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A NOTE ON QUASI-$*$-INVERTIBLE AND $*$-INVERTIBLE IDEALS

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Abstract. Let $*$ be a star-operation on an integral domain R. We show that if R is a $*_w$-Noetherian domain, then quasi-$*_w$-invertible prime $*_w$-ideals of R are minimal, and prime ideals of R minimal over a $*_w$-invertible $*_w$-ideal are minimal.

1. Introduction

The concept of quasi-invertibility of a prime ideal was first introduced by Krull in [8]. A prime ideal P is said to be *quasi-invertible* if $P \subset PP^{-1}$. In [2], Butts showed that if every proper prime ideal of R is quasi-invertible, then R is a Dedekind domain. Also in [9], Perić showed that in a Noetherian domain, quasi-invertible prime ideals are minimal, and prime ideals minimal over an invertible ideal are minimal. Recently in [3], Chang gave a characterization of Krull domains, which is analogous to Butts' result. Finally in [7], Kim and Park generalized the notion of quasi-invertibility for prime ideals to that in the setting equipped with arbitrary star-operations and any (nonzero) ideals. Analogously to Butts' result, they also characterized Krull domains. In this article, we generalize and unify the results due to Perić ([9]).

Let R be an integral domain with quotient field K and $\mathfrak{F}(R)$ be the set of nonzero fractional ideals of R. A mapping $A \mapsto A_*$ of $\mathfrak{F}(R)$ into $\mathfrak{F}(R)$ is called a *star-operation* on R if the following conditions are satisfied for all $a \in K \setminus \{0\}$ and $A, B \in \mathfrak{F}(R)$:

(i) $(aR)_* = aR, (aA)_* = aA_*$;

(ii) $A \subseteq A_*$, if $A \subseteq B$ then $A_* \subseteq B_*$; and

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(iii) $(A_*)_* = A_*$.

Let $*$ be any star-operation on R. It is easy to show that for all $A, B \in \mathfrak{F}(R)$, $(AB)_* = (AB_*)_* = (A_*B_*)_*$. An $A \in \mathfrak{F}(R)$ is called a $*$-*ideal* if $A = A_*$; A is said to be $*$-*invertible* if $(AA^{-1})_* = R$, where $A^{-1} = \{x \in K | xA \subseteq R\}$; and A is said to be of *finite type* if $A_* = B_*$ for some finitely generated ideal B of R.

Given any star-operation $*$ on R, we can construct two new star-operations $*_f$ and $*_w$ induced by $*$. For all $A \in \mathfrak{F}(R)$, the $*_f$-operation is defined by $A_{*_f} = \bigcup\{B_* \mid B \in \mathfrak{F}(R), B \text{ is finitely generated, and } B \subseteq A\}$ and the $*_w$-operation is defined by $A_{*_w} = \{x \in K \mid Bx \subseteq A \text{ for some } B \in GV^*(R)\}$, where $GV^*(R)$ is the set of nonzero finitely generated ideals B of R with $B_* = R$. Clearly, if $A \in \mathfrak{F}(R)$ is finitely generated, then $A_* = A_{*_f}$. We say that $*$ is *of finite character* if $* = *_f$ and that $*_f$ is the *finite character star-operation induced by* $*$. It is known that the $*_w$-operation is also a finite character star-operation on R ([1, Theorem 2.7]). It is well known that if $* = *_f$, then each prime ideal minimal over a $*$-ideal is a prime $*$-ideal (in particular, each height-one prime ideal is a prime $*$-ideal).

The most important examples of star-operations are (1) the d-operation defined by $A_d := A$, (2) the v-operation defined by $A_v := (A^{-1})^{-1}$, (3) the t-operation defined by $A_t := A_{v_f}$, and (4) the w-operation defined by $A_w := A_{v_w}(= \{x \in K \mid Bx \subseteq A \text{ for some finitely generated ideal } B \text{ of } R \text{ with } B^{-1} = R\})$ for all $A \in \mathfrak{F}(R)$. Note that all star-operations above except for v are of finite character. For any star-operation $*$ on R and for any $A \in \mathfrak{F}(R)$, we have that $A \subseteq A_* \subseteq A_v$, $A \subseteq A_{*_w} \subseteq A_{*_f} \subseteq A_t$, and $A_{*_w} \subseteq A_w$; so $(A_*)_v = A_v = (A_v)_*$ and $(A_{*_f})_t = A_t = (A_t)_{*_f}$. In particular, a v-ideal (resp., t-ideal) is a $*$-ideal (resp., $*_f$-ideal).

Let $*$ be a star-operation on R. Then R is called a $*$-*Noetherian domain* if R has the ascending chain condition on integral $*$-ideals of R. It is well known that R is a $*$- Noetherian domain if and only if every integral $*$-ideal of R is of finite type and that if R is a $*$-Noetherian domain, then $*= *_f$ ([11, Theorem 1.1]). Recall that a $*$-Noetherian domain R is a Mori domain when $* = v$ (or $* = t$); R is a strong Mori domain (SM domain) when $* = w$; and R is just the usual Noetherian domain when $* = d$. For any two star-operations $*_1$ and $*_2$ on R, $*_1 \leq *_2$ means that $A_{*_1} \subseteq A_{*_2}$ for all $A \in \mathfrak{F}(R)$. Note that $d \leq w \leq t \leq v$. It is clear that if R is a $*_1$-Noetherian domain, then R is also a $*_2$-Noetherian domain for any star-operations $*_1 \leq *_2$ on R. Thus we have the following implications: Noetherian domain $\Rightarrow$ SM domain $\Rightarrow$ Mori domain.

In this article, we show that for a star-operation $*$ on an integral domain R, if R is a $*_w$-Noetherian domain, then quasi-$*_w$-invertible prime $*_w$-ideals of R are minimal, and prime ideals of R minimal over a $*_w$-invertible $*_w$-ideal are minimal (quasi-$*$-invertibility will be defined in Section 2). As corollaries, we obtain the results [9, Satzs 1 and 2] done by Perić and the similar results on an SM domain.

General references for any undefined terminology or notation are [4, 6].

2. Main results

Let $*$ be a star-operation. Recall from [7] that a nonzero proper ideal I is said to be *quasi-$*$-invertible* if $I_* \subset (II^{-1})_*$. Clearly any $*$-invertible $*$-ideal is quasi-$*$-invertible.

Hereafter we let $*$ be a star-operation on an integral domain such that $* = *_w$.

THEOREM 2.1. *Let R be a $*$-Noetherian domain and let P be a quasi-$*$-invertible prime $*$-ideal of R. Then P is minimal.*

Proof. Since R is a $*$-Noetherian domain, by [5, Corollary 2.13] R_P is a Noetherian domain. Since P is quasi-$*$-invertible and R_P is a local ring with the maximal ideal PR_P, we have

$$R_P = (PP^{-1})R_P = (PR_P)(P^{-1}R_P) \subseteq (PR_P)(PR_P)^{-1} \subseteq R_P.$$

Thus PR_P is invertible, and so principal. Thus by Krull's principal ideal theorem ([6, Theorem 142]), PR_P is height-one. Since $\mathrm{ht}(P) = \mathrm{ht}(PR_P)$, P is minimal. □

Without using localization technique, Theorem 2.1 can be seen as follows:

Put $C := (PP^{-1})_*$. For any $0 \neq p \in P$, we have

$$pC = p(PP^{-1})_* = (BP)_*, \text{ where } B := P^{-1}p \subseteq R. \tag{2.1}$$

Since P is quasi-$*$-invertible, one can choose an element $p \in P$ such that $B \not\subseteq P$. Since $p \in P$, P contains a minimal prime $*$-ideal P' over Rp, i.e., $P' \subseteq P$. If $P' \neq P$, then by (2.1) we have $B \subseteq P' \subseteq P$, because $BP \subseteq Rp \subseteq P'$ and P' is a prime ideal of R. This is a contradiction to the condition that $B \not\subseteq P$. Thus we have $P' = P$, i.e., P is a minimal prime ideal over Rp. Therefore by [1, Corollary 3.7], P is minimal.

Applying Theorem 2.1 to the cases when $* = d$ and $* = w$, we can get the following:

Corollary 2.2. 1. ([9, Satz 1]) *Let P be a quasi-invertible prime ideal of a Noetherian domain R. Then P is minimal.*

2. *Let P be a quasi-w-invertible prime w-ideal of an SM domain R. Then P is minimal.*

The generalized principal ideal theorem (GPIT) states that in a Noetherian domain R, if P is a prime ideal of R minimal over an ideal generated by n elements, then $\mathrm{ht}(P) \leq n$ ([6, Theorem 152]). When $n = 1$, this theorem is well known as Krull's principal ideal theorem (PIT). This was generalized to SM domains by Wang and McCasland in [10, Corollary 1.12]. They proved that in an SM domain R, a prime ideal of R minimal over a w-ideal $(a_1, \ldots, a_n)_w$ has height at most n. By Anderson and Cook, it was shown that $*_w$-Noetherian domains also satisfy the GPIT ([1, Corollary 3.7]).

Now we give a variant of the GPIT for $*_w$-Noetherian domains. We first introduce the following simple lemma.

Lemma 2.3. *Let A be a $*$-invertible $*$-ideal in a $*$-Noetherian domain R and let P be a prime $*$-ideal of R. Then there is an element $a \in A$ such that $aA^{-1} \not\subseteq P$.*

Proof. Suppose that P is a prime $*$-ideal of R and A is a $*$-invertible $*$-ideal of R. If $AA^{-1} \subseteq P$, then $R = (AA^{-1})_* = P_* = P$, a contradiction. So $AA^{-1} \not\subseteq P$, which implies that there exists an element $a \in A$ such that $aA^{-1} \not\subseteq P$. □

Note that Lemma 2.3 holds without the assumption that R is a $*$-Noetherian domain.

Theorem 2.4. *Let R be a $*$-Noetherian domain and A be a $*$-invertible $*$-ideal of R. Then every prime ideal of R minimal over A is minimal.*

Proof. Let R be a $*$-Noetherian domain and A be a $*$-invertible $*$-ideal of R. Since $* = *_w$ is of finite character, we note that a prime ideal of R minimal over a $*$-ideal of R is also a $*$-ideal. Suppose that P is a prime ideal of R which is minimal over A. Then P is also a $*$-ideal. By Lemma 2.3, we can choose an element $a \in A$ such that $aA^{-1} \not\subseteq P$. Put $B := aA^{-1}$. Then $(BA)_* = a(AA^{-1})_* = aR \subseteq A \subseteq P$; so P contains a prime ideal Q of R which is minimal over aR. Note that since R is a $*$-Noetherian domain, every prime ideal of R which is minimal over aR is minimal by [1, Corollary 3.7]. Hence Q is minimal. Since $BA \subseteq (BA)_* \subseteq Q \subseteq P$ and $B \not\subseteq P$, we have $B \not\subseteq Q$. Thus $A \subseteq Q$.

Obviously, Q is a minimal prime ideal over A, and so $Q = P$. Thus P is minimal. □

Note that Theorem 2.4 can be proved by localization technique as in the proof of Theorem 2.1.

Applying Theorem 2.4 to the case when $* = d$ or $* = w$, we can get the following:

Corollary 2.5. 1. ([9, Satz 2]) *Every prime ideal of a Noetherian domain which is minimal over an invertible ideal is minimal.*
2. *Every prime ideal of an SM domain which is minimal over a w-invertible w-ideal is minimal.*

Lemma 2.3 plays an important role in the proof of Theorem 2.4. In fact, we can get a stronger result than Lemma 2.3 as follows:

Proposition 2.6. *Let A be a $*$-invertible $*$-ideal in a $*$-Noetherian domain R and let P_i $(i = 1, \ldots, n)$ be prime $*$-ideals of R. Then there is a $*$-ideal B which is not contained in any P_i $(i = 1, \ldots, n)$, so that*

$$(BA)_* = Ra \tag{2.2}$$

with a suitable $a \in A$.

Proof. Note that for each $a \in A$, Equation (2.2) follows from $(AA^{-1})_* = R$ by multiplying with a. Now we show that there is an element $a \in A$ such that $A^{-1}a \not\subseteq P_i$ for any $i = 1, \ldots, n$ by using the induction on n. First, assume that $n = 1$. By Lemma 2.3, it is true. Suppose that the assertion is true for $n-1$ $(n > 1)$, i.e., there is an $a' \in A$ such that $A^{-1}a' \not\subseteq P_i$ $(i = 1, \ldots, n-1)$. Let $\{P_i \mid i = 1, 2, \ldots, n\}$ be a set of prime $*$-ideals of R. Obviously, we can assume that $P_i \not\subseteq P_j$ for $i \neq j$. Under this assumption, for each $i = 1, \ldots, n$, there is a $p_i \in P_i$ with the property that $p_i \notin P_j$ for all $j \neq i$. Now we take $a = p_n a' + pa'$, where $p = p_1 \cdots p_{n-1}$. It is easy to see that for this a, the condition $A^{-1}a \not\subseteq P_i$ $(i = 1, \ldots, n)$ has been assured. □

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