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Transient and Stationary Analyses of the Surplus in a Risk Model

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Abstract

The surplus process in a risk model is stochastically analyzed. We obtain the characteristic function of the level of the surplus at a finite time, by establishing and solving an integro-differential equation for the distribution function of the surplus. The characteristic function of the stationary distribution of the surplus is also obtained by assuming that an investment of the surplus is made to other business when the surplus reaches a sufficient level. As a consequence, we obtain the first and second moments of the surplus both at a finite time and in an infinite horizon (in the long-run).

Keywords: Risk model, surplus process, characteristic function, integro-differential equation, stationary distribution.

1. Introduction

In this paper, we consider a classical risk model in which the surplus at time $t > 0$ is given by

$$U(t) = u + ct - \sum_{i=1}^{N(t)} Y_i,$$

where, $u = U(0)$, c is the premium rate, $N(t)$ is the number of claims by time t, and Y_i is the amount of the ith claim. $\{N(t),\ t > 0\}$ is assumed to be a Poisson process of rate $\lambda > 0$ and Y_i's are assumed to follow, independently and identically, distribution function G with mean $\mu > 0$. The premium rate c is usually assumed to be larger than $\lambda\mu$ which is the expected total amount of claims per unit time.

This risk model has been studied by many authors by assuming that a ruin occurs when the surplus becomes negative. They have obtained the ruin probability of the surplus and some related characteristics of the risk model. The core result on the ruin probability is well summarized in Klugman *et al.* (2004). The first passage time of the surplus to a certain level was introduced by Gerber (1990), thereafter, Gerber and Shiu (1997) obtained the joint distribution of the time of ruin, the surplus immediately before ruin and the deficit at ruin. Dickson and Willmot (2005) calculated the density of the time to ruin by an inversion of its Laplace transform.

However, until now, most works have been concentrated on the ruin probability of the surplus and its related characteristics. In this paper, we assume that the surplus process continues though it becomes negative and analyze stochastically the level of the surplus in the risk model. In Section 2, we derive the characteristic function of the level of the surplus at finite time $t > 0$, by establishing a partial

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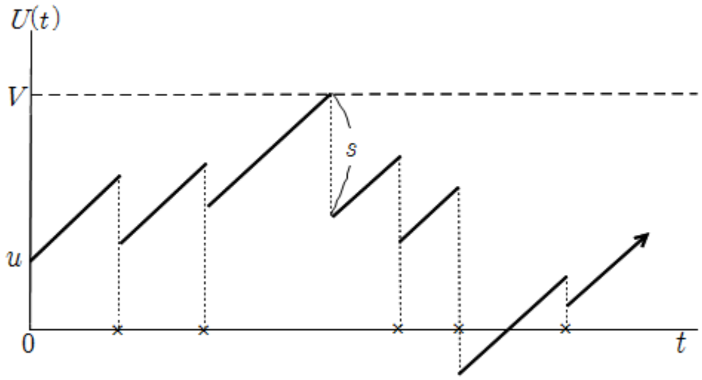

Figure 1: *A sample path of the surplus process with investment*

integro-differential equation for the distribution function of the surplus and solving the equation for it. As a consequence, we obtain the first and second moments of the surplus at finite time $t > 0$.

Since the premium rate is larger than the loss rate $\lambda\mu$, the surplus process goes eventually to infinity in the classical risk model. To analyze stochastically the level of the surplus in an infinite horizon, we assume that an investment of the surplus to other business is made by an amount s ($0 < s < V$), if the level of the surplus reaches a sufficient level $V > u$. This is a very practical assumption, since the large stock of the surplus increases the opportunity cost. The concept of the investment of the surplus was considered in Jeong *et al.* (2009) and Jeong and Lee (2010). They studied some optimal policies related to managing the surplus in the risk model.

The surplus process in the risk model with investments is illustrated in Figure 1.

In Section 3, we obtain the characteristic function of the stationary distribution of the surplus in the modified risk model by establishing and solving an ordinary integro-differential equation which does not depend on time t. By differentiating the characteristic function, we obtain the first and second moments of the surplus in the long-run (in an infinite horizon).

2. Transient Analysis of the Surplus

Let $F(x, t)$ be the distribution function of $U(t)$, the surplus at time $t > 0$, that is,

$$F(x, t) = P\{U(t) \leq x\}, \quad \text{for } -\infty < x < \infty.$$

Conditioning on whether a claim arrives in a small interval $[t, t + \Delta t]$, we can have the following relations between $U(t)$ and $U(t + \Delta t)$:

(i) If no claims arrive, then

$$U(t + \Delta t) = U(t) + c\Delta t.$$

(ii) If a claim arrives, then

$$U(t + \Delta t) = U(t) + c\Delta t - Y.$$

Hence, we can obtain the following equation for $F(x, t)$:

$$F(x, t + \Delta t) = [1 - \lambda\Delta t + o(\Delta t)]\, F(x - c\Delta t, t) + [\lambda\Delta t + o(\Delta t)]\, P\{U(t) - Y \leq x - c\Delta t\} + o(\Delta t).$$

Note that two or more claims arrive in a small interval $[t, t + \Delta t]$ is $o(\Delta t)$.

Observe that conditioning on Y yields

$$P\{U(t) - Y \leq x - c\Delta t\} = \int_0^\infty F(x - c\Delta + y, t)dG(y)$$

and applying Taylor series expansion on $F(x - c\Delta t, t)$ gives

$$F(x - c\Delta t, t) = F(x, t) - c\Delta t \frac{\partial}{\partial x} F(x, t) + o(\Delta t).$$

Inserting these two into the above equation, we have

$$F(x, t + \Delta t) = F(x, t) - c\Delta t \frac{\partial}{\partial x} F(x, t) - \lambda \Delta t F(x, t) + \lambda \Delta t \int_0^\infty F(x - c\Delta t + y, t)dG(y) + o(\Delta t).$$

Subtracting $F(x, t)$ from both sides of the equation, dividing by Δt, and letting $\Delta t \to 0$, we have the following integro-differential equation for $F(x, t)$:

$$\frac{\partial}{\partial t} F(x, t) = -c \frac{\partial}{\partial x} F(x, t) - \lambda F(x, t) + \lambda \int_0^\infty F(x + y, t)dG(y). \tag{2.1}$$

Let $\phi(r, t) = \int_{-\infty}^\infty e^{irx} dF(x, t)$ be the characteristic function of $F(x, t)$. Multiplying both sides of (2.1) by e^{irx} and taking Stieltjes integral with respect to x give

$$\frac{\partial}{\partial t} \phi(r, t) = icr\phi(r, t) - \lambda \phi(r, t) + \lambda \phi(r, t) \overline{\phi_Y}(r), \tag{2.2}$$

where $\overline{\phi_Y}(r) = \int_0^\infty e^{-iry} dG(y)$. Solving (2.2) for $\phi(r, t)$ with initial condition $\phi(r, 0) = e^{iru}$, we, finally, have

$$\phi(r, t) = \exp\left\{i(ct + u)r - \lambda t + \lambda t \overline{\phi_Y}(r)\right\}. \tag{2.3}$$

Differentiating $\phi(r, t)$ with respect to r, we can obtain the moments of $U(t)$. For examples,

$$E[U(t)] = u + (c - \lambda\mu)t,$$
$$E\left[U^2(t)\right] = [(c - \lambda\mu)t]^2 + \lambda t\left(\mu^2 + \sigma^2\right),$$

where $\sigma^2 = \mathrm{Var}(Y)$.

3. Stationary Analysis of the Surplus

In this section, after assuming that an investment of the surplus to other business is made by an amount s $(0 < s < V)$, if the level of the surplus reaches a sufficient level $V > u$, we obtain the characteristic function of the stationary distribution of the surplus.

Conditioning on whether a claim arrives in a small interval $[t, t + \Delta t]$, we can have the following relations between $U(t)$ and $U(t + \Delta t)$:

(i) If no claims arrive, then

$$U(t + \Delta t) = \begin{cases} U(t) + c\Delta t, & \text{when } U(t) \leq V - c\Delta t, \\ V - s + c\Delta t, & \text{when } V - c\Delta t < U(t) \leq V. \end{cases}$$

(ii) If a claim arrives, then

$$U(t+\Delta t) = U(t) + c\Delta t - Y.$$

From these relations, we can obtain the following equation for $x \le V$:

$$\begin{aligned} P\{U(t+\Delta t) \le x\} = [1-\lambda\Delta t + o(\Delta t)]\,[P\{U(t) \le x - c\Delta t, U(t) \le V - c\Delta t\} \\ +P\{V - s \le x - c\Delta t, V - c\Delta t < U(t) \le V\}] \\ + [\lambda\Delta t + o(\Delta t)]\,P\,[U(t) - Y \le x - c\Delta t] + o(\Delta t). \end{aligned}$$

Let $F(x,t) = P\{U(t) \le x\}$, for $x \le V$. When $V - s < x \le V$, if Δt is sufficiently small, the equation becomes

$$\begin{aligned} F(x, t+\Delta t) = [1-\lambda\Delta t + o(\Delta t)]\,[F(x - c\Delta t, t) + F(V,t) - F(V - c\Delta t, t)] \\ + [\lambda\Delta t + o(\Delta t)]P\,[U(t) - Y \le x - c\Delta t] + o(\Delta t). \end{aligned}$$

Observe again that by conditioning on Y,

$$P\{U(t) - Y \le x - c\Delta t\} = \int_0^\infty F(x - c\Delta t + y, t)dG(y)$$

and by the Taylor series expansion,

$$F(x - c\Delta t, t) = F(x,t) - c\Delta t\frac{\partial}{\partial x}F(x,t) + o(\Delta t).$$

Inserting these two into the above equation, subtracting $F(x,t)$ from both sides of the equation, dividing by Δt, and letting $\Delta t \to 0$, we can derive the following integro-differential equation for $F(x,t)$:

$$\frac{\partial}{\partial t}F(x,t) = -c\frac{\partial}{\partial x}F(x,t) - \lambda F(x,t) + cf(V,t) + \lambda\int_0^\infty F(x+y,t)dG(y), \tag{3.1}$$

where $f(V,t) = (\partial/\partial x)F(x,t)|_{x=V}$.

Now, when $x \le V - s$, by an argument similar to the foregoing, we can obtain the integro-differential equation for $F(x,t)$ as follows:

$$\frac{\partial}{\partial t}F(x,t) = -c\frac{\partial}{\partial x}F(x,t) - \lambda F(x,t) + \lambda\int_0^\infty F(x+y,t)dG(y). \tag{3.2}$$

Since $F(x,t) = 1$, for $x \ge V$, the last term in (3.1) and (3.2) can be written as

$$\int_0^\infty F(x+y,t)dG(y) = \int_0^{V-x} F(x+y,t)dG(y) + \overline{G}(V-x),$$

where $\overline{G}(y) = 1 - G(y)$. Inserting this into (3.1) and (3.2) and combining them together, we have the following integro-differential equation for $F(x,t)$:

$$\frac{\partial}{\partial t}F(x,t) = -c\frac{\partial}{\partial x}F(x,t) - \lambda F(x,t) + cf(V,t)I_{\{x>V-s\}} + \lambda\int_0^{V-s} F(x+y,t)dG(y) + \lambda\overline{G}(V-x), \tag{3.3}$$

where I_A is the indicator of A.

Let $F(x)$ be the stationary distribution function of $U(t)$. Since $(\partial/\partial t)F(x,t) = 0$ in the stationary case, $F(x)$, now, satisfies

$$0 = -c\frac{\partial}{\partial x}F(x) - \lambda F(x) + cf(V)I_{\{x>V-s\}} + \lambda\int_0^{V-x} F(x+y)dG(y) + \lambda\overline{G}(V-x). \tag{3.4}$$

Let $\phi(r) = \int_{-\infty}^{V} e^{irx}dF(x)$ be the characteristic function of $F(x)$. Multiplying both sides of (3.4) by e^{irx} and taking Stieltjes integral with respect to x give

$$0 = -c\left\{f(V)e^{irV} - ir\phi(r)\right\} - \lambda\phi(r) + cf(V)e^{ir(V-s)} + \lambda\phi(r)\overline{\phi_Y}(r), \tag{3.5}$$

where $\overline{\phi_Y}(r) = \int_0^{\infty} e^{-iry}dG(y)$. To obtain (3.5), we make use of the following identities:

$$\int_{-\infty}^{V} e^{irx}dI_{\{x>V-s\}} = e^{ir(V-s)}$$

and

$$\begin{aligned}
&\int_{-\infty}^{V} e^{irx}d\left[\int_0^{V-x} F(x+y)dG(y)\right] \\
&= \int_{-\infty}^{V}\int_0^{V-x} e^{irx}f(x+y)dG(y)dx - \int_{-\infty}^{V} e^{irx}dG(V-x) \\
&= \int_0^{\infty}\int_{-\infty}^{V-y} e^{irx}f(x+y)dxdG(y) - \int_{-\infty}^{V} e^{irx}dG(V-x) \quad \text{(by interchanging integral signs)} \\
&= \int_0^{\infty} e^{-iry}\int_{-\infty}^{V} e^{irz}f(z)dzdG(y) - \int_{-\infty}^{V} e^{irx}dG(V-x) \quad \text{(by putting } z = x+y\text{)} \\
&= \phi(r)\overline{\phi_Y}(r) - \int_{-\infty}^{V} e^{irx}d\overline{G}(V-x), \quad \text{since } dG(V-x) = d\overline{G}(V-x) \text{ with respect to } x.
\end{aligned}$$

Solving (3.5) for $\phi(r)$, we have

$$\phi(r) = \frac{cf(V)\left\{e^{irV} - e^{ir(V-s)}\right\}}{icr - \lambda + \lambda\overline{\phi_Y}(r)}. \tag{3.6}$$

Since $\phi(0) = 1$, we can obtain $f(V)$ by applying l'Hôpital's rule to (3.6), which is given by $f(V) = (c-\lambda\mu)/(cs)$.

Observe that the points where the surplus reaches V form embedded regeneration points of the surplus process. If we define a cycle as the period between two successive regeneration points, the expected length of a cycle will be $s/(c-\lambda\mu)$. Hence, we can obtain the same formula for $f(V)$ by applying the level crossing argument of Brill and Posner (1977), since the surplus reaches V once during a cycle and the slope of the surplus process is c.

Differentiating $\phi(r)$ with respect to r, we can obtain the moments of U, the surplus in the long-run (in an infinite horizon). For examples,

$$\begin{aligned}
E[U] &= \frac{2V-s}{2} - \frac{\lambda\mu_2}{2(c-\lambda\mu)}, \\
E\left[U^2\right] &= \frac{(\lambda\mu_2)^2}{2(c-\lambda\mu)^2} - \frac{(6V-3s)\lambda\mu_2 + 2\lambda\mu_3}{6(c-\lambda\mu)} + \frac{(s^2-3Vs+3V^2)}{3},
\end{aligned}$$

where $\mu = E(Y)$, $\mu_2 = E(Y^2)$ and $\mu_3 = E(Y^3)$.

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