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PRIME AND POISSON PRIME IDEALS IN SKEW EXTENSIONS DETERMINED BY DERIVATIONS

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Abstract. Let R be a commutative noetherian algebra and let δ be a nonzero derivation. Here we determine the prime ideals of the skew polynomial algebra $R[z;\delta]$ and the Poisson prime ideals of $R[z;\delta]_p$

1. Introduction

It is very difficult to find prime ideals of $R[z;\alpha,\delta]$ and Poisson prime ideals of $R[z;\alpha,\delta]_p$ generally. Here we determine the prime ideals of $R[z;\delta]$ and the Poisson prime ideals of $R[z;\delta]_p$ in the case that R is a commutative noetherian algebra and δ is a nonzero derivation of R. If $\delta=0$ then $R[z;\delta]=R[z]=R[z;\delta]_p$ and thus the prime ideals of $R[z]$ are determined by commutative algebra theory. So we concentrate on the case $\delta\neq 0$.

Assume throughout the paper that the base field is the complex number field $\mathbb{C}$ and that all algebras considered have unities.

Definition 1.1. (1) Given an automorphism α on a $\mathbb{C}$-algebra R, a $\mathbb{C}$-linear map δ is said to be a *(left) α-derivation* on R if $\delta(ab)=\delta(a)b+\alpha(a)\delta(b)$ for all $a,b\in R$. For such pair (α,δ), there exists a skew polynomial algebra $R[z;\alpha,\delta]$.

(2) A commutative $\mathbb{C}$-algebra S is said to be a *Poisson algebra* if there exists a bilinear product $\{-,-\}$ on S, called a *Poisson bracket*, such that $(S,\{-,-\})$ is a Lie algebra and $\{ab,c\}=a\{b,c\}+\{a,c\}b$ for all $a,b,c\in S$.

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We recall [5, 1.1]. A derivation α on S is said to be a *Poisson derivation* if

$$\alpha(\{a,b\}) = \{\alpha(a), b\} + \{a, \alpha(b)\}$$

for all $a, b \in S$. Let α and δ be a Poisson derivation and a derivation on S respectively. The pair (α, δ) is said to be a *skew Poisson derivation* if

$$\delta(\{a,b\}) - \{\delta(a), b\} - \{a, \delta(b)\} = \alpha(a)\delta(b) - \delta(a)\alpha(b)$$

for all $a, b \in S$. In such case, the commutative polynomial algebra $S[z]$ becomes a Poisson algebra with Poisson bracket $\{z, a\} = \alpha(a)z + \delta(a)$ for all $a \in S$ and is denoted by $S[z; \alpha, \delta]_p$. (In [5, 1.1], $\{z, a\}$ is defined by $\{z, a\} = -\alpha(a)z - \delta(a)$ for all $a \in S$.) If $\alpha = 0$ then we write $S[z; \delta]_p$ for $S[z; 0, \delta]_p$ and if $\delta = 0$ then we write $S[z; \alpha]_p$ for $S[z; \alpha, 0]_p$.

(3) An ideal I of a Poisson algebra S is said to be a *Poisson ideal* if $\{I, S\} \subseteq I$. A Poisson ideal P is said to be *Poisson prime* if, for all Poisson ideals I and J, $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. If S is noetherian then a Poisson prime ideal of S is a prime ideal by [3, Lemma 1.1(d)].

Definition 1.2. (1) A derivation δ on an algebra R is said to be *inner* if there exists an element $a \in R$ such that $\delta(b) = ab - ba$ for all $b \in R$. If R is commutative then every nonzero derivation on R is not inner.

(2) A derivation δ on a Poisson algebra S is said to be *inner* if there exists an element $a \in S$ such that δ is a Hamiltonian determined by a, that is, $\delta(b) = \{a, b\}$ for all $b \in S$. Note that every inner derivation on a Poisson algebra is a Poisson derivation and that if the Poisson bracket of S is trivial then every nonzero derivation on S is not inner.

Lemma 1.3. *(1) Let δ be a nonzero derivation on a simple algebra R. If δ is not inner then the skew polynomial algebra $R[z; \delta]$ is simple.*

(2) Let δ be a nonzero derivation on a Poisson simple algebra S. If δ is not inner then the Poisson polynomial algebra $S[z; \delta]_p$ is Poisson simple.

Proof. (1) Let I be a nonzero ideal of $R[z; \delta]$ and let g be a nonzero element of I such that the degree n of g is minimal among nonzero elements in I. Let J be the set of all leading coefficients of elements in I with degree n, together with zero. Then J is a nonzero ideal of R and thus $1_R \in J$. Hence we may assume that the leading coefficient of g is 1_R. If $n = 0$ then there is nothing to prove. Assume that $n > 0$ and set $g = z^n + az^{n-1} + (\dagger)$, where $(\dagger)$ is a polynomial with degree less than $n - 1$. For any $b \in R$, $gb - bg \in I$ has degree less than n, and thus

$gb = bg$. Comparing the coefficients of z^{n-1} in gb and bg, we have that $\delta(b) = (-n^{-1}a)b - b(-n^{-1}a)$ for all $b \in R$. This is a contradiction since δ is not inner.

(2) Let I be a nonzero Poisson ideal of $S[z;\delta]$ and let g be a nonzero element of I such that the degree n of g is minimal among nonzero elements in I. Let J be the set of all leading coefficients of elements in I with degree n, together with zero. Then J is a nonzero Poisson ideal of S and thus $1_S \in J$. It follows that the leading coefficient of g is 1_S. If $n = 0$ then there is nothing to prove. Assume that $n > 0$ and set $g = z^n + az^{n-1} + (\dagger)$, where $(\dagger)$ is a polynomial with degree less than $n-1$. For any $b \in S$, $\{g, b\} \in I$ has degree less than n, and thus $\{g, b\} = 0$. Considering the coefficients of z^{n-1} in $\{g, b\}$, we have that $\delta(b) = \{-n^{-1}a, b\}$ for all $b \in S$. This is a contradiction since δ is not inner. $\square$

THEOREM 1.4. *Let δ be a derivation on a commutative noetherian $\mathbb{C}$-algebra R.*

(1) Let P be a nonzero prime ideal of the skew polynomial algebra $R[z;\delta]$. Then $P \cap R$ is a prime ideal that is δ-stable. If the derivation $\overline{\delta}$ in $R/P \cap R$ induced by δ is non-zero then $P = (P \cap R)R[z;\delta]$ and $P \cap R \neq 0$.

(2) The commutative polynomial algebra $S = R[z]$ becomes a Poisson algebra with Poisson bracket $\{-,-\}_\delta$ defined by $\{a, b\}_\delta = 0, \{z, a\}_\delta = \delta(a)$ for all $a, b \in R$. That is, S is the Poisson polynomial algebra $R[z;\delta]_p$. Let P be a nonzero Poisson prime ideal of S. Then $P \cap R$ is δ-Poisson prime. If the derivation $\overline{\delta}$ in $R/P \cap R$ induced by δ is non-zero then $P = (P \cap R)S$ and $P \cap R \neq 0$.

Proof. (1) Since $z(P \cap R) = (P \cap R)z + \delta(P \cap R)$, $\delta(P \cap R)$ is contained in $P \cap R$ and thus $P \cap R$ is δ-stable. Let I and J be δ-ideals of R such that $IJ \subseteq P \cap R$. Then $(IR[z;\delta])(JR[z;\delta])$ is contained in P and thus $I \subseteq P$ or $J \subseteq P$. It follows that $P \cap R$ is δ-prime. Let Q be a minimal prime ideal of $P \cap R$. Then the largest δ-ideal contained in Q is a prime ideal containing $P \cap R$ by [1, 3.3.2], and thus every minimal prime ideal of $P \cap R$ is δ-stable. Since a finite product of minimal prime ideals of $P \cap R$ is contained in $P \cap R$ by [4, Theorem 2.4] and $P \cap R$ is δ-prime, $P \cap R$ is a prime ideal.

Note that $(P \cap R)R[z;\delta]$ is an ideal of $R[z;\delta]$ and $R[z;\delta]/(P \cap R)R[z;\delta]$ is isomorphic to $(R/P \cap R)[z;\overline{\delta}]$. Let D be the quotient field of $R/P \cap R$ and let $\overline{\delta}'$ be the extension of $\overline{\delta}$ to D, which exists uniquely by [2, Lemma 1.3]. Since D is commutative and $\overline{\delta}'$ is non-zero, $\overline{\delta}'$ is not inner. Hence

$D[z;\overline{\delta}']$ is simple by Lemma 1.3(1), and thus the ideal of $D[z;\overline{\delta}']$ induced by P is zero. It follows that $P = (P\cap R)R[z;\delta]$. In particular, $P\cap R \neq 0$ since $P \neq 0$.

(2) If $\delta(R) \subseteq P$ then $P \cap R$ is δ-stable. If $\delta(R) \nsubseteq P$ then $z \notin P$ since $\{z, R\}_\delta = \delta(R)$, and thus $P \cap R$ is δ-stable by [6, 2.4]. Hence $P \cap R$ is δ-Poisson prime, $(P \cap R)S$ is a Poisson ideal of S by [6, 2.2] and $S/(P\cap R)S$ is Poisson isomorphic to the Poisson polynomial algebra $(R/P\cap R)[z;\overline{\delta}]$. Let D be the quotient field of $R/P\cap R$ and let $\overline{\delta}'$ be the extension of $\overline{\delta}$. Then $\overline{\delta}'$ is not inner since $\overline{\delta}'$ is non-zero and the Poisson bracket of D is trivial. Hence $D[z;\overline{\delta}']$ is Poisson simple by Lemma 1.3(2), and thus the Poisson ideal of $D[z;\overline{\delta}']$ induced by P is zero. It follows that $P = (P \cap R)S$. In particular, $P \cap R \neq 0$ since $P \neq 0$. □

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