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A FAMILY OF CHARACTERISTIC CONNECTIONS

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Abstract. The characteristic connection is a good substitute for Levi-Civita connection in studying non-integrable geometries. In this paper we consider the homogeneous space $U(3)/(U(1)\times U(1)\times U(1))$ with a one-parameter family of Hermitian structures. We prove that the one-parameter family of Hermtian structures admit a characteristic connection. We also compute the torsion of the characteristic connecitons.

1. Introduction

The non-integrable geometries are studied by many mathematicians ([5], [6], [9]) and a very important tool in studying non-integrable geometries is the characteristic connection ([7]). The characteristic connection is a metric connection with a skew symmetric torsion which preserves a given G-structure. So, the characteristic connection is a good substitute for the Levi-Civita connection on a manifold with a non-integrable geometry where the holonomy group with respect to the Levi-Civita connection is the whole group $\mathrm{SO}(n)$ and the geometric structure is not preserved by the Levi-Civita connection.

Recently, many geometric properties with respect to the characteristic connection are discussed. Eigenvalues of the generalised Dirac operators, Dirac operators with respect to the characteristic connection, and other geometric properties concerning the eigenvalue estimates are investigated ([3], [4], [10], [11]). In discussing the above geometric properties, examples of manifolds with geometric structures admitting a characteristic connection are needed. But not every geometric structure admits a characteristic connection.

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In [4], [12] , an example of 6-dimensional homogeneous manifold with a characteristic connection is given, which is in fact a nearly kähler manifold. Furthermore, this manifold admits a split holonomy, a geometric structure concerning the condition of the torsion and the holonomy. We then use the generalized Dirac eigenvalue estimate for the split holonomy ([4]).

In this paper we prove that the homogeneous space $U(3)/(U(1) \times U(1) \times U(1))$ admits a one-parameter family of Hermitian structures with characteristic connection and compute the characteristic connections concretely.

It is well known that the difference of the characteristic connection, denoted by ∇^{ch}, from the Levi-Civita connection, denoted by ∇^{g}, is the torsion of the characteristic connection ([8]):

$$\nabla^{ch}_X Y = \nabla^{g}_X Y + \frac{1}{2} T(X, Y).$$

So, it suffices to compute the Torsion T for the characteristic connection ∇^{ch}. For the torsion T we use the following formula which is available for 6-dimensional almost hermitian manifolds (Theorem 4.2 [2]).

$$T(X, Y, -) = N(X, Y) + d\Omega(JX, JY, J-). \tag{1.1}$$

In Section 2, we we prove that the homogeneous space $U(3)/(U(1) \times U(1) \times U(1))$ admits a one-parameter family of Hermitian structures with characteristic connection.

In section 3, we compute the characteristic connections ∇^{ch} of (M, g_t, J).

2. Characteristic connections

We begin with a well-known metric family for a homogeneous reductive space. We refer to [1] and [12] for more informations.

Let $G := U(3)$ and $H := U(1) \times U(1) \times U(1) \subset G$ diagonally embedded. Then $M := G/H$ is a 6-dimensional manifold with

$$\mathfrak{g} = \mathfrak{u}(3) = \{A \in M_3(\mathbb{C}) : A + \bar{A}^t = 0\}, \ \ \mathfrak{h} = \{A \in \mathfrak{u}(3) : \ \text{A is diagonal}\}.$$

We define an Ad (G)-invariant inner product $\beta := -\frac{1}{2}\mathrm{Re}(\mathrm{tr} AB), \ \ A, B \in \mathfrak{u}(3)$ and decompose $\mathfrak{m} = \mathfrak{h}^{\perp}$ into

$$\mathfrak{m}_1 := \left\{ \begin{bmatrix} 0 & a & b \\ -\bar{a} & 0 & 0 \\ -\bar{b} & 0 & 0 \end{bmatrix} : a, b \in \mathbb{C} \right\}, \quad \mathfrak{m}_2 := \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -\bar{c} & 0 \end{bmatrix} : c \in \mathbb{C} \right\}.$$

Then an Ad (H)-invariant inner product on $\mathfrak{m}$ defined by

$$\beta_t := \beta|_{\mathfrak{m}_1\times\mathfrak{m}_1} + 2t\beta|_{\mathfrak{m}_2\times\mathfrak{m}_2}$$

induces a left invariant metric g_t on G/H or each $t > 0$.

Let $D_{kl} = (d_{ij})$ be the $n \times n$ matrix with zero entries except that its (k,l)-entry is 1. Furthermore, let $E_{kl} := D_{kl} - D_{lk}$ for $k \neq l$ and $S_{kl} := i(D_{kl} + D_{lk})$. Then

$$\{e_1 := E_{12},\, e_2 := S_{12},\, e_3 := E_{13},\, e_4 := S_{13},\, e_5 := \frac{1}{\sqrt{2t}}E_{23},\, e_6 := \frac{1}{\sqrt{2t}}S_{23}\}$$

is an orthonormal basis of $\mathfrak{m}$ with respect to β_t. As basis for $\mathfrak{h}$ we take $H_k = S_{kk}/2, k = 1, 2, 3$.

We now recall the Nijeunhuis tensor

$$N(X,Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y]$$

where $J^2 = -Id$ in an almost complex structure. Then it holds

$$\begin{aligned} &N(X,Y) = N(Y,X), \\ &N(X,JY) = -JN(X,Y) = N(JX,Y). \end{aligned} \tag{2.1}$$

The Nijenhuis tensor N as $(2,1)$-tensor field is already skew-symmetric from the definition, so the N-tensor as $(3,0)$-tensor is totally skew-symmetric if

$$N(X,Y,Z) = -N(X,Z,Y), \text{ for } X, Y, Z \in TM^{2n}.$$

We now consider a 2-form Ω and J on G/H as follows:

$$\Omega(X,Y) := e_{12} + e_{34} - e_{56} =: g_t(JX,Y) \text{ with } J^2 = -Id, \tag{2.2}$$

where $e_{ij} = e_i \wedge e_j$.

Then by computation,

$$\begin{aligned} &J(e_1) = e_2, J(e_2) = -e_1, J(e_3) = e_4, \\ &J(e_4) = -e_3, J(e_5) = -e_6, J(e_6) = e_5. \end{aligned} \tag{2.3}$$

THEOREM 2.1. *On* $M = U(3)/(U(1) \times U(1) \times U(1))$ *we consider a metric family* g_t *and an almost complex structure* J *as above. Then the characteristic connection exists for all* $t > 0$.

Proof. It is well known that a 6-dimensional almost Hermitian manifold (M^6, g, J) admits a characteristic connection if and only if its Nijenhuis tensor N is totally skew-symmetric ([2] Theorem 4.2). We will actually show that in our case the tensor $N \equiv 0$.

For $X \in TM^{2n}$, (2.1) implies

$$N(X, JX) = -JN(X,X) = 0,$$

so by (2.3) we have

$$N(e_1,e_2)=N(e_3,e_4)=N(e_5,e_6)=0.$$

Now we compute

$$\begin{aligned}
N(e_1,e_3) &= [Je_1,Je_3]-J[e_1,Je_3]-J[Je_1,e_3]-[e_1,e_3]\\
&= [e_2,e_4]-J[e_1,e_4]-J[e_2,e_3]-[e_1,e_3]\\
&= -\sqrt{2t}e_5-J(-\sqrt{2t}e_6)-J(\sqrt{2t}e_6)+\sqrt{2t}e_5\\
&= 0,
\end{aligned}$$

$$\begin{aligned}
N(e_1,e_5) &= [Je_1,Je_5]-J[e_1,Je_5]-J[Je_1,e_5]-[e_1,e_5]\\
&= -[e_2,e_6]+J[e_1,e_6]-J[e_2,e_5]-[e_1,e_5]\\
&= \frac{1}{\sqrt{2t}}e_3+J(\frac{1}{\sqrt{2t}}e_4)-J(\frac{1}{\sqrt{2t}}e_4)-\frac{1}{\sqrt{2t}}e_3\\
&= 0,
\end{aligned}$$

$$\begin{aligned}
N(e_3,e_5) &= [Je_3,Je_5]-J[e_3,Je_5]-J[Je_3,e_5]-[e_3,e_5]\\
&= -[e_4,e_6]+J[e_3,e_6]-J[e_4,e_5]-[e_3,e_5]\\
&= \frac{1}{\sqrt{2t}}e_1+J(\frac{1}{\sqrt{2t}}e_2)+J(\frac{1}{\sqrt{2t}}e_2)+\frac{1}{\sqrt{2t}}e_1\\
&= 0.
\end{aligned}$$

Here we use the following relations:

$$[e_2,e_4]=[e_1,e_3]=-\sqrt{2t}e_5 \text{ and } [e_1,e_4]=-[e_2,e_3]=-\sqrt{2t}e_6,$$
$$[e_1,e_5]=-[e_2,e_6]=\frac{1}{\sqrt{2t}}e_3 \text{ and } [e_1,e_6]=[e_2,e_5]=\frac{1}{\sqrt{2t}}e_4,$$
$$[e_4,e_6]=[e_3,e_5]=-\frac{1}{\sqrt{2t}}e_1 \text{ and } -[e_3,e_6]=[e_4,e_5]=-\frac{1}{\sqrt{2t}}e_2.$$

By (2.1), (2.3) we obtain

$$\begin{aligned}
N(e_1,e_3)&=-JN(e_1,e_4)=N(e_2,e_4)=JN(e_2,e_3)=0,\\
N(e_1,e_5)&=JN(e_1,e_6)=-N(e_2,e_6)=JN(e_2,e_5)=0,\\
N(e_3,e_5)&=JN(e_3,e_6)=N(e_4,e_6)=-JN(e_4,e_5)=0.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
N(e_1,e_4)&=N(e_2,e_3)=0,\\
N(e_1,e_6)&=N(e_2,e_5)=0,\\
N(e_3,e_6)&=N(e_4,e_5)=0.
\end{aligned}$$

So, we actually have $N\equiv 0$ and N is totally skew-symmetric. □

3. The torsion of the characteristic connection

For the further computations we recall the following (X.2 [13]):

- The map Λ_t, which implies the Levi-Civita connection, is uniquely characterized by (X.2 [13]),

$$\Lambda_t(X)Y - \Lambda_t(Y)X = [X,Y]_{\mathfrak{m}}, \tag{3.1}$$
$$\beta_t(\Lambda_t(X)Y,Z) + \beta_t(Y,\Lambda_t(X)Z) = 0. \tag{3.2}$$

- For the metric $g_t, t > 0$, the map $\Lambda_t : \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$ is defined by

$$\begin{aligned} \Lambda_t(X)Y &= \frac{1}{2}[X,Y]_{\mathfrak{m}_2}, \qquad \Lambda_t(X)B = t[X,B], \\ \Lambda_t(A)Y &= (1-t)[A,Y], \qquad \Lambda_t(A)B = 0, \end{aligned} \tag{3.3}$$

  for $X, Y \in \mathfrak{m}_1, A, B \in \mathfrak{m}_2$. By direct computaions we can check that the map $\Lambda_t(X)$ definded as (3.3) satisfies the conditions (3.1) and (3.2).

Let (M,g) be a manifold with a characteristic connection. We denote the Levi-Civita connection and the characteristic connection by ∇^g and ∇^{ch}, respectively. Then, it is well known that for $X, Y \in TM$ (see [8])

$$\nabla^{ch}_X Y = \nabla^g_X Y + \frac{1}{2}T(X,Y)$$

for some $(2,1)$-tensor T which is known to be the torsion of the characteristic connection. So, it suffices to compute the above torsion T for the characteristic connection ∇^{ch}.

Furthermore, in a 6−dimensional almost Hermitian manifold (M,g,J), the torsion for ∇^{ch} satisfies (Theorem 4.2 [2])

$$T(X,Y,-) = N(X,Y) + d\Omega(JX,JY,J-). \tag{3.4}$$

Here the $(2,1)$-tensors T, Ω are considered as $(3,0)$-tensors. That is, for T we define $T(X,Y,Z) = g(T(X,Y),Z)$, similarly for N.

For g_t as above we now compute the Levi-Civita connection ∇^{g_t} using (3.3). The map Λ_{g_t} (see (3.3)) is simply denoted by Λ_t and we consider E_{ij} with respect to the orthonormal basis e_i of $\mathfrak{m}$. So, E_{ij} actually maps e_i to $-e_j$. We recall the following lemma ([1], [12]).

Lemma 3.1. *We identify $\mathfrak{m}$ with $\mathbb{R}^6$ and take E_{ij} defined above as basis of $\mathfrak{so}(\mathfrak{m})$. Then*

$$\Lambda_t(e_1) = \sqrt{t/2}(E_{35} + E_{46}),\ \ \Lambda_t(e_2) = \sqrt{t/2}(E_{45} - E_{36}),$$
$$\Lambda_t(e_3) = \sqrt{t/2}(E_{26} - E_{15}),\ \ \Lambda_t(e_4) = -\sqrt{t/2}(E_{16} + E_{25}),$$
$$\Lambda_t(e_5) = \frac{1-t}{\sqrt{2t}}(E_{13} + E_{24}),\ \ \Lambda_t(e_6) = \frac{1-t}{\sqrt{2t}}(E_{14} - E_{23}).$$

Proof. We compute $\Lambda(e_1)$. By (3.3) $\Lambda_t(e_1)e_i = \frac{1}{2}[e_1, e_i]_{\mathfrak{m}^2}$ for $i = 1, \cdots, 4$ and $\Lambda_t(e_1)e_j = t[e_1, e_j]$ for $j = 5, 6$. Hence,

$$\Lambda_t(e_1)e_1 = 0$$
$$\Lambda_t(e_1)e_2 = \frac{1}{2}[e_1, e_2]_{\mathfrak{m}^2} = 0,$$
$$\Lambda_t(e_1)e_3 = \frac{1}{2}[e_1, e_3]_{\mathfrak{m}^2} = -\sqrt{t/2}e_5,$$
$$\Lambda_t(e_1)e_4 = \frac{1}{2}[e_1, e_4]_{\mathfrak{m}^2} = -\sqrt{t/2}e_6,$$
$$\Lambda_t(e_1)e_5 = t[e_1, e_5] = \sqrt{t/2}e_3,$$
$$\Lambda_t(e_1)e_6 = t[e_1, e_6] = \sqrt{t/2}e_4.$$

We consider E_{ij} which maps e_i of $\mathfrak{m}$ and e_j to e_i. Then we have

$$\Lambda(e_1) = \sqrt{t/2}(E_{35} + E_{46}).$$

We obtain the other results by similar computations. □

Theorem 3.1. *The manifold $(M, g_t, J), t > 0$ as above admits a characteristic connection*

$$\nabla^{ch}_X Y = \nabla^t_X Y + \frac{1}{2}T(X, Y, -),$$

with $T = (\sqrt{2t} - \sqrt{t/2} + \frac{1-t}{\sqrt{2t}})(e_{145} - e_{235}) - \sqrt{2t}(e_{136} + e_{246})$, where e_{ijk} means $e_i \wedge e_j \wedge e_k$ and $\nabla^t := \nabla^{g_t}$.

Proof. By (3.4), we need to compute $d\Omega$ and N in (M, g, J).

i) First $d\Omega$ is given by

$$d\Omega = \sum_i e_i \wedge \nabla^g_{e_i}\Omega,$$

so we compute $\nabla^g_{e_i}\Omega$, $\Omega = e_{12} + e_{34} - e_{56}$, $i = 1, \cdots, 6$. It is well known that the three 2-forms $w = e_{12}$, e_{34}, e_{56} are invariant under the isotropy representation. It is well known that([12])

$$\nabla^g_{e_i} w = \Lambda_t(e_i)w = \sum_j (e_j \lrcorner \Lambda_t(e_i)) \wedge (e_j \lrcorner w). \tag{3.5}$$

Note that E_{ij} maps e_i to $-e_j$, so E_{ij} can be identified with the two form $-e_{ij}$.

From Lemma 3.1 $\Lambda_t(e_1) = \sqrt{t/2}(E_{35}+E_{46})$ identified with $-\sqrt{t/2}(e_{35}+e_{46})$, so $e_j \lrcorner \Lambda_t(e_1) = 0$ for $j = 1, 2$ and (3.5) implies

$$\nabla^t_{e_1} e_{12} = \nabla^t_{e_2} e_{12} = 0.$$

Similarly

$$\nabla^t_{e_3} e_{34} = \nabla^t_{e_4} e_{34} = 0$$

and

$$\nabla^t_{e_5} e_{56} = \nabla^t_{e_6} e_{56} = 0.$$

Now

$$\begin{aligned}
\nabla^t_{e_1} e_{34} &= \sum_j (e_j \lrcorner \Lambda(e_1)) \wedge (e_j \lrcorner e_{34}) \\
&= -\sqrt{t/2} \sum_j (e_j \lrcorner (e_{35} + e_{46})) \wedge (e_j \lrcorner e_{34}) \\
&= -\sqrt{t/2}\Big((e_3 \lrcorner (e_{35} + e_{46})) \wedge (e_3 \lrcorner e_{34}) \\
&\qquad + (e_4 \lrcorner (e_{35} + e_{46})) \wedge (e_4 \lrcorner e_{34})\Big) \\
&= -\sqrt{t/2}(e_5 \wedge e_4 - e_6 \wedge e_3) \\
&= \sqrt{t/2}(e_{45} - e_{36})
\end{aligned}$$

and

$$\begin{aligned}
\nabla^t_{e_1} e_{56} &= \sum_j (e_j \lrcorner \Lambda(e_1)) \wedge (e_j \lrcorner e_{56}) \\
&= -\sqrt{t/2} \sum_j (e_j \lrcorner (e_{35} + e_{46})) \wedge (e_j \lrcorner e_{56}) \\
&= -\sqrt{t/2}\Big((e_5 \lrcorner (e_{35} + e_{46})) \wedge (e_5 \lrcorner e_{56}) \\
&\qquad +(e_6 \lrcorner (e_{35} + e_{46})) \wedge (e_6 \lrcorner e_{56})\Big) \\
&= -\sqrt{t/2}(-e_3 \wedge e_6 + e_4 \wedge e_5) \\
&= \sqrt{t/2}(e_{36} - e_{45}).
\end{aligned}$$

Similarly,

$$\nabla^t_{e_2} e_{34} = -\sqrt{t/2}\sum_j (e_j \lrcorner (e_{45} - e_{36})) \wedge (e_j \lrcorner e_{34}) = -\sqrt{t/2}(e_{46} + e_{35}).$$

$$\nabla^t_{e_2} e_{56} = -\sqrt{t/2}\sum_j (e_j \lrcorner (e_{45} - e_{36})) \wedge (e_j \lrcorner e_{56}) = \sqrt{t/2}(e_{46} + e_{35}).$$

$$\nabla^t_{e_3} e_{12} = -\sqrt{t/2}\sum_j (e_j \lrcorner (e_{26} - e_{15})) \wedge (e_j \lrcorner e_{12}) = -\sqrt{t/2}(e_{16} + e_{25}).$$

$$\nabla^t_{e_3} e_{56} = -\sqrt{t/2}\sum_j (e_j \lrcorner (e_{26} - e_{15})) \wedge (e_j \lrcorner e_{56}) = -\sqrt{t/2}(e_{16} + e_{25}).$$

$$\nabla^t_{e_4} e_{12} = \sqrt{t/2}\sum_j (e_j \lrcorner (e_{16} + e_{25})) \wedge (e_j \lrcorner e_{12}) = \sqrt{t/2}(e_{15} - e_{26}).$$

$$\nabla^t_{e_4} e_{56} = \sqrt{t/2}\sum_j (e_j \lrcorner (e_{16} + e_{25})) \wedge (e_j \lrcorner e_{56}) = \sqrt{t/2}(e_{15} - e_{26}).$$

$$\nabla^t_{e_5} e_{12} = -\sqrt{t/2}\sum_j (e_j \lrcorner (e_{13} + e_{24})) \wedge (e_j \lrcorner e_{12}) = \sqrt{t/2}(e_{23} - e_{14}).$$

$$\nabla^t_{e_5} e_{34} = -\frac{1-t}{\sqrt{2t}}\sum_j (e_j \lrcorner (e_{13} + e_{24})) \wedge (e_j \lrcorner e_{34}) = \frac{1-t}{\sqrt{2t}}(e_{14} - e_{23}).$$

$$\nabla^t_{e_6} e_{12} = -\frac{1-t}{\sqrt{2t}}\sum_j (e_j \lrcorner (e_{14} - e_{23})) \wedge (e_j \lrcorner e_{12}) = \frac{1-t}{\sqrt{2t}}(e_{13} + e_{24}).$$

$$\nabla^t_{e_6} e_{34} = -\frac{1-t}{\sqrt{2t}}\sum_j (e_j \lrcorner (e_{14} - e_{23})) \wedge (e_j \lrcorner e_{34}) = -\frac{1-t}{\sqrt{2t}}(e_{13} + e_{24}).$$

Note that

$$\nabla^t_{e_1} e_{34} + \nabla^t_{e_1} e_{56} = \nabla^t_{e_2} e_{34} + \nabla^t_{e_2} e_{56} = 0,$$

$$\nabla^t_{e_3} e_{12} = \nabla^t_{e_3} e_{56}, \;\; \nabla^t_{e_4} e_{12} = \nabla^t_{e_4} e_{56}.$$

So, we have

$$
\begin{aligned}
d\Omega &= \sum_i e_i \wedge \nabla^t_{e_i}\Omega \\
&= \sum_i e_i \wedge \nabla^t_{e_i}(e_{12} + e_{34} - e_{56}) \\
&= e_1 \wedge \nabla^t_{e_1} e_{34} - e_1 \wedge \nabla^t_{e_1} e_{56} + e_2 \wedge \nabla^t_{e_2} e_{34} - e_2 \wedge \nabla^t_{e_2} e_{56} \\
&\quad + e_3 \wedge \nabla^t_{e_3} e_{12} - e_3 \wedge \nabla^t_{e_3} e_{56} + e_4 \wedge \nabla^t_{e_4} e_{12} - e_4 \wedge \nabla^t_{e_4} e_{56} \\
&\quad + e_5 \wedge \nabla^t_{e_5} e_{12} + e_5 \wedge \nabla^t_{e_5} e_{34} + e_6 \wedge \nabla^t_{e_6} e_{12} + e_6 \wedge \nabla^t_{e_6} e_{34} \\
&= 2(e_1 \wedge \nabla^t_{e_1} e_{34} + e_2 \wedge \nabla^t_{e_2} e_{34}) \\
&\quad + e_5 \wedge \nabla^t_{e_5} e_{12} + e_5 \wedge \nabla^t_{e_5} e_{34} + e_6 \wedge \nabla^t_{e_6} e_{12} + e_6 \wedge \nabla^t_{e_6} e_{34} \\
&= 2(\sqrt{t/2} e_1 \wedge (e_{45} - e_{36}) - \sqrt{t/2} e_2 \wedge (e_{46} + e_{35})) \\
&\quad + \sqrt{t/2} e_5 \wedge (e_{23} - e_{14}) + \frac{1-t}{\sqrt{2t}} e_5 \wedge (e_{14} - e_{23}) \\
&\quad + \frac{1-t}{\sqrt{2t}} e_6 \wedge (e_{13} + e_{24}) - \frac{1-t}{\sqrt{2t}} e_6 \wedge (e_{13} + e_{24}) \\
&= \sqrt{2t}(e_{145} - e_{136} - e_{246} - e_{235}) + \sqrt{t/2}(e_{235} - e_{145}) \\
&\quad + \frac{1-t}{\sqrt{2t}}(e_{145} - e_{235}) \\
&= (\sqrt{2t} - \sqrt{t/2} + \frac{1-t}{\sqrt{2t}})(e_{145} - e_{235}) - \sqrt{2t}(e_{136} + e_{246})
\end{aligned}
$$

And from (2.3)

$$d\Omega(J) = (\sqrt{2t} - \sqrt{t/2} + \frac{1-t}{\sqrt{2t}})(e_{236} - e_{146}) + \sqrt{2t}(e_{245} + e_{135}).$$

In the proof of Theorem 2.1 we see that $N \equiv 0$ which means

$$T = N + d\Omega(J) = (\sqrt{2t} - \sqrt{t/2} + \frac{1-t}{\sqrt{2t}})(e_{145} - e_{235}) - \sqrt{2t}(e_{136} + e_{246}).$$

□

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