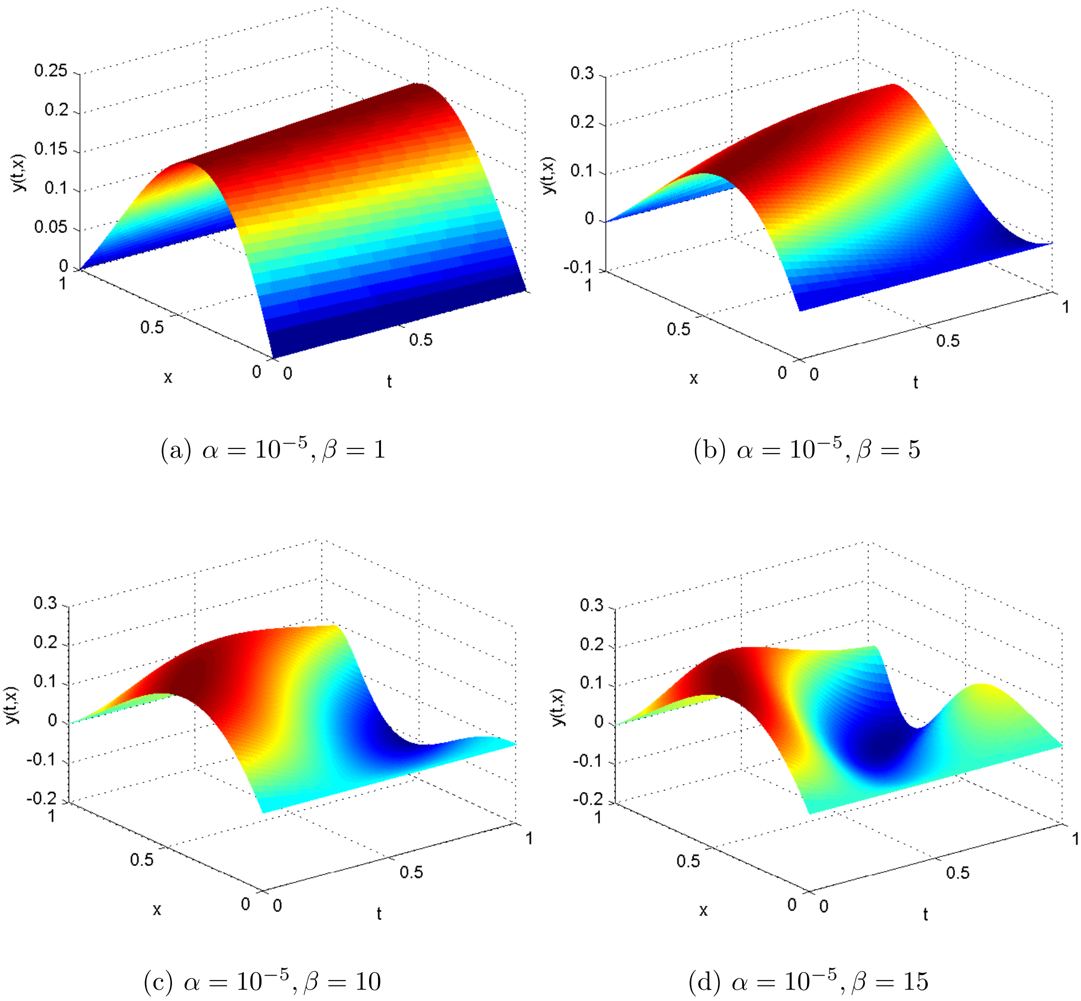


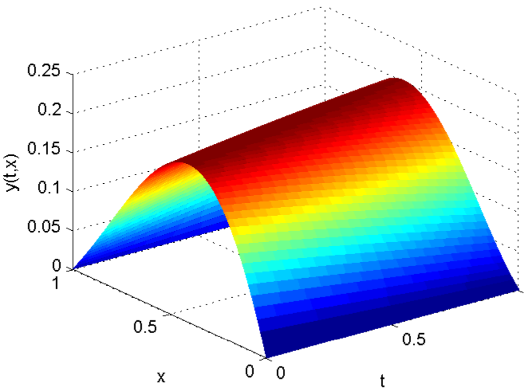
(a) $\alpha = 10^{-5}, \beta = 1$

(b) $\alpha = 10^{-5}, \beta = 5$

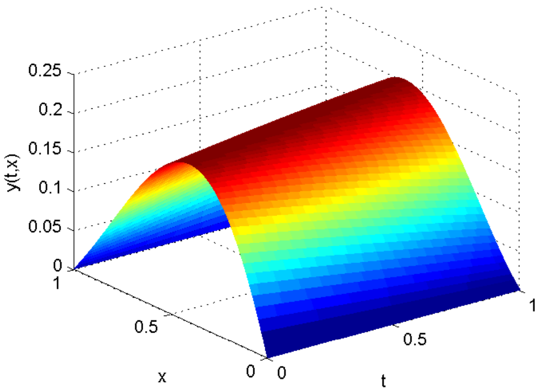
(c) $\alpha = 10^{-5}, \beta = 10$

(d) $\alpha = 10^{-5}, \beta = 15$

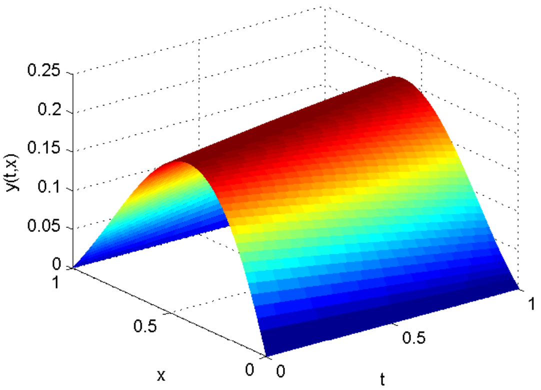
FIGURE 1. Approximate solution of Example 2 with a fixed α and various β.



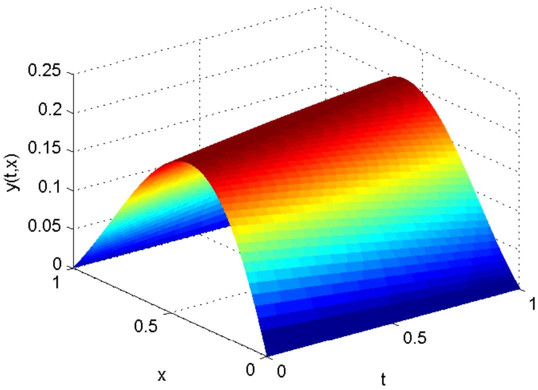
(a) $\alpha = 10^{-2}, \beta = 2$



(b) $\alpha = 10^{-3}, \beta = 2$



(c) $\alpha = 10^{-4}, \beta = 2$



(d) $\alpha = 10^{-5}, \beta = 2$

FIGURE 2. Approximate solution of Example 2 with a fixed β and various α.