JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **26**, No. 4, November 2013
http://dx.doi.org/10.14403/jcms.2013.26.4.915

2-NORM MIDPOINTS AND 2-NORMED EQUALITIES IN 2-NORMED SPACES

Sang-Cho Chung*

Abstract. In this paper, we investigate some properties of 2-norm midpoints and 2-normed equalities in 2-normed spaces.

1. Introduction and preliminaries

We assume that every space is a linear space over the field $\mathbb{R}$ of real numbers.

In the 1960's, the concept of 2-normed spaces was introduced by S. Gähler [1, 2] and many mathematicians studied on this subject.

In this paper, under the 2-normed spaces we give easy solutions of Theorem 2.1 of [3] in Theorem 2.3 and investigate some properties of 2-normed equalities in Theorem 2.5.

Let give us some definitions and lemmas for our main results.

Definition 1.1. Let $\mathcal{X}$ be a linear space over $\mathbb{R}$ with $\dim \mathcal{X} > 1$ and let $\|\cdot,\cdot\| : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be a function satisfying the following properties:

(2N1) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
(2N2) $\|x, y\| = \|y, x\|$,
(2N3) $\|\alpha x, y\| = |\alpha|\|x, y\|$,
(2N4) $\|x, y + z\| \le \|x, y\| + \|x, z\|$

for all $x, y, z \in \mathcal{X}$ and $\alpha \in \mathbb{R}$. Then the mapping $\|\cdot,\cdot\|$ is called a 2-*norm* on $\mathcal{X}$ and the pair $(\mathcal{X}, \|\cdot,\cdot\|)$ is called a *linear* 2*-normed space*. Sometimes the condition (2N4) called the *triangle inequality*.

Remark 1.2. We have some basic properties for a linear 2-normed space $\mathcal{X}$ over $\mathbb{R}$ with $\dim \mathcal{X} > 1$.

Received September 31, 2013; Accepted October 30, 2013.
2010 Mathematics Subject Classification: Primary 46B06, 41A65; Secondary 30L99.
Key words and phrases: 2-normed space, 2-metric space, 2-norm midpoint, 2-normed equality.

(1) For all x, y in $\mathcal{X}$, we have $0 \leq \|x, y\|$.
(2) For all α in $\mathbb{R}$ and x, y in $\mathcal{X}$, we have $\|x, y\| = \|x, y + \alpha x\|$.
(3) For all x, y, z in $\mathcal{X}$, we have

$$|\,\|x, z\| - \|y, z\|\,| \leq \|x - y, z\| \leq \|x, z\| + \|y, z\|.$$

and

$$|\,\|x, z\| - \|y, z\|\,| \leq \|x + y, z\| \leq \|x, z\| + \|y, z\|.$$

In particular, if $m = \min\{\|x, z\|, \|y, z\|\}$, then

$$-2m \leq \|x + y, z\| - \|x - y, z\| \leq 2m.$$

Proof. (1) and (2) follow from the definitions of 2-normed spaces.

(3) For all x, y, z in $\mathcal{X}$, we have $\|x, z\| = \|x - y + y, z\| \leq \|x - y, z\| + \|y, z\|$. Hence we have $\|x, z\| - \|y, z\| \leq \|x - y, z\|$.

On the other hand, $\|y, z\| = \|y - x + x, z\| \leq \|y - x, z\| + \|x, z\|$, or $\|y, z\| - \|x, z\| \leq \|y - x, z\| = \|x - y, z\|$. Therefore we get

$$|\,\|x, z\| - \|y, z\|\,| \leq \|x - y, z\|.$$

The other parts follows from (2N2), (2N3) and (2N4). □

Definition 1.3. A sequence $\{x_n\}$ in a linear 2-normed space $\mathcal{X}$ is called a *convergent sequence* if there is a point $x \in \mathcal{X}$ such that $\lim_{n\to\infty} \|x_n - x, y\| = 0$ for all $y \in \mathcal{X}$. If $\{x_n\}$ converges to x, write $x_n \to x$ as $n \to \infty$ and call x the *limit* of $\{x_n\}$. In this case, we also write $\lim_{n\to\infty} x_n = x$.

Theorem 1.4. *Let $\mathcal{X}$ be a linear 2-normed space with $dim\mathcal{X} = r$. Suppose that $\{x_n\}$ is a sequence in $\mathcal{X}$ and $\{y_1, y_2, \cdots, y_r\}$ is a basis of $\mathcal{X}$. Then for a point $x \in \mathcal{X}$ we have the following.*

(1) $\lim_{m,n\to\infty} \|x_n - x_m, y\| = 0$ *for all* $y \in \mathcal{X}$ if and only if $\lim_{m,n\to\infty} \|x_n - x_m, y_i\| = 0$ for $i = 1, 2, \cdots, r$.
(2) $\lim_{m,n\to\infty} \|x_n - x, y\| = 0$ for all $y \in \mathcal{X}$ if and only if $\lim_{m,n\to\infty} \|x_n - x, y_i\| = 0$ for $i = 1, 2, \cdots, r$.

Proof. (1) ($\Rightarrow$) It is clear.

($\Leftarrow$) For all $y \in \mathcal{X}$, there are numbers $\alpha_1, \alpha_2, \cdots, \alpha_r \in \mathbb{R}$ such that $y = \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_r y_r$. Hence we have

$$\begin{aligned}\|x_n - x_m, y\| &= \|x_n - x_m, \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_r y_r\| \\ &\leq \|x_n - x_m, \alpha_1 y_1\| + \cdots + \|x_n - x_m, \alpha_r y_r\| \\ &= |\alpha_1|\|x_n - x_m, y_1\| + \cdots + |\alpha_r|\|x_n - x_m, y_r\|.\end{aligned}$$

Therefore we have $\lim_{m,n\to\infty} \|x_n - x_m, y\| = 0$ for all $y \in \mathcal{X}$.

(2) The proof is similar to (1). □

LEMMA 1.5. *Let $\mathcal{X}$ be a linear 2-normed space over $\mathbb{R}$ with $\dim \mathcal{X} > 1$. Let $\{x_n\}$ be a sequence in $\mathcal{X}$ and x be a vector of $\mathcal{X}$. Then the following are equivalent.*

(1) *The vector x is a limit of $\{x_n\}$. That is, for all y in $\mathcal{X}$,* $\lim_{n\to\infty} \|x_n - x, y\| = 0$.
(2) *For all a, y in $\mathcal{X}$,* $\lim_{n\to\infty} \|a - x_n, y\| = \|a - x, y\|$.
(3) *For all a, y in $\mathcal{X}$,* $\lim_{n\to\infty} \|a - x_n, y - x_n\| = \|a - x, y - x\|$.
(4) *For all y in $\mathcal{X}$,* $\lim_{n\to\infty} \|x_n - y, x - y\| = 0$.

Proof. (1) $\Rightarrow$ (2) By remark 1.2(3), for all a, y in $\mathcal{X}$, we have the following.

$$|\, \|a - x_n, y\| - \|a - x, y\| \,| \le \|x_n - x, y\|.$$

Hence we have $\lim_{n\to\infty} \|a - x_n, y\| = \|a - x, y\|$ for all a, y in $\mathcal{X}$.

(2) $\Rightarrow$ (3) For all a, y in $\mathcal{X}$, we have $\lim_{n\to\infty} \|a - x_n, y - x_n\|$ $= \lim_{n\to\infty} \|a - x_n, y - a\| = \|a - x, y - a\| = \|a - x, y - x\|$.

(3) $\Rightarrow$ (4) Replacing a by y and y by x, we have

$$\lim_{n\to\infty} \|y - x_n, x - y\| = \lim_{n\to\infty} \|y - x_n, x - x_n\| = \|y - x, x - x\| = 0$$

for all y in $\mathcal{X}$.

(4) $\Rightarrow$ (1) For all y in $\mathcal{X}$, we have the following.

$$\|x_n - x, y\| = \|x_n - x + y, y\| = \|x_n - (x - y), x - (x - y)\|.$$

Hence we have $\lim_{n\to\infty} \|x_n - x, y\| = 0$ for all y in $\mathcal{X}$. □

THEOREM 1.6 (cf. [4] Lemma 1.6). *For a convergent sequence $\{x_n\}$ in a linear 2-normed space $\mathcal{X}$, we have* $\lim_{n\to\infty} \|x_n, y\| = \|x, y\| = \|\lim_{n\to\infty} x_n, y\|$ *for all $y \in \mathcal{X}$.*

Proof. In Lemma 1.5(2), take $a = 0$. □

The following lemma has fewer conditions than Lemma 1.2 of [4]. We need only two linearly independent vectors.

LEMMA 1.7 (cf. [4] Lemma 1.2). *Let $(\mathcal{X}, \|\cdot,\cdot\|)$ be a linear 2-normed space with $\dim \mathcal{X} > 1$. If $\|x, y\| = \|x, z\| = 0$ for linearly independent $y, z \in \mathcal{X}$, then $x = 0$.*

In particular, If $\|x, y\| = 0$ for all $y \in \mathcal{X}$, then $x = 0$.

Proof. By the hypothesis, x and y are linearly dependent, and also x and z are linearly dependent. Then since y and z are not zero, there exist non-zero scalars α and α' such that $\alpha x + \beta y = 0$ and $\alpha' x + \beta' z = 0$ for some scalars β and β'. Hence we have

$$x = -\frac{\beta}{\alpha} y \text{ and } x = -\frac{\beta'}{\alpha'} z.$$

Thus we have

$$-\frac{\beta}{\alpha}y + \frac{\beta'}{\alpha'}z = 0.$$

Since y and z are linearly independent, we have $\beta = \beta' = 0$. Therefore we have $x = 0$. □

2. Main results

Firstly, we define the 2-metric space.

DEFINITION 2.1. A *2-metric space* is a space $\mathcal{X}$ with a real-valued nonnegative function d defined on $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$ which the following conditions:

(2M1) For each pair of elements x, y in $\mathcal{X}$ with $x \neq y$, there exists an element z in $\mathcal{X}$ such that $d(x, y, z) \neq 0$,
(2M2) $d(x, y, z) = 0$ whenever at least two of the points x, y, z are equal,
(2M3) $d(x, y, z) = d(x, z, y) = d(y, z, x)$,
(2M4) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$,

for all x, y, z, w in $\mathcal{X}$. d is called a *2-metric* the space $\mathcal{X}$ and $(\mathcal{X}, d)$ is called a *2-metric space*.

From the condition (2M3), we can easily show that $d(x, y, z) = d(x, z, y) = d(y, z, x) = d(y, x, z) = d(z, x, y) = d(z, y, x)$.

If $(\mathcal{X}, d)$ is a linear 2-normed space, then the function $d(x, y, z) = \|x - z, y - z\|$ defines a 2-metric on $\mathcal{X}$. Therefore every 2-normed space will be considered to be a 2-metric space with the 2-metric defined in this sense.

Three or more points $p_1, p_2, p_3, \cdots$ are said to be *collinear* if they lie on a single straight line, that is, for each $i = 3, 4, 5, \cdots$, if $p_1 \neq p_2$ and $p_1 \neq p_i$, then there is a real number t_i such that $p_1 - p_2 = t_i(p_1 - p_i)$.

DEFINITION 2.2. A point p in a linear 2-normed space $\mathcal{X}$ is called *2-norm midpoint* of 3 non-collinear points x, y, z in $\mathcal{X}$ if $d(x, y, p) = d(x, p, z) = d(p, y, z) = \frac{1}{3}d(x, y, z)$.

For non-collinear points x, y, z in $\mathcal{X}$, let $T(x, y, z) = \{w \in \mathcal{X} : d(x, y, z) = d(x, y, w) + d(x, w, z) + d(w, y, z)\}$. $T(x, y, z)$ will be called the *triangle* with vertices x, y and z. Furthermore, we will designate the area of $T(x, y, z)$ to be $d(x, y, z)$. A point p of $\mathcal{X}$ will be a *center* of $T(x, y, z)$ if p is a 2-norm midpoint of x, y and z.

The following theorem was proved in [3]. We give another easy solutions.

THEOREM 2.3 ([3] THEOREM 2.1). *Suppose that $\mathcal{X}$ is a linear 2-normed space.*

(1) $x \in T(a,b,c)$ *if and only if* $x - y \in T(a-y, b-y, c-y)$.
(2) $T(a+p, b+p, c+p) = T(a,b,c) + p$.
(3) *For a real number* α, *we have*

$$\alpha T(a,b,c) = T(\alpha a, \alpha b, \alpha c).$$

(4) *Let a sequence* $\{x_n\}$ *in* $\mathcal{X}$ *converge to a point* x *in* $\mathcal{X}$. *If* $\{x_n\}$ *is a sequence in* $T(a,b,c)$ *for some non-collinear points of* $\mathcal{X}$, *then* x *is a point in* $T(a,b,c)$.

Proof. (1) Suppose that $x \in T(a,b,c)$. Then we have

$$d(a,b,c) = d(a,b,x) + d(a,x,c) + d(x,b,c)$$

or

$$\|a-b, a-c\| = \|a-x, b-x\| + \|b-x, c-x\| + \|c-x, a-x\|$$

if and only if

$$\begin{aligned} &\|a-y-(b-y), a-y-(c-y)\| \\ &= \|a-y-(x-y), b-y-(x-y)\| \\ &\quad + \|b-y-(x-y), c-y-(x-y)\| \\ &\quad + \|c-y-(x-y), a-y-(x-y)\| \end{aligned}$$

for all $y \in \mathcal{X}$. Therefore we have $x - y \in T(a-y, b-y, c-y)$.

(2) Suppose that $x \in T(a+p, b+p, c+p)$. Then we have

$$\begin{aligned} &d(a+p, b+p, c+p) \\ &= d(a+p, b+p, x) + d(a+p, x, c+p) + d(x, b+p, c+p) \end{aligned}$$

or

$$\begin{aligned} &\|a+p-(b+p), a+p-(c+p)\| \\ &= \|a+p-x, b+p-x\| \\ &\quad + \|b+p-x, c+p-x\| + \|c+p-x, a+p-x\| \end{aligned}$$

if and only if

$$\begin{aligned} \|a-b, a-c\| = &\|a-(x-p), b-(x-p)\| \\ &+ \|b-(x-p), c-(x-p)\| \\ &+ \|c-(x-p), a-(x-p)\|. \end{aligned}$$

Therefore we have $x - p \in T(a,b,c)$ or $x \in T(a,b,c) + p$.

(3) We may assume that α is not zero. For all $y \in \alpha T(a,b,c)$, there is a point $x \in T(a,b,c)$ such that $y = \alpha x$. Therefore we have

$$d(a,b,c) = d(a,b,x) + d(a,x,c) + d(x,b,c)$$

or

$$\|a-b, a-c\| = \|a-x, b-x\| + \|b-x, c-x\| + \|c-x, a-x\|.$$

Hence from multiplying both sides by $|\alpha|^2$, we have

$$\begin{aligned} &\|\alpha a - \alpha b, \alpha a - \alpha c\| \\ = \; &\|\alpha a - \alpha x, \alpha b - \alpha x\| + \|\alpha b - \alpha x, \alpha c - \alpha x\| + \|\alpha c - \alpha x, \alpha a - \alpha x\|. \end{aligned}$$

Thus we have $y = \alpha x \in T(\alpha a, \alpha b, \alpha c)$.

On the other hand, for all $x \in T(\alpha a, \alpha b, \alpha c)$, we have

$$d(\alpha a, \alpha b, \alpha c) = d(\alpha a, \alpha b, x) + d(\alpha a, x, \alpha c) + d(x, \alpha b, \alpha c)$$

or

$$\begin{aligned} &\|\alpha a - \alpha b, \alpha a - \alpha c\| \\ = \; &\|\alpha a - x, \alpha b - x\| + \|\alpha b - x, \alpha c - x\| + \|\alpha c - x, \alpha a - x\|. \end{aligned}$$

Hence from dividing both sides by $|\alpha|^2$, we have

$$\|a-b, a-c\| = \left\|a - \frac{x}{\alpha}, b - \frac{x}{\alpha}\right\| + \left\|b - \frac{x}{\alpha}, c - \frac{x}{\alpha}\right\| + \left\|c - \frac{x}{\alpha}, a - \frac{x}{\alpha}\right\|.$$

Thus we have $\dfrac{x}{\alpha} \in T(a,b,c)$ or $x \in \alpha T(a,b,c)$.

(4) Assume that $x_n \in T(a,b,c)$ and $x_n \to x$ as $n \to \infty$. Then we have $d(a,b,c) = d(a,b,x_n) + d(a,x_n,c) + d(x_n,b,c)$. Since $d(a,b,x_n) = \|a-x_n, b-x_n\|$, by lemma 1.5 we have $\lim_{n\to\infty} d(a,b,x_n) = d(a,b,x)$. Hence we have

$$\begin{aligned} d(a,b,c) &= \lim_{n\to\infty} d(a,b,c) \\ &= \lim_{n\to\infty} (d(a,b,x_n) + d(a,x_n,c) + d(x_n,b,c)) \\ &= d(a,b,x) + d(a,x,c) + d(x,b,c). \end{aligned}$$

Therefore $x \in T(a,b,c)$. □

Definition 2.4. Let $\mathcal{X}$ be a linear 2-normed space. For two points b, c in $\mathcal{X}$, let $E(b,c) = \{x \in \mathcal{X} : \|x, b+c\| = \|x,b\| + \|x,c\|\}$. We will call $E(b,c) (= E(c,b))$ the *2-norm equality* with respect to b and c.

If the set $\{x,\ b\}$ or the set $\{x,\ c\}$ is linearly dependent, then $x \in E(b,c)$. Hence $E(b,c)$ is a non-empty set.

THEOREM 2.5. *Let b and c be points in a linear 2-normed space $\mathcal{X}$. Then we have the following.*

(1) *For all $b \in \mathcal{X}$ and non-negative real number α, we have $E(b, \alpha b) = \mathcal{X}$. For all non-zero $b \in \mathcal{X}$ and negative real number α, we have $E(b, \alpha b) = \{\beta b : \beta \in \mathbb{R}\}$.*

(2) *For all non-zero $\alpha \in \mathbb{R}$ and $x \in E(b, c)$, we have*

$$E(b, c) = \alpha E(b, c) = E(\alpha b, \alpha c) = E(b + \alpha x, c) = E(b, c + \alpha x).$$

(3) *If $\|x, b + c\| = 0$ for a non-zero point $x \in E(b, c)$, then x, b and c are pairwise linearly dependent.*

Therefore if b and c are linearly independent and a non-zero point $x \in E(b, c)$, then $\|x, b + c\| \neq 0$ or x and $b + c$ are linearly independent.

(4) *The points b and c are linearly dependent if and only if $E(b, c)$ is a subspace of $\mathcal{X}$.*

In this case, the dimension of $E(b, c)$ over $\mathbb{R}$ is 1 or $\dim \mathcal{X}$.

(5) *A sequence $\{x_n\}$ in $\mathcal{X}$ converges to a point x in $\mathcal{X}$. If $\{x_n\}$ is a sequence in $E(b, c)$, then x is a point in $E(b, c)$.*

Proof. (1) For all $x \in \mathcal{X}$, we have $\|x, b + \alpha b\| = (1 + \alpha)\|x, b\| = \|x, b\| + \|x, \alpha b\|$. Therefore we have $E(b, \alpha b) = \mathcal{X}$.

Next suppose that α is a negative real number and $x \in E(b, \alpha b)$. Assume that x and b are linearly independent. Then we have

$$|1 + \alpha|\|x, b\| = \|x, b + \alpha b\| = \|x, b\| + \|x, \alpha b\| = (1 + |\alpha|)\|x, b\|.$$

Since $\|x, b\| \neq 0$, we have $|1 + \alpha| = 1 + |\alpha|$.

In case $-1 \leq \alpha < 0$, we have $1 + \alpha = 1 - \alpha$ or $\alpha = 0$. This is a contradiction. The other case $\alpha < -1$, we have $-1 - \alpha = 1 - \alpha$ or $-1 = 1$. These contradictions imply that x and b are linearly dependent. Since b is not zero, there is a real number β_0 such that $x = \beta_0 b$. Thus $E(b, \alpha b) \subset \{\beta b : \beta \in \mathbb{R}\}$.

On the other hand, for all $\beta \in \mathbb{R}$ and all $b \in \mathcal{X}$, we have $\|\beta b, b + \alpha b\| = 0 = \|\beta b, b\| + \|\beta b, \alpha b\|$. Thus $\{\beta b : \beta \in \mathbb{R}\} \subset E(b, \alpha b)$. Therefore we have $E(b, \alpha b) = \{\beta b : \beta \in \mathbb{R}\}$.

(2) For all $x \in E(b, c)$, we have

$$\begin{aligned}
& |x, b + c\| = \|x, b\| + \|x, c\| \\
\Leftrightarrow \quad & \|\alpha^{-1}x, b + c\| = \|\alpha^{-1}x, b\| + \|\alpha^{-1}x, c\| \\
\Leftrightarrow \quad & \|x, \alpha(b + c)\| = \|x, \alpha b\| + \|x, \alpha c\| \\
\Leftrightarrow \quad & \|x, b + \alpha x + c\| = \|x, b + \alpha x\| + \|x, c\|
\end{aligned}$$

$$\Leftrightarrow \quad \|x, b+c+\alpha x\| = \|x, b\| + \|x, c+\alpha x\|.$$

Therefore we have

$$E(b,c) = \alpha E(b,c) = E(\alpha(b,c)) = E(b+\alpha x, c) = E(b, c+\alpha x).$$

(3) By the hypothesis we have $0 = \|x, b+c\| = \|x, b\| + \|x, c\|$. Hence we have $0 = \|x, b\| = \|x, c\|$. Therefore x, b and c are pairwise linearly dependent by (2N1) and Lemma 1.7 .

(4) Let b and c be linearly dependent. Then $E(b,c)$ is $\mathcal{X}$ or $\{\beta b : \beta \in \mathbb{R}\}$ by (1). Hence $E(b,c)$ is a subspace of $\mathcal{X}$.

On the other hand, let $E(b,c)$ be a subspace of $\mathcal{X}$. Since $b, c \in E(b,c)$, we have $b+c \in E(b,c)$. Then we have

$$0 = \|b+c, b+c\| = \|b+c, b\| + \|b+c, c\| = 2\|b, c\|.$$

Hence b and c are linearly dependent.

(5) Assume that $x_n \in E(b,c)$ and $x_n \to x$ as $n \to \infty$. Then we have $\|x_n, b+c\| = \|x_n, b\| + \|x_n, c\|$.

By lemma 1.5 we have

$$\begin{aligned} \|x, b+c\| &= \lim_{n\to\infty} \|x_n, b+c\| \\ &= \lim_{n\to\infty} (\|x_n, b\| + \|x_n, c\|) \\ &= \|x, b\| + \|x, c\| \end{aligned}$$

Therefore we have $x \in E(b,c)$. □

References

[1] S. Gähler, 2-*metrische Räume und ihre topologische Struktur*, Math. Nachr. **26** (1963), 115-148.

[2] S. Gähler, *Lineare* 2-*normierte Räumen*, Math. Nachr. **28** (1964), 1-43.

[3] A. White Jr. and Y. J. Cho, *Linear mapping on linear 2-normed spaces*, Bull. Korean Math. Soc. **21** (1984), no. 1, 1-6.

[4] W. G. Park, *Approximate additive mapping in* 2-*Banach spaces and related topics*, J. Math. Anal. Appl. **376** (2011), 193-202.

*

Department of Mathematics Education
Mokwon University
Daejeon 302-729, Republic of Korea
E-mail: math888@naver.com