Management Science and Financial Engineering
Vol 19, No 2, November 2013, pp.43-47
ISSN 2287-2043 | EISSN 2287-2361 |
http://dx.doi.org/10.7737/MSFE.2013.19.2.043

When Do the Unemployed Jump in the Workforce?

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(Received: October 24, 2013 / Revised: November 6, 2013 / Accepted: November 8, 2013)

ABSTRACT

This paper studies an optimal consumption and portfolio choice problem for unemployed people who have an option to work. Our problem is to find optimal consumption, risky investment, and workforce re-entry strategies for the unemployed. We find a closed form of the critical wealth level to re-enter the workforce. We show that the unemployed with a higher disutility of labor or a larger relative risk aversion are more reluctant to re-enter the workforce.

Keywords: Optimal Consumption, Optimal Risky Investment, Optimal Re-entry Time, Critical Wealth Level, Dynamic Programming

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1. INTRODUCTION

After the seminal paper of Merton (1969), many works have dealt with optimal consumption and portfolio choice problems with several applications for individuals. Especially, many researchers have been interested in the problem with labor income (see, e.g., Bodie *et al*. (1992) and Viceira (2001)). Farhi and Panageas (2007) and Dybvig and Liu (2010) examine individuals' optimal consumption, portfolio selection, and retirement, and model the labor supply for an individual as an optimal stopping problem. Recently, Jang *et al*. (2013) investigate the impact of unemployment risks on individuals' optimal behaviors. However, none of the existing literature has studied the workforce re-entry problem for an individual after voluntary or involuntary retirement.

This paper studies an optimal consumption and portfolio choice problem for voluntarily-unemployed people who have an option to work. When they exercise the option to work, they can receive labor income because they re-enter the workforce. We consider an unemployed individual who has a disutility of labor: she experiences utility loss from labor and utility gain from labor income after re-entering the workforce. Hence, her decision to re-enter the workforce is associated with the tradeoff between such utility loss and gain. Our model permits the unemployed to determine a critical wealth level, below which they are willing to re-enter the workforce. We further figure out the optimal re-entry time to the workforce, at which the utility gain from labor income is equivalent to the utility loss due to a disutility of labor.

Under this setup we obtain some interesting results concerning optimal consumption and investment strategies for unemployed individuals who have an option to work. The findings include: in optimum,

- the unemployed with a higher disutility of labor tend to re-enter the workforce at a lower wealth level, i.e., they are more reluctant to re-enter the workforce,
- the unemployed with a larger risk aversion tend to re-enter the workforce later,
- the unemployed with lower wealth (but still higher

than the critical wealth level) tend to reduce their consumption and risky investment less for a unit decrease of their wealth, and

- both consumption and risky investment decrease as a disutility of labor or a relative risk aversion increases.

2. THE MODEL

We consider an optimal consumption and portfolio choice problem for voluntarily-unemployed people with an option to enter the workforce. Their objective is to find optimal consumption, investment policies, and re-entry time to work, the following lifetime utility over consumption:

$$U = E\left[\int_0^{\tau} e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt + e^{-\beta t}\int_{\tau}^{\infty} e^{-\beta(t-\tau)}\left(\frac{c_t^{1-\gamma}}{1-\gamma} - l\right)dt\right],$$

where E is the expectation taken at time 0, τ is the optimal re-entry time to work for the unemployed, β is the subjective discount rate, γ is the coefficient of constant relative risk aversion, and l is the constant disutility due to labor.

We assume two broad classes of asset are traded in a financial market: a bond (or a risk-free asset) and a stock (or a risky asset). The bond price S_t^0 follows

$$dS_t^0 = rS_t^0 dt,$$

where $r > 0$ is a risk-free interest rate. The stock price S_t evolves according to the following equation:

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

where $\mu > r$ is the expected rate of stock return, $\sigma > 0$ is the stock volatility, and B_t is a standard Brownian motion defined on a suitable probability space. We also assume that the unemployed have an irreversible option to re-enter the workforce at some time τ and obtain a constant income stream I forever after re-entry. At re-entry time τ, we assume the unemployed should pay a job searching cost η. To guarantee the positivity of the option value, we take the assumption of $\frac{I}{r} > \eta$.

The wealth process W_t of the unemployed with initial wealth $W_0 = w$ should satisfy

$$dW_t = \begin{cases} (rW_t + \pi_t(\mu - r) - c_t)dt + \pi_t\sigma dB_t & 0 \le t < \tau, \\ (rW_t + \pi_t(\mu - r) - c_t + I)dt + \pi_t\sigma dB_t & t \ge \tau, \end{cases}$$

where π_t is the dollar amount invested in the stock, and I is the income rate after re-entering the workforce. At re-entry time τ, there is a lump-sum wealth loss η, from which we obtain the relationship of $W_\tau = W_{\tau-} - \eta$.

Letting

$$U_2(W_\tau) = \max_{(c,\pi)} E\left[\int_{\tau}^{\infty} e^{-\beta(t-\tau)}\left(\frac{c_t^{1-\gamma}}{1-\gamma} - l\right)dt\right],$$

U_2 must be

$$U_2(W_\tau) = K\frac{(W_\tau + I/r)^{1-\gamma}}{1-\gamma} - \frac{l}{\beta},$$

(see Merton (1969)), where

$$K = \left(\frac{1}{\delta}\right)^{\gamma},\ \ \delta = \frac{\gamma-1}{\gamma}\left(r + \frac{\theta^2}{2\gamma}\right) + \frac{\beta}{\gamma},\ \ \theta = \frac{\mu - r}{\sigma}.$$

By the principle of dynamic programming, our problem is equivalent to the problem of finding the following value function $V(w)$:

$$V(w) \equiv \max_{(c,\pi,\tau)} E\left[\int_0^{\tau} e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt + e^{-\beta\tau-}\left(K\frac{(W_{\tau-} + I/r - \eta)^{1-\gamma}}{1-\gamma} - \frac{l}{\beta}\right)\right].$$

The value function satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$(rw + \pi(\mu - r) - c)V'(w) + \frac{1}{2}\pi^2\sigma^2 V''(w) - \beta V(w) + \frac{c^{1-\gamma}}{1-\gamma} = 0,$$

for $0 \le t < \tau$. Furthermore, it should satisfy the value-matching and smooth-pasting conditions at a critical wealth level $\underline{w}$, below which it is optimal to reenter the workforce:

$$V(\underline{w}) = K\frac{(\underline{w} + I/r - \eta)^{1-\gamma}}{1-\gamma} - \frac{l}{\beta},$$
$$V'(\underline{w}) = K(\underline{w} + I/r - \eta)^{-\gamma}. \tag{1}$$

By utilizing the first order conditions of c and π, we can rewrite the HJB equation as

$$rwV'(w) + \frac{\gamma}{1-\gamma}(V'(w))^{1-\frac{1}{\gamma}} - \frac{1}{2}\theta^2\frac{\{V'(w)\}^2}{V''(w)} - \beta V(w) = 0. \tag{2}$$

3. OPTIMAL CONSUMPTION, INVESTMENT, AND RE-ENTRY

We utilize the convex-duality method (see Farhi and Panageas (2007), Dybvig and Liu (2010)) and show analytic solutions for optimal strategies in terms of a dual variable of initial wealth *w*.

Theorem 3.1: *The optimal consumption c and portfolio* π *prior to re-entering the workforce are given as:*

$$c_t = \delta\left[w + n_1 A_1 \left(x^*(w)\right)^{n_1-1} \right], \quad (3)$$

$$\pi_t = \frac{\theta}{\sigma}\left[\frac{1}{\gamma} w + n_1 A_1 \left(x^*(w)\right)^{n_1-1} (n_1 - 1 + \frac{1}{\gamma}) \right], \quad (4)$$

where $n_1 > 1$ *and* A_1 *are positive constants satisfying*

$$A_1 = \frac{1}{n_1}\left(\frac{I}{r} - \eta\right)\left(\frac{n_1 l}{\beta(n_1-1)(I/r-\eta)}\right)^{1-n_1},$$

$$\equiv \frac{1}{n_1}\left(\frac{I}{r} - \eta\right)\bar{x}^{\,1-n_1} \text{ for } \bar{x} > 0,$$

and

$$n_1 = \frac{-\left(\beta - r - \frac{1}{2}\theta^2\right) + \sqrt{\left(\beta - r - \frac{1}{2}\theta^2\right)^2 + 2\theta^2\beta}}{\theta^2}$$

Here, $0 < x^*(w) \le \bar{x}$ *is a decreasing function with respect to initial wealth w, and is the solution to the following equation:*

$$w = -n_1 A_1 \left(x^*(w)\right)^{n_1-1} + \left(x^*(w)\right)^{-1/\gamma}\frac{1}{\delta}. \quad (5)$$

Moreover, the optimal entry time τ *is given by*

$$\tau = \inf\left\{ t \ge 0 : W_t \le \underline{w} \right\},$$

where

$$\underline{w} = K^{1/\gamma}\left(\frac{n_1 l}{\beta(n_1-1)(I/r-\eta)}\right)^{-1/\gamma} - \frac{I}{r} + \eta. \quad (6)$$

Proof: Following Farhi and Panageas (2007) and Dybvig and Liu (2010), we utilize the following convex dual function $\phi(x)$ of the value function $V(w)$:

$$V(w) = \inf_x \left\{\phi(x) + xw\right\} = \phi(x) - x\phi'(x).$$

Then equation (2) and the conditions in (1) can be rewritten as:

$$\frac{1}{2}\theta^2 x^2 \phi''(x) + (\beta - r)x\phi'(x)$$

$$-\beta\phi(x) + \frac{\gamma}{1-\gamma}x^{1-\frac{1}{\gamma}} = 0, \quad 0 < x \le \bar{x}, \quad (7)$$

$$\phi(\bar{x}) = K^{1/\gamma}\frac{\gamma}{1-\gamma}\bar{x}^{\,1-1/\gamma} + \left(\frac{I}{r} - \eta\right)\bar{x} - \frac{l}{\beta},$$

$$\phi'(x) = -K^{1/\gamma}\bar{x}^{-1/\gamma} + \frac{I}{r} - \eta. \quad (8)$$

The general solution of (7) has a closed-form of

$$\phi(x) = A_1 x^{n_1} + A_2 x^{n_2} + \frac{\gamma}{1-\gamma}x^{1-\frac{1}{\gamma}}\frac{1}{\delta}, \quad 0 < x \le \bar{x}, \quad (9)$$

where A_1 and A_2 are constants to be determined and $n_1 > 1$ and $n_2 < 0$ are the two solutions to

$$\frac{1}{2}\theta^2 n^2 + \left(\beta - r - \frac{1}{2}\theta^2\right)n - \beta = 0.$$

The fact that $\phi(x)$ should satisfy equation (7) for $0 < x \le \bar{x}$ makes $A_2 = 0$. Now it remains to determine two unknown constants A_1 and *x*, which are derived from the two equations in (8). Substituting the solution $\phi(x)$ in (9) into the equations in (8) yields:

$$\bar{x} = \frac{n_1 l}{\beta(n_1-1)(I/r-\eta)}, \text{ and } A_1 = \frac{1}{n_1}\left(\frac{I}{r} - \eta\right)\bar{x}^{\,1-n_1}.$$

The optimal consumption *c* and portfolio π are directly derived from the first order conditions.

Finally, equation (5) and the critical wealth level to re-enter the workforce $\underline{w}$ are derived from the definition of the convex dual function $\phi(x)$. Further, by differentiating equation (5) with respect to the initial wealth *x*, we get the decreasing property of $x^*(w)$. **Q.E.D.**

Theorem 3.1 asserts that it is optimal to re-enter the workforce as soon as the wealth of unemployed people approaches the critical wealth level $\underline{w}$ from above. If their wealth is larger than $\underline{w}$, it is optimal to remain unemployed. Intuitively, this is obvious because the unemployed with sufficient wealth are not willing to bear the unhappiness of labor. Moreover, from relation (6), we observe that $\underline{w}$ is a decreasing function with respect to *l*. This implies that a higher disutility of labor induces the unemployed to postpone the re-entry into the workforce. Notice that the critical wealth level $\underline{w}$ could be negative for a sufficiently large disutility.

The optimal consumption c_t described in equation (3) prior to re-entering the workforce consists of two parts; the first part is the optimal consumption policy obtained from the classical models such as Merton (1969), and the second is associated with an option to work. Since $n_1 > 1$ and A_1 are positive, the unemployed in our model are willing to consume more than those who do not have the option to work. Notice that as

$x^*(w)$ approaches zero, optimal consumption approaches the classical consumption level, implying that the unemployed with sufficient wealth do not care much about the option to work. However, when their wealth approaches the critical wealth level $\underline{w}$, $x^*(w)$ increases to the maximum value of $\bar{x}$. Thus, the option to work comes to have a large (positive) impact on their optimal consumption behaviors. Intuitively, since an income stream is guaranteed by the option to work after the re-entry into the workforce, the unemployed with wealth greater than the critical level by just a small amount will consume more to acquire some immediate utility.

Similarly, the optimal risky investment π_t before working is comprised of two parts; the first part of the right hand side in equation (4) is the conventional optimal risky investment of Merton (1969), and the second is associated with the option to work. The option to work affects the optimal risky investment behaviors of the unemployed in the second part, and this effect becomes increasingly crucial as their wealth approaches $\underline{w}$. This is because the unemployed can get labor income if only they re-enter the workforce, so the option can provide an effective hedge against market risks. As expected, the option to work is poorly attractive for the unemployed who have already accumulated wealth sufficiently.

4. NUMERICAL IMPLICATIONS

4.1 Parameter Values

The risk-free interest rate is fixed as $r = 3.71\%$, which is the annual rate of return from rolling over 1-month T-bills during the time period of 1926-2009. We can obtain the data from Bureau of Labor Statistics. We assume β has the same value as r. We utilize $\mu =$ 11.23% and $\sigma = 19.54\%$, which are the return and standard deviation of the world's large stocks during the time period of 1926-2009. The data is included in p.170 of Bodie *et al*. (2010). We set $l = 0.01$, $\gamma = 2$, $I = 1$, and $\eta = 10$, following Miao and Wang (2007).

4.2 Wealth at Re-entering the Workforce

Figure 1 shows the critical wealth level $\underline{w}$ to re-enter the workforce for unemployed people with respect to a disutility l and a relative risk aversion γ. A higher disutility of labor and a higher relative risk aversion induce the lower critical wealth levels. It is quite intuitive that the unemployed with a higher preference of leisure tend to re-enter the workforce later. Interestingly, our model shows that the unemployed with a larger relative risk aversion also tend to re-enter the workforce later. This is because the assumption of a constant disutility during post-retirement induces the relatively smaller utility gain obtained from consumption for the unemployed with a larger relative risk aversion.

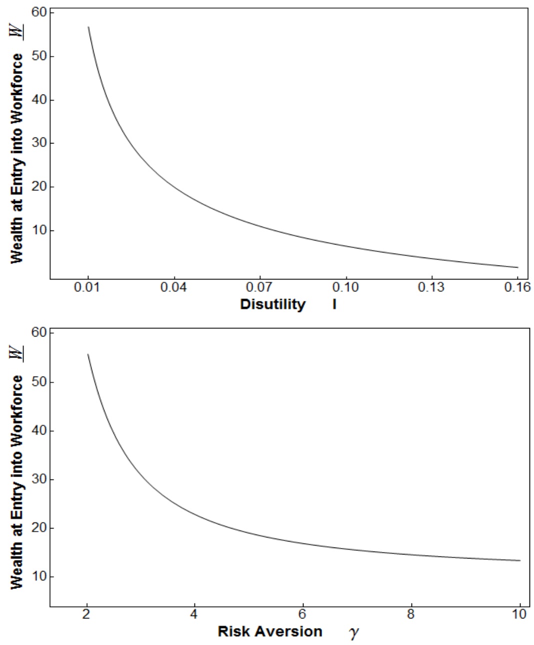

Figure 1. Critical wealth levels to re-enter the workforce, $\underline{w}$, as a function of a disutility l and a relative risk aversion γ : Parameter values are set as follows: $\beta = 0.0371$, $r = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $\gamma = 2$, $\eta = 10$, and $I = 1$

4.3 Optimal Consumption and Risky Investment Policies

Figure 2 depicts the optimal consumption c_t and risky investment π_t for unemployed people as a function of initial wealth w with several disutility values. For each disutility, both optimal consumption and risky investment policies are convex functions with respect to initial wealth and decrease as a disutility value increases. The convexity of the optimal consumption and risky investment implies the *marginal propensity* to consume/invest in the risky asset out of wealth decreases as wealth approaches the critical wealth level. Intuitively, the unemployed with lower wealth (but still higher than the critical wealth level) tend to reduce their consumption and risky investment less for a unit decrease of their wealth, since they can get the certain wealth gain of $(\frac{I}{r} - \eta)$ after re-entering the workforce. On the other hand, it is obvious that a higher disutility of labor induces the lower value of the optimal consumption and risky investment, since the unemployed with much unhappiness from labor are more reluctant to enter the workforce. This implies that they are willing to keep up their current (unemployed) status by reducing their consumption and risky investment.

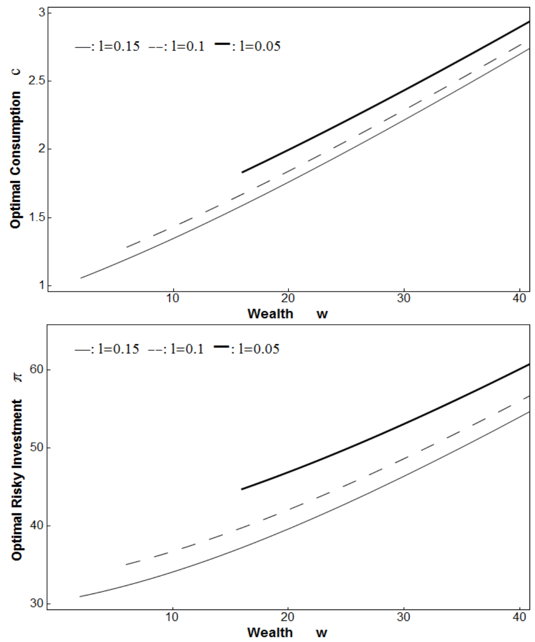

Figure 2. Optimal consumption and risky investment policies for the unemployed as a function of initial wealth w with several disutility values l: Parameter values are set as follows: β = 0.0371, r = 0.0371, μ = 0.1123, σ = 0.1954, l = 0.01, η = 10, and $I = 1$

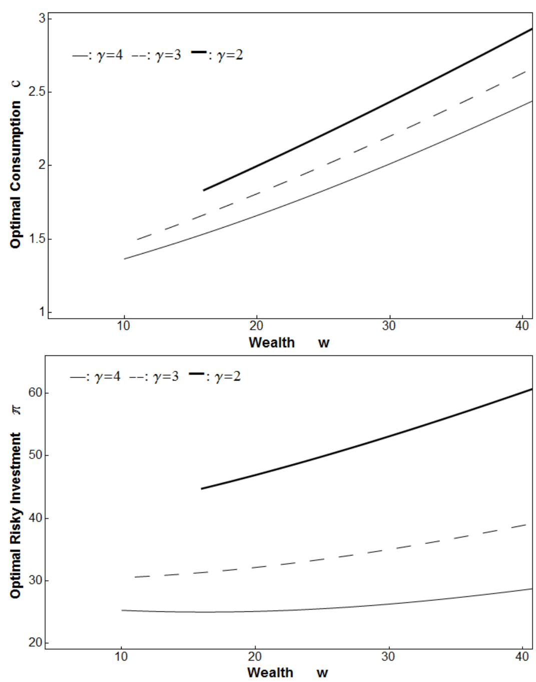

Figure 3. Optimal consumption and risky investment policies for the unemployed as a function of initial wealth w with several relative risk aversions γ: Parameter values are set as follows: $\beta = 0.0371$, $r = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $l = 0.01$, $\eta = 10$, and $I = 1$

Figure 3 shows the optimal consumption c_t and risky investment π_t for unemployed people with several relative risk aversions γ. For each risk aversion, both optimal consumption and risky investment policies decrease as a relative risk aversion increases. It follows from the fact that the unemployed with a larger relative risk aversion are more reluctant to enter the workforce because they feel as if they had a relatively higher disutility.

5. CONCLUSIONS

This paper studies an optimal consumption and portfolio choice problem for unemployed people with an option to work. Our problem is to find optimal consumption, risky investment policies, and workforce reentry strategies for the unemployed. This paper finds a closed form of the critical wealth level to re-enter the workforce for the unemployed and shows some interesting results concerning changes in a disutility of labor and a relative risk aversion.

ACKNOWLEDGEMENTS

This research in the paper was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (NRF-2012 R1A1A2038735).

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