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CONGRUENCE PROPERTIES OF COEFFICIENTS OF MODULAR FORMS FOR $\Gamma_0^+(5)$

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Abstract. We find congruence properties on the coefficients of modular forms for $\Gamma_0^+(5)$ generated by $\Gamma_0(5)$ and a Fricke involution $\begin{pmatrix} 0 & -1 \\ 5 & 0 \end{pmatrix}$.

1. Introduction

The study of the arithmetic properties of modular forms with integers is an interesting branch in the theory of modular forms (see [3]). Choie, Kohnen and Ono (see [1]) obtained congruence properties for coefficients of modular forms for $SL_2(\mathbb{Z})$. In this paper we discover congruence properties on the coefficients of modular forms for $\Gamma_0^+(5)$ which is generated $\Gamma_0(5)$ and a Fricke involution $\begin{pmatrix} 0 & -1 \\ 5 & 0 \end{pmatrix}$. Let k be an even integer. Let $M_k(\Gamma_0^+(5))$ the vector space of modular forms for $\Gamma_0^+(5)$ and $r := \dim M_k(\Gamma_0^+(5))$. Indeed, we have the following.

(1) $M_2(\Gamma_0^+(5)) = \{0\}$.
(2) $\dim M_k(\Gamma_0^+(5)) = (k-2)/4$ if $k \equiv 2 \pmod 4$ and $\dim M_k(\Gamma_0^+(5)) = k/4 + 1$ otherwise. (See Theorem 2.5.2 in [2]).

As usual, we let $\mathbb{H}$ be the complex upper half plane and $q = e^{2\pi i z}$ $(z \in \mathbb{H})$ and

$$E_k = 1 - \frac{2k}{B_k} \sum_{n \geq 0} \sigma_{k-1}(n) q^n$$

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be an Eisenstein series of weight k, where $\sigma_{k-1}(n)$ is the sum of $(k-1)$-st powers of the positive divisors of n and B_k is Bernoulli number. For instance,

$E_4(z) = 1+240q+2160q^2+\cdots$ and $E_6(z) = 1-504q+-16632q^2+\cdots$.

We are ready to state our main theorem.

THEOREM 1.1. *Let $k > 4r-4$ be an even positive integer such that $k \equiv 0 \pmod 4$. For any $f = \sum_{n\geq 0} a_f(n)q^n \in M_k(\Gamma_0^+(5)) \cap \mathbb{Z}[[q]]$, we have that for each positive integer b,*

$$a_{fE_6}(5^b) \equiv -a_f(0) \pmod 5.$$

2. Proof of Theorem 1.1

For each positive even integer $k > 2$, let

$$E_k^+(z) = E_k + 5^{k/2}E_k(5z),$$
$$E_2(z) = 1 - 24\sum_{n>0}\sigma_1(n)q^n, \quad E_2^+(z) = E_2 - 5E_2(5z),$$

then $E_k^+(z)$ is a modular form for $\Gamma_0^+(5)$ of weight k and $E_2^+(z)$ is a modular form for $\Gamma_0(5)$ (see [5, page 88]) whose the sign of the Fricke involution is -1. Consequently $(E_2^+(z))^2$ is a modular form for $\Gamma_0^+(5)$ of weight 4

Specially we have the following Fourier expansions:

$$E_4^+(z) = 26 + 240q + \cdots, \quad (E_2^+(z))^2 = 16 + 192q + \cdots .$$

Thus

$$\Delta_5^+(z) := \frac{13(E_2^+(z))^2 - 8E_4^+(z)}{1576} = q + \cdots$$

is a normalized cusp form for $\Gamma_0^+(5)$ of weight 4. The below proposition guarantees that $\Delta_5^+(z)$ has no zero on $\mathbb{H}$.

PROPOSITION 2.1. *Let f be a modular form for $\Gamma_0^+(5)$ of weight k, which is not identically zero. We have*

$$\sum_{p\in\Gamma_0^+(5)\backslash\mathbb{H}} e_p v_p(f) + v_\infty(f) = \frac{k}{4},$$

where $1/e_p$ is the cardinality of $\Gamma_0^+(5)_p$ and $v_p(f)$ is the order of a modular form f at a point p.

Proof. See [4, Proposition 2.1]. □

We define a Hauptmodul $j_5^+(z)$ for $\Gamma_0^+(5)$ which plays an important role in this paper as follows

$$j_5^+(z) := \frac{E_4^+(z)}{\Delta_5^+(z)} = \frac{1}{q} + \cdots .$$

For any $f \in M_k(\Gamma_0^+(5))$, we define

$$W(f) = \frac{f}{(\Delta_5^+)^{r-1}}.$$

To prove Theorem 1.1 we need the following proposition.

PROPOSITION 2.2. *W is a vector space isomorphism from $M_k(\Gamma_0^+(5))$ onto the space R of polynomials in j_5^+ of degree less than r.*

Proof. For $d = 0, 1, ..., r-1$ the functions $(j_5^+)^d(\Delta_5^+)^{r-1} \in M_k(\Gamma_0^+(5))$. Since $W((j_5^+)^d(\Delta_5^+)^{r-1}) = (j_5^+)^d$, W carries the subspace Q of $M_k(\Gamma_0^+(5))$ generated by the modular forms $(j_5^+)^d(\Delta_5^+)^{r-1}$ isomorphically onto R. Hence dim$Q = r$ which implies that $Q = M_k(\Gamma_0^+(5))$.

We are ready to prove Theorem 1.1. We note that two functions

$$\frac{-1}{2\pi i}\frac{dj_5^+(z)}{dz} = \frac{26}{q} + ...$$

and

$$\frac{E_6^+(z)}{\Delta_5^+(z)} = \frac{126}{q} + \cdots$$

are weakly holomorphic modular forms for $\Gamma_0^+(5)$ of weight 2. We note that $M_2(\Gamma_0^+(5)) = \{0\}$. These imply that

$$\frac{-63}{26\pi i}\frac{dj_5^+(z)}{dz} = \frac{E_6^+(z)}{\Delta_5^+(z)}.$$

Moreover, we have that

$$j^m\frac{dj_5^+(z)}{dz} = \frac{1}{m+1}\frac{d(j_5^+(z))^{m+1}}{dz} \qquad (m \in \mathbb{Z}, m \geq 0).$$

Since the constant term in the Fourier expansion of $\frac{d(j_5^+(z))^{m+1}}{dz}$ is zero, by linearity it follows that

$$(j_5^+)^{5^b-r}\frac{-63f}{26\pi i(\Delta_5^+)^{r-1}}\frac{dj_5^+}{dz}$$

has constant term zero. Thus we have that the constant term of

$$\begin{aligned}(j_5^+)^{5^b-r}\frac{-63f}{26\pi i(\Delta_5^+)^{r-1}}\frac{dj_5^+}{dz} &\equiv \frac{fE_6}{\Delta_5^+(5^b z)} \\ &\equiv (\sum_{n\geq 0} a_{fE_6}(n))(q^{-5^b}+1+...) \\ &\equiv \cdots + (a_{fE_6}(5^b)+a_{fE_6}(0)) + \cdots \pmod 5\end{aligned}$$

is zero modulo 5 which means

$$a_{fE_6}(5^b) \equiv -a_{fE_6}(0) \equiv -a_f(0) \pmod 5.$$

□

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