JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **26**, No. 4, November 2013
http://dx.doi.org/10.14403/jcms.2013.26.4.879

CONVERGENCE THEOREMS FOR THE HENSTOCK DELTA INTEGRAL ON TIME SCALES

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Abstract. In this paper, we define an extension $f^* : [a, b] \to \mathbb{R}$ of a function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ for a time scale $\mathbb{T}$ and prove the convergence theorems for the Henstock delta integral on time scales.

1. Introduction and preliminaries

The Henstock delta integral on time scales was introduced by Allan Peterson and Bevan Thompson [7].

In this paper, we investigate the convergence theorems for the Henstock delta integral on time scales.

First, we introduce some concepts related to the notion of time scales. A time scale $\mathbb{T}$ is any closed nonempty subset of $\mathbb{R}$, with the topology inherited from the standard topology on the real numbers $\mathbb{R}$. For each $t \in \mathbb{T}$, we define the forward jump operator $\sigma(t)$ by

$$\sigma(t) = \inf\{z > t : z \in \mathbb{T}\}$$

and the backward jump operator $\rho(t)$ by

$$\rho(t) = \sup\{z < t : z \in \mathbb{T}\}$$

where $\inf \phi = \sup \mathbb{T}$ and $\sup \phi = \inf \mathbb{T}$.

If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. If $\sigma(t) = t$, we say that t is right-dense,while if $\rho(t) = t$, we say that t is left-dense. The forward graininess function $\mu(t)$ is defined by $\mu(t) = \sigma(t) - t$, and the backward graininess function $\nu(t)$ is defined by $\nu(t) = t - \rho(t)$.

Received September 16, 2013; Accepted October 11, 2013.

2010 Mathematics Subject Classification: Primary 26A39; Secondary 26E70.

Key words and phrases: time scales, Henstock delta integral, $\triangle$-gauge.

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For $a, b \in \mathbb{T}$, we define the time scale interval in $\mathbb{T}$ by

$$[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \le t \le b\}.$$

2. Convergence theorems for the Henstock delta integral on time scales

Definition 2.1. ([6]) $\delta = (\delta_L, \delta_R)$ is a $\triangle$-gauge on $[a,b]_{\mathbb{T}}$ if $\delta_L(t) > 0$ on $(a,b]_{\mathbb{T}}$, $\delta_R(t) > 0$ on $[a,b)_{\mathbb{T}}$, $\delta_L(a) \ge 0, \delta_R(b) \ge 0$, and $\delta_R(t) \ge \mu(t)$ for each $t \in [a,b)_{\mathbb{T}}$.

Definition 2.2. ([6]) A collection $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i]_{\mathbb{T}})\}_{i=1}^n$ of tagged intervals is a Henstock partition of $[a,b]_{\mathbb{T}}$ if $\bigcup_{i=1}^n [t_{i-1}, t_i]_{\mathbb{T}} = [a,b]_{\mathbb{T}}$, $[t_{i-1}, t_i]_{\mathbb{T}} \subset [\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i)]$ and $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}}$ for each $i = 1, 2, \cdots, n$.

For a Henstock partition $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n$, we write

$$S(f, \mathcal{P}) = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}),$$

whenever $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$.

Definition 2.3. ([6]). A function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is Henstock delta integrable (or $H_\triangle$-integrable) on $[a,b]_{\mathbb{T}}$ if there exists a number A such that for each $\epsilon > 0$ there exists a $\triangle$-gauge δ on $[a,b]_{\mathbb{T}}$ such that

$$\left| S(f, \mathcal{D}) - A \right| < \epsilon$$

for every δ-fine Henstock partition $\mathcal{D}$ of $[a,b]_{\mathbb{T}}$. The number A is called the $H_\triangle$-integral of f on $[a,b]_{\mathbb{T}}$, and we write $A = (H_\triangle) \int_a^b f$.

Recall that $f : [a,b] \to \mathbb{R}$ is Henstock integrable (or H-integrable) on $[a,b]$ if there exists a number A such that for each $\epsilon > 0$ there exists a gauge $\delta : [a,b] \to \mathbb{R}^+$ on $[a,b]$ such that

$$\left| S(f, \mathcal{P}) - A \right| < \epsilon$$

for every $\delta-$fine Henstock partition $\mathcal{P}$ of $[a,b]$.

Theorem 2.4. ([5]) *A function $f : [a,b] \to \mathbb{R}$ is H-integrable on $[a,b]$ if and only if f is $H_\triangle$-integrable on $[a,b]$.*

Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a function on $[a,b]_{\mathbb{T}}$, and let $\{(a_k,b_k)\}_{k=1}^{\infty}$ be the sequence of intervals contiguous to $[a,b]_{\mathbb{T}}$ in $[a,b]$.

Define a function $f^* : [a,b] \to \mathbb{R}$ on $[a,b]$ by

$$f^*(t) = \begin{cases} f(a_k) & \text{if} \quad t \in (a_k,b_k) \quad \text{for some } k \\ f(t) & \text{if} \quad t \in [a,b]_{\mathbb{T}} \end{cases}$$

Theorem 2.5. ([6]) *A function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is $H_\Delta-$ integrable if and only if $f^* : [a,b] \to \mathbb{R}$ is Henstock integrable. In this case,*

$$(H_\Delta) \int_a^b f = (H) \int_a^b f^*.$$

We next verify the basic properties of the Henstock delta integral.

Theorem 2.6. *Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ and let $c \in [a,b]$.*

(1) *If f is $H_\Delta-$integrable on $[a,b]_{\mathbb{T}}$, then f is $H_\Delta-$integrable on every subinterval of $[a,b]_{\mathbb{T}}$.*
(2) *If f is $H_\Delta-$integrable on each of the intervals $[a,c]_{\mathbb{T}}$ and $[c,b]_{\mathbb{T}}$, then f is $H_\Delta-$integrable on $[a,b]_{\mathbb{T}}$ and*

$$(H_\Delta) \int_a^b f = (H_\Delta) \int_a^c f + (H_\Delta) \int_c^b f.$$

Proof. (1) If f is $H_\Delta-$integrable on $[a,b]_{\mathbb{T}}$, then f^* is $H-$integrable on $[a,b]$. Since f^* is $H-$integrable on every subinterval of $[a,b]$, f is $H_\Delta-$integrable on every subinterval of $[a,b]_{\mathbb{T}}$ by Theorem 2.5.

(2) If f is $H_\Delta-$integrable on each of $[a,c]_{\mathbb{T}}$ and $[c,b]_{\mathbb{T}}$, then f^* is $H-$integrable on each of $[a,c]$ and $[c,b]$. By the property of the Henstock integral, f^* is $H-$integrable on $[a,b]$ and

$$(H) \int_a^b f^* = (H) \int_a^c f^* + (H) \int_c^b f^*.$$

By Theorem 2.5, f is $H_\Delta-$integrable on $[a,c]_{\mathbb{T}}$ and $[c,b]_{\mathbb{T}}$ and

$$(H_\Delta) \int_a^b f = (H_\Delta) \int_a^c f + (H_\Delta) \int_c^b f.$$

□

Theorem 2.7. *Let $f,g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be $H_\Delta-$ integrable on $[a,b]_{\mathbb{T}}$. Then $\alpha f + \beta g$ is $H_\Delta-$integrable on $[a,b]_{\mathbb{T}}$ and*

$$(H_\Delta) \int_a^b (\alpha f + \beta g) = \alpha (H_\Delta) \int_a^b f + \beta (H_\Delta) \int_a^b g.$$

Proof. If f and g are $H_\Delta-$integrable on $[a,b]_\mathbb{T}$, then f^* and g^* are $H-$integrable on $[a,b]$. Hence $\alpha f^* + \beta g^*$ is $H-$integrable on $[a,b]$, and $(H)\int_a^b(\alpha f^* + \beta g^*) = \alpha(H)\int_a^b f + \beta(H)\int_a^b g$. By Theorem 2.5, $\alpha f + \beta g$ is $H_\Delta-$integrable on $[a,b]_\mathbb{T}$ and

$$(H_\Delta)\int_a^b(\alpha f + \beta g) = \alpha(H_\Delta)\int_a^b f + \beta(H_\Delta)\int_a^b g.$$

□

Definition 2.8. ([7]) A subset $\mathcal{S}$ of a time scale $\mathbb{T}$ has delta measure zero if $\mathcal{S}$ contain no right-scattered points and $\mathcal{S}$ has Lebesgue measure zero. A property $\mathcal{A}$ holds delta almost everywhere ($\Delta - a.e$) on $\mathbb{T}$ if there is a subset $\mathcal{S}$ of $\mathbb{T}$ such that $\mathcal{A}$ holds for all $t \in \mathbb{T} - \mathcal{S}$ and $\mathcal{S}$ has delta measure zero.

Theorem 2.9. *If f and g are $H_\Delta-$ integrable on $[a,b]_\mathbb{T}$ and $f(t) \leq g(t)$ $\Delta - a.e$ on $[a,b]_\mathbb{T}$, then $(H_\Delta)\int_a^b f \leq (H_\Delta)\int_a^b g$.*

Proof. If f and g are $H_\Delta-$integrable on $[a,b]_\mathbb{T}$ and $f(t) \leq g(t)$ $\Delta - a.e$ on $[a,b]_\mathbb{T}$, then f^* and g^* are $H-$integrable on $[a,b]$ and $f^*(t) \leq g^*(t)$ $a.e$ on $[a,b]$. Hence, $(H)\int_a^b f^* \leq (H)\int_a^b g^*$. By Theorem 2.5,

$$(H_\Delta)\int_a^b f \leq (H_\Delta)\int_a^b g.$$

□

Theorem 2.10. *(Monotone Convergence Theorem)*
Let $f_n, f : [a,b]_\mathbb{T} \to \mathbb{R}$ and assume that
(a) *f_n is $H_\Delta-$integrable for each $n \in \mathbb{N}$.*
(b) *$f_n \to f$ $\Delta - a.e$ on $[a,b]_\mathbb{T}$.*
(c) *$f_n \leq f_{n+1}$ $\Delta - a.e$ on $[a,b]_\mathbb{T}$ for each $n \in \mathbb{N}$.*
(d) *$\lim_{n\to\infty}(H_\Delta)\int_a^b f_n = I$.*
Then f is $H_\Delta-$integrable on $[a,b]_\mathbb{T}$ and

$$(H_\Delta)\int_a^b f = \lim_{n\to\infty}(H_\Delta)\int_a^b f_n.$$

Proof. By Theorem 2.5, we have
(a) f_n^* is $H-$integrable for each $n \in \mathbb{N}$
(b) $f_n^* \to f^*$ a.e on $[a,b]$
(c) $f_n^* \leq f_{n+1}^*$ a.e on $[a,b]$ for each $n \in \mathbb{N}$
(d) $\lim_{n\to\infty}(H)\int_a^b f_n^* = I^*$.

By the Monotone Convergence Theorem for the Henstock integral, f^* is $H-$integrable on $[a,b]$ and

$$(H)\int_a^b f^* = \lim_{n\to\infty}(H)\int_a^b f_n^*.$$

By Theorem 2.5, f is $H_\Delta-$integrable on $[a,b]_{\mathbb{T}}$ and

$$(H_\Delta)\int_a^b f = \lim_{n\to\infty}(H_\Delta)\int_a^b f_n.$$

□

THEOREM 2.11. *(Dominated Convergence Theorem) Assume that*

(a) $f_n \to f\ \Delta - a.e$ *on* $[a,b]_{\mathbb{T}}$
(b) $g \le f_n \le h\ \Delta - a.e$ *on* $[a,b]_{\mathbb{T}}$
(c) f_n, g *and* h *are* $H_\Delta-$*integrable on* $[a,b]$.

Then f *is* $H_\Delta-$*integrable on* $[a,b]_{\mathbb{T}}$ *and*

$$(H_\Delta)\int_a^b f = \lim_{n\to\infty}(H_\Delta)\int_a^b f_n.$$

Proof. By Theorem 2.5, we have
(a) $f_n^* \to f^*\ a.e$ on $[a,b]$
(b) $g^* \le f_n^* \le h^*\ a.e$ on $[a,b]$
(c) f_n^*, g^* and h^* are $H-$integrable on $[a,b]$.

By the Dominated Convergence Theorem for the Henstock integral, f^* is $H-$integrable on [a,b] and

$$(H)\int_a^b f^* = \lim_{n\to\infty}(H)\int_a^b f_n^*.$$

By theorem 2.5, f is $H_\Delta-$ integrable on $[a,b]_{\mathbb{T}}$ and

$$(H_\Delta)\int_a^b f = \lim_{n\to\infty}(H_\Delta)\int_a^b f_n.$$

□

References

[1] R. Agarwal and M. Bohner, *Basic calculus on time scales and some of its applications*, Results Math. **35** (1999), 3-22.
[2] S. Avsec, B. Bannish, B. Johnson, and S. Meckler, *The Henstock-Kurzweil delta integral on unbounded time scales*, PanAmerican Math. J. Vol. **16** (2006), no. 3, 77-98.

[3] G. Sh. Guseinov, *Intergration on time scales*, J. Math. Anal. Appl. **285** (2003), 107-127.

[4] G. Sh. Guseinov and B. Kaymakcalan, *Basics of Riemann delta and nabla integration on time scales*, J. Difference Equations Appl. **8** (2002), 1001-1027.

[5] J. M. Park, D. H. Lee, J. H. Yoon, and J. T. Lim, *The Henstock and Henstock delta Integrals,* J. Chungcheong Math. Soc. **26** (2013), no. 2, 291-298.

[6] J. M. Park, D. H. Lee, J. H. Yoon, and J. T. Lim, *The relation between Henstock integral and Henstock delta integral on time scales,* J. Chungcheong Math. Soc. **26** (2013), no. 3, 625-630.

[7] A. Peterson and B. Thompson, *Henstock-Kurzweil Delta and Nabla Integral,* J. Math. Anal. Appl. **323** (2006), 162-178.

[8] C. W. Swartz and D. S. Kurtz, *Theories of Integration: The Integrals of Riemann Lebesgue, Henstock-Kurzweil, and Mcshane,* World Scientific, 2004.

[9] B. S. Thomson, *Henstock Kurzweil integtals on time scales*, PanAmerican Math. J. Vol. **18** (2008), no. 1, 1-19.

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