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# UNIFORM LIPSCHITZ AND ASYMPTOTIC STABILITY FOR PERTURBED DIFFERENTIAL SYSTEMS

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Abstract. In this paper, we investigate uniform Lipschitz and asymptotic stability for perturbed differential systems using integral inequalities.

## 1. Introduction

Dannan and Elaydi [7] introduced a new notion of stability, which is called uniform Lipschitz stability (ULS), for systems of differential equations. For linear systems, the notions of uniform Lipschitz stability and that of uniform stability are equivalent. However, for nonlinear systems, the two notions are quite distinct. This notion of ULS lies somewhere between uniform stability on one side and the notions of asmptotic stability in variation of Brauer[3] and uniform stability in variation of Brauer and Strauss[2] on the other side. Also, Elaydi and Farran [8] introduced the notion of exponential asymptotic stability(EAS) which is a stronger notion than that of ULS. They studied the properties of EAS dynamical systems on a compact Riemannian manifold, and gave some analytic criteria for an autonomous differential system and its perturbed systems to be EAS. Gonzalez and Pinto[9] investigated the asymptotic behavior and boundedness of the solutions of nonlinear differential systems.

In this paper, we investigate uniform Lipschitz and asymptotic stability for solutions of the nonlinear differential systems using integral inequalities. The method incorporating integral inequalities takes an important place among the methods developed for the qualitative analysis of solutions to linear and nonlinear system of differential equations.

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In the presence the method of integral inequalities is as efficient as the direct Lyapunov's method.

## 2. Preliminaries

We consider the nonlinear nonautonomous differential system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{2.1}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean $n$-space. We assume that the Jacobian matrix $f_x = \partial f/\partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. Also, consider the perturbed differential system of (2.1)

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s), Ty(s))ds,\ y(t_0) = y_0, \tag{2.2}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ , $g(t, 0, 0) = 0$ and $T$ is a continuous operator mapping from $C(\mathbb{R}^+, \mathbb{R}^n)$ into $C(\mathbb{R}^+, \mathbb{R}^n)$. The symbol $|\cdot|$ donotes arbitrary vector norm in $\mathbb{R}^n$.

In our subsequent discussion we assume that for any two continuous functions $u, v \in C(I)$ where $I$ is the closed interval, the operator $T$ satisfies the following property:

$$u(t) \le v(t), 0 \le t \le t_1, t_1 \in I,$$

imples $Tu(t) \le Tv(t), 0 \le t \le t_1$, and $|Tu| \le T|u|$.

Let $x(t, t_0, x_0)$ be denoted the unique solution of (2.1) through $(t_0, x_0)$ in $\mathbb{R}^+ \times \mathbb{R}^n$ with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (2.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t),\ v(t_0) = v_0 \tag{2.3}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t),\ z(t_0) = z_0. \tag{2.4}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (2.4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (2.3).

Before giving further details, we recall some notions of stability that we need in the sequel[8].

DEFINITION 2.1. The system (2.1) (the zero solution $x = 0$ of (2.1)) is called
(S)*stable* if for any $\epsilon > 0$ and $t_0 \geq 0$, there exists $\delta = \delta(t_0, \epsilon) > 0$ such that if $|x_0| < \delta$ , then $|x(t)| < \epsilon$ for all $t \geq t_0 \geq 0$,
(US)*uniformly stable* if the $\delta$ in (S) is independent of the time $t_0$,
(ULS)*uniformly Lipschitz stable* if there exist $M > 0$ and $\delta > 0$ such that $|x(t)| \leq M|x_0|$ whenever $|x_0| \leq \delta$ and $t \geq t_0 \geq 0$,
(ULSV)*uniformly Lipschitz stable in variation* if there exist $M > 0$ and $\delta > 0$ such that $|\Phi(t, t_0, x_0) \leq M$ for $|x_0| \leq \delta$ and $t \geq t_0 \geq 0$,
(EAS) *exponentially asymptotically stable* if there exist constants $K > 0$, $c > 0$, and $\delta > 0$ such that

$$|x(t)| \leq K\,|x_0|e^{-c(t-t_0)}, 0 \leq t_0 \leq t$$

provided that $|x_0| < \infty$,
(EASV) *exponentially asymptotically stable in variation* if there exist constants $K > 0$ and $c > 0$ such that

$$|\Phi(t, t_0, x_0)| \leq K\,e^{-c(t-t_0)}, 0 \leq t_0 \leq t$$

provided that $|x_0| < \infty$.

We give some related properties that we need in the sequel.

We need Alekseev formula to compare between the solutions of (2.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y),\ y(t_0) = y_0, \tag{2.5}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (2.5) passing through the point $(t_0, y_0)$ in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

LEMMA 2.2. *If $y_0 \in \mathbb{R}^n$, then for all $t$ such that $x(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

LEMMA 2.3. [6] *Let $u, f, g, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in $u$ and $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$. If ,for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^{t} f(s)u(s)ds + \int_{t_0}^{t} f(s)\Big\{\int_{t_0}^{s} g(\tau)w(u(\tau))d\tau\Big\}ds,\ \ t \geq t_0 \geq 0,$$

*then*

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} g(s)ds\Big] \exp\Big(\int_{t_0}^{t} f(s)ds\Big),\ t_0 \leq t < b_1,$$

*where* $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $u > 0$, $u_0 > 0$, $W^{-1}(u)$ *is the inverse of* $W(u)$ *and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} g(s)ds \in \mathrm{domW}^{-1}\Big\}.$$

Lemma 2.4. [12] *Let* $u, f, g, h \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* $u$ ,$u \leq w(u)$ . *Suppose that for some* $c > 0$,

$$u(t) \leq c + \int_{t_0}^{t} f(s)w(u(s))ds + \int_{t_0}^{t} g(s)\Big(\int_{t_0}^{s} h(\tau)u(\tau)d\tau\Big)ds, \ \ 0 \leq t_0 \leq t.$$

*Then*

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t}(f(s) + g(s)\int_{t_0}^{s} h(\tau)d\tau))ds\Big], \ t_0 \leq t < b_1, \tag{2.6}$$

*where* $W$, $W^{-1}$ *are the same functions as in Lemma 2.3 and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t}(f(s) + g(s)\int_{t_0}^{s} h(\tau)d\tau))ds \in \mathrm{domW}^{-1}\Big\}.$$

Lemma 2.5. [4] *Let* $u, f, g, h \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* $u$. *Suppose that for some* $c > 0$,

$$u(t) \leq c + \int_{t_0}^{t} f(s)w(u(s))ds + \int_{t_0}^{t} g(s)(\int_{t_0}^{s} h(\tau)w(u(\tau))d\tau)ds, \ \ 0 \leq t_0 \leq t.$$

*Then*

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t}(f(s) + g(s)\int_{t_0}^{s} h(\tau))ds\Big], \ t_0 \leq t < b_1,$$

*where* $W$, $W^{-1}$ *are the same functions as in Lemma 2.3 and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t}(f(s) + g(s)\int_{t_0}^{s} h(\tau))ds \in \mathrm{domW}^{-1}\Big\}.$$

Lemma 2.6. [10] *Let* $u, p, q, w, and\ r \in C(\mathbb{R}^+)$ *and suppose that, for some* $c \geq 0$, *we have*

$$u(t) \leq c + \int_{t_0}^{t} p(s)\int_{t_0}^{s}[q(\tau)u(\tau) + w(\tau)\int_{t_0}^{\tau} r(a)u(a)da]d\tau ds, \ t \geq t_0. \tag{2.7}$$

*Then*

$$u(t) \leq c\exp\Big(\int_{t_0}^{t} p(s)\int_{t_0}^{s}[q(\tau) + w(\tau)\int_{t_0}^{\tau} r(a)da]d\tau ds\Big), \ t \geq t_0. \tag{2.8}$$

Lemma 2.7. [12] *Let* $u, p, q, w, r \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* $u$. *Suppose that for some* $c \geq 0$,

(2.9)
$$u(t) \leq c + \int_{t_0}^{t} (p(s) \int_{t_0}^{s} (q(\tau)w(u(\tau)) + v(\tau) \int_{t_0}^{\tau} r(a)u(a)da)d\tau)ds,\ t \geq t_0.$$

*Then*

(2.10)
$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} (p(s) \int_{t_0}^{s} (q(\tau) + v(\tau) \int_{t_0}^{\tau} r(a)da)d\tau)ds\Big],\ t_0 \leq t < b_1,$$

*where* $W$, $W^{-1}$ *are the same functions as in Lemma 2.3 and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} (p(s) \int_{t_0}^{s} (q(\tau) + v(\tau) \int_{t_0}^{\tau} r(a)da)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Lemma 2.8. [14] *Let* $u(t)$, $f(t)$ *,and* $g(t)$ *be real-valued nonnegative continuous functions defined on* $\mathbb{R}^+$, *for which the inequality*

$$u(t) \leq u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s)\Big(\int_0^s g(\tau)u(\tau)d\tau\Big)ds,\ t \in \mathbb{R}^+,$$

*holds, where* $u_0$ *is a nonnegative constant. Then,*

$$u(t) \leq u_0(1 + \int_0^t f(s) \exp\Big(\int_0^s (f(\tau) + g(\tau))d\tau)\Big)ds,\ t \in \mathbb{R}^+.$$

Lemma 2.9. [5] *Let the following condition hold for functions* $u(t)$, $v(t) \in C(\mathbb{R}^+)$ *and* $k(t,u,v) \in C(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$:

$$u(t) - \int_{t_0}^{t} k(s, u(s), Tu(s))ds \leq v(t) - \int_{t_0}^{t} k(s, v(s), Tv(s))ds,$$

$t \geq t_0 \geq 0$ *and* $k(t,u,v)$ *is monotone nondecreasing in* $u$ *and* $v$ *for each fixed* $t \geq 0$. *If* $u(t_0) < v(t_0)$, *then* $u(t) < v(t)$, $t \geq t_0 \geq 0$.

## 3. Main results

In this section, we investigate uniform Lipschitz and asymptotic stability for solutions of the nonlinear perturbed differential systems using integral inequalities.

Theorem 3.1. *Suppose that $x = 0$ of (2.1) is ULS. Let the following condition hold for (2.2):*

$$\int_{t_0}^{t} |g(s, y(s), Ty(s))|ds \le W(t, |y|, T|y|), 0 \le t_0 \le t,$$

*where $W(t, u, v) \in C(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ is monotone nondecreasing in $u$ and $v$ for each fixed $t \ge 0$ with $W(t, 0, 0) = 0$. Assume that $u(t)$ is any solution of the scalar differential equation*

$$u'(t) = KW(t, u, Tu), u(t_0) = u_0 > 0, K \ge 1, \tag{3.1}$$

*existing on $\mathbb{R}^+$ such that $m(t_0) < u(t_0)$. If $u = 0$ of (3.1) is ULS, then $y = 0$ of (2.2) is also ULS whenever $K|y_0| < u_0$.*

*Proof.* Let $y(t) = y(t, t_0, y_0)$ be any solution of (2.2). By Lemma 2.2, we have

$$|y(t)| \le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| \int_{t_0}^{s} |g(\tau, y(\tau), Ty(\tau))|d\tau ds,$$

where $\Phi(t, t_0, y_0)$ is the fundemental matrix of (2.4). Since $x = 0$ of (2.1) is ULS, it is ULSV by Corollary 3.6[7]. Therefore there exist $K > 0$ and $\delta > 0$ such that $|\Phi(t, t_0, y_0)| \le K$ for $t \ge t_0 \ge 0$. Thus, by the assmption, we obtain

$$\begin{aligned} &|y(t)| - K \int_{t_0}^{t} W(s, |y(s)|, T|y(s)|)ds \\ &\le K|y_0| < u_0 = u(t) - K \int_{t_0}^{t} W(s, u(s), Tu(s))ds. \end{aligned}$$

Hence $|y(t)| < u(t)$ by Lemma 2.9. Since $u = 0$ of (3.1) is ULS, it easily follows that $y = 0$ of (2.2) is ULS. □

Corollary 3.2. *Suppose that $x = 0$ of (2.1) is ULS. Consider the scalar differential equation*

$$u'(t) = KW(t, u, Tu) = Ka(t)[u + \int_{t_0}^{t} k(s)u(s)ds], \tag{3.2}$$

*where $u_0 \ge 1, K \ge 1$ and $a, k \in C(\mathbb{R}^+)$ satisfy the conditions:*

(a) $\int_{t_0}^{t} |g(s, y(s), Ty(s))|ds \le W(t, |y|, T|y|)$,where $\int_{t_0}^{t} g(s, y(s), Ty(s))ds$ *is in (2.2),*

(b) $M(t_0) = (1 + K \int_{t_0}^{\infty} a(s) \exp(\int_{t_0}^{s}(Ka(\tau) + k(\tau))d\tau)ds) < \infty$ *and* $b_1 = \infty$.

*Then $y = 0$ of (2.2) is ULS.*

*Proof.* Let $u(t) = u(t, t_0, u_0)$ be any solution of (3.2). Then, by Lemma 2.8 , we obtain

$$|u(t)| \leq u_0\Big(1 + K\int_{t_0}^{t} a(s)\exp\Big(\int_{t_0}^{s}(Ka(\tau) + k(\tau))d\tau\Big)ds\Big) \leq M(t_0)|u_0|,$$

Hence $u = 0$ of (3.2) is ULS. By Theorem 3.1, the solution $y = 0$ of (2.2) is ULS . □

Remark 3.1. In corollary 3.2, it is needed that $b_1 = \infty$. The condition $W(\infty) = \infty$ is too strong and it represents situations which are not stable. For example, if $w(u) = u^\alpha$, then only $\alpha \leq 1$ satisfies $W(\infty) = \infty$ and $\alpha < 1$ is not stable. See [17].

Corollary 3.3. *Suppose that $x = 0$ of (2.1)is ULS. Consider the scalar differential equation*

$$u'(t) = KW(t, u, Tu) = Ka(t)\Big[u + \int_{t_0}^{t} k(s)w(u(s))ds\Big], \tag{3.3}$$

*where $u_0 \geq 1, K \geq 1$, $u, w \in C(\mathbb{R}^+)$, $w(u)$ is nondecreasing in $u$ and $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$, and $a, k \in C(\mathbb{R}^+)$ satisfy the conditions:*

(a) $\int_{t_0}^{t} |g(s, y(s), Ty(s))|ds \leq W(t, |y|, T|y|)$,
*where $\int_{t_0}^{t} g(s, y(s), Ty(s))ds$ is in (2.2),*

(b) $M(t_0) = W^{-1}[W(u_0) + \int_{t_0}^{\infty} k(s)ds] \cdot \exp(\int_{t_0}^{\infty} Ka(s)ds) < \infty$, $b_1 = \infty$,*and* $a, k \in L_1(\mathbb{R}^+)$ .

*Then $y = 0$ of (2.2) is ULS.*

*Proof.* Let $u(t) = u(t, t_0, u_0)$ be any solution of (3.3). Then, by Lemma 2.3, we have

$$\begin{aligned}|u(t)| &\leq W^{-1}\Big[W(u_0) + \int_{t_0}^{t} k(s)ds\Big] \cdot \exp\Big(\int_{t_0}^{t} Ka(s)ds\Big)\\ &= M(t_0) \leq M(t_0)|u_0|,\end{aligned}$$

Hence $u = 0$ of (3.3) is ULS, and so by Theorem 3.1, the solution $y = 0$ of (2.2) is ULS . □

Corollary 3.4. *Suppose that $x = 0$ of (2.1) is ULS. Consider the scalar differential equation*

$$u'(t) = KW(t, u, Tu) = K\Big[a(t)w(u(t)) + b(t)\int_{t_0}^{t} k(s)u(s)ds\Big], \tag{3.4}$$

*where $w \in C((0, \infty)$, $w(u)$ is nondecreasing on $u$ and $u \leq w(u)$, $u_0 \geq 1$, $K \geq 1$ and $a, b, k \in C(\mathbb{R}^+)$ satisfy the conditions:*

(a) $\int_{t_0}^{t} |g(s, y(s), Ty(s))|ds \le W(t, |y|, T|y|)$,
*where* $\int_{t_0}^{t} g(s, y(s), Ty(s))ds$ *is in (2.2),*

(b) $M(t_0) = W^{-1}\Big[W(u_0) + K\int_{t_0}^{\infty}(a(s) + b(s)\int_{t_0}^{s} k(s)ds)\Big] < \infty$,
$b_1 = \infty$, *and* $a, b, k \in L_1(\mathbb{R}^+)$ .

*Then* $y = 0$ *of (2.2) is ULS.*

*Proof.* Let $u(t) = u(t, t_0, u_0)$ be any solution of (3.4). Then, Lemma 2.4, we obtain

$$
\begin{aligned}
|u(t)| &\le W^{-1}\Big[W(u_0) + K\int_{t_0}^{t}(a(s) + b(s)\int_{t_0}^{s} k(s)ds)\Big] \\
&\le M(t_0) \le M(t_0)|u_0|,
\end{aligned}
$$

Hence $u = 0$ of (3.4) is ULS. This implies that the solution $y = 0$ of (2.2) is ULS by Theorem 3.1. □

Theorem 3.5. *For the perturbed (2.2), we suppose that*

$$
\int_{t_0}^{t} |g(s, y(s), Ty(s))|ds \le a(t)w(|y(t)|) + b(t)\int_{t_0}^{t} k(s)|y(s)|ds, \tag{3.5}
$$

*where* $a, b, k \in C(\mathbb{R}^+)$,$a, b, k \in L_1(\mathbb{R}^+)$, $w \in C((0, \infty)$, *and* $w(u)$ *is nondecreasing in* $u$, $u \le w(u)$, *and* $\frac{1}{v}w(u) \le w(\frac{u}{v})$ *for some* $v > 0$,

$$
M(t_0) = W^{-1}\Big[W(K) + K\int_{t_0}^{\infty}(a(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau)ds\Big], \tag{3.6}
$$

*where* $M(t_0) < \infty$ *and* $b_1 = \infty$. *If the zero solution of (2.1) is ULSV, the zero solution of (2.2) is ULS.*

*Proof.* Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Since $x = 0$ of (2.1) is ULSV, there exist $K > 0$ and $\delta > 0$ such that $|\Phi(t, t_0, y_0)| \le K$ for $t \ge t_0 \ge 0$. In view of Lemma 2.2 and (3.5), we obtain

$$
\begin{aligned}
|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))|\left|\int_{t_0}^{s} g(\tau, y(\tau), Ty(\tau))d\tau\right|ds \\
&\le K|y_0| + \int_{t_0}^{t} K|y_0|a(s)w(\frac{|y(s)|}{|y_0|})ds \\
&\quad + \int_{t_0}^{t} K|y_0|b(s)\int_{t_0}^{s} k(\tau)\frac{|y(\tau)|}{|y_0|}d\tau ds.
\end{aligned}
$$

Set $u(t) = |y(t)||y_0|^{-1}$. Then, by Lemma 2.4, we have

$$|y(t)| \leq |y_0| W^{-1}\Big[W(K) + K\int_{t_0}^{t}(a(s)+b(s)\int_{t_0}^{s}k(\tau)d\tau)ds\Big].$$

The above estimation yields the desired result. Hence the proof is complete. □

THEOREM 3.6. *For the perturbed (2.2), we suppose that*

$$|g(t,y(t),Ty(t))| \leq a(t)w(|y(t)|) + b(t)\int_{t_0}^{t}k(s)|y(s)|ds, \tag{3.7}$$

*where* $a, b, k \in C(\mathbb{R}^+)$, $a, b, k \in L_1(\mathbb{R}^+)$, $w \in C((0,\infty)$, *and* $w(u)$ *is nondecreasing in* $u$, $u \leq w(u)$, *and* $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ *for some* $v > 0$,

$$M(t_0) = W^{-1}\Big[W(K) + K\int_{t_0}^{\infty}\int_{t_0}^{s}(a(\tau)+b(\tau)\int_{t_0}^{\tau}k(r)dr)d\tau ds\Big], \tag{3.8}$$

*where* $M(t_0) < \infty$ *and* $b_1 = \infty$. *If the zero solution of (2.1) is ULSV, the zero solution of (2.2) is ULS.*

*Proof.* Let $x(t) = x(t,t_0,y_0)$ and $y(t) = y(t,t_0,y_0)$ be solutions of (2.1) and (2.2), respectively. Using Lemma 2.2, ULSV condition of $x = 0$ of (2.1), and (3.7), we obtain

$$\begin{aligned} |y(t)| &\leq |x(t)| + \int_{t_0}^{t}|\Phi(t,s,y(s))|\int_{t_0}^{s}|g(\tau,y(\tau),Ty(\tau))|d\tau ds \\ &\leq K|y_0| + \int_{t_0}^{t}K|y_0|\int_{t_0}^{s}[a(\tau)w(\frac{|y(\tau)|}{|y_0|})d\tau ds \\ &\quad + \int_{t_0}^{t}K|y_0|\int_{t_0}^{s}b(\tau)\int_{t_0}^{\tau}k(r)\frac{|y(r)|}{|y_0|}drd\tau]ds. \end{aligned}$$

Set $u(t) = |y(t)||y_0|^{-1}$. Then, an application of Lemma 2.7 yields

$$|y(t)| \leq |y_0| W^{-1}\Big[W(K) + K\int_{t_0}^{t}\int_{t_0}^{s}(a(\tau)+b(\tau)\int_{t_0}^{\tau}k(r)dr)d\tau ds\Big].$$

Thus we have $|y(t)| \leq M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$. This completes the proof. □

THEOREM 3.7. *Let the solution $x = 0$ of (2.1) be EASV. Assume that the perturbing term $\int_{t_0}^{t} g(s, y(s), Ty(s))ds$ satisfies*

$$\begin{aligned} &\int_{t_0}^{t} |g(s, y(s), Ty(s))|ds \\ &\le e^{-\alpha t}\Big(a(t)w(|y(t)|) + b(t)\int_{t_0}^{t} k(s)w(|y(s)|)ds\Big), \end{aligned} \tag{3.9}$$

*where $\alpha > 0$, $a, b, k, w \in C(\mathbb{R}^+)$, $a, b, k \in L_1(\mathbb{R}^+)$ and $w(u)$ is nondecreasing in u. If*

(3.10)

$$M(t_0) = W^{-1}\Big[W(c) + K\int_{t_0}^{\infty}(a(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau)ds\Big] < \infty, b_1 = \infty,$$

*where $c = K|y_0|e^{\alpha t_0}$, then all solutions of (2.2) approch zero as $t \to \infty$*

*Proof.* Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Using Lemma 2.2 and (3.9), we obtain

$$\begin{aligned} |y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| \Big| \int_{t_0}^{s} g(\tau, y(\tau), Ty(\tau))d\tau \Big| ds \\ &\le K|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Ke^{-\alpha(t-s)}[e^{-\alpha s}a(s)w(|y(s)|) \\ &\quad + Kb(s)e^{-\alpha s}\int_{t_0}^{s} k(\tau)w(|y(\tau)|)d\tau]ds. \end{aligned}$$

Set $u(t) = |y(t)|e^{\alpha t}$. Then, since $w(u)$ is nondecreasing, by Lemma 2.5 we obtain

$$|y(t)| \le e^{-\alpha t}W^{-1}\Big[W(c) + K\int_{t_0}^{t}(a(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau)ds\Big], \ t \ge t_0 \ge 0,$$

where $c = K|y_0|e^{\alpha t_0}$. The above estimation yields the desired result. □

THEOREM 3.8. *Let the solution $x = 0$ of (2.1) be EASV. Assume that the perturbing term $\int_{t_0}^{t} g(s, y(s), Ty(s))ds$ satisfies*

$$|g(t, y(t), Ty(t))| \le e^{-\alpha t}\Big(a(t)|y(t)| + b(t)\int_{t_0}^{t} k(s)|y(s)|ds\Big), \tag{3.11}$$

*where $\alpha > 0$, $a, b, k \in C(\mathbb{R}^+)$, $a, b, k \in L_1(\mathbb{R}^+)$, $w(u)$ is nondecreasing in u,and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$. If*

$$M(t_0) = c\exp(\int_{t_0}^{\infty} Ke^{\alpha s}\int_{t_0}^{s} e^{-\alpha\tau}[a(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr]d\tau ds) < \infty,\ t \geq t_0 \geq 0. \tag{3.12}$$

*where $c = K|y_0|e^{\alpha t_0}$, then all solutions of (2.2) approch zero as $t \to \infty$*

*Proof.* Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. By the assumption, we have $|\Phi(t, t_0, y_0)| \leq Ke^{-c(t-t_0)}$ for some $K > 0$ and $c > 0$. Using Lemma 2.2 and (3.11), we obtain

$$\begin{aligned} |y(t)| &\leq |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| \int_{t_0}^{s} |g(\tau, y(\tau), Ty(\tau))| d\tau ds \\ &\leq K|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Ke^{-\alpha(t-s)} \int_{t_0}^{s} e^{-\alpha\tau}[a(\tau)|y(\tau)| \\ &\quad + b(\tau)\int_{t_0}^{\tau} k(r)|y(r)|dr]d\tau ds. \end{aligned}$$

Set $u(t) = |y(t)|e^{\alpha t}$. Then, since $e^{\alpha t} \geq 1$, an application of Lemma 2.6 obtains

$$\begin{aligned} |y(t)| &\leq ce^{-\alpha t}\exp\Big(\int_{t_0}^{t} Ke^{\alpha s}\int_{t_0}^{s} e^{-\alpha\tau}[a(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr]d\tau ds\Big) \\ &\leq ce^{-\alpha t}M(t_0),\ t \geq t_0 \geq 0, \end{aligned}$$

where $c = K|y_0|e^{\alpha t_0}$. From the above estimation, we obtain the desired result. □

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