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ON RIGHT DERIVATIONS OF INCLINE ALGEBRAS

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Abstract. In this paper, we introduce the concept of a right derivation in incline algebras and give some properties of incline algebras. Also, the concept of d-ideal is introduced in an incline algebra with respect to right derivation.

1. Introduction

Z. Q. Cao, K. H. Kim and F. W. Roush [4] introduced the notion of incline algebras in their book and later it was developed by some authors [1, 2, 3, 5]. Ahn et al [1] introduced the notion of quotient incline and obtained the structure of incline algebras. N. O. Alshehri [3] introduced the notion of derivation in incline algebra. Incline algebra is a generalization of both Boolean and fuzzy algebra and it is a special type of semiring. It has both a semiring structure and a poset structure. It can also be used to represent automata and other mathematical systems, to study inequalities for non-negative matrices of polynomials. In this paper, we introduce the concept of a right derivation in incline algebras and give some properties of incline algebras. Also, the concept of d-ideal is introduced in an incline algebra with respect to right derivation.

2. Preliminaries

An *incline algebra* is a set K with two binary operations denoted by " $+$ " and " $*$ " satisfying the following axioms:

(K1) $x + y = y + x$,

(K2) $x + (y + z) = (x + y) + z$,

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(K3) $x * (y * z) = (x * y) * z$,
(K4) $x * (y + z) = (x * y) + (x * z)$,
(K5) $(y + z) * x = (y * x) + (z * x)$,
(K6) $x + x = x$,
(K7) $x + (x * y) = x$,
(K8) $y + (x * y) = y$,

for all $x, y, z \in K$. For convenience, we pronounce " $+$ " (resp. " $*$ ") as addition (resp. multiplication). Every distributive lattice is an incline algebra. An incline algebra is a distributive lattice if and only if $x*x = x$ for all $x \in K$. Note that $x \leq y \Leftrightarrow x + y = y$ for all $x, y \in K$. It is easy to se that " $\leq$ " is a partial order on K and that for any $x, y \in K$, the element $x + y$ is the least upper bound of $\{x, y\}$. We say that $\leq$ is induced by operation $+$.

In an incline algebra K, the following properties hold.

(K9) $x * y \leq x$ and $y * x \leq x$ for all $x, y \in K$,
(K10) $y \leq z$ implies $x * y \leq x * z$ and $y * x \leq z * x$, for all $x, y, z \in K$,
(K11) If $x \leq y$ and $a \leq b$, then $x + a \leq y + b$, and $x * a \leq y * b$ for all $x, y, a, b \in K$.

Furthermore, an incline algebra K is said to be *commutative* if $x * y = y * x$ for all $x, y \in K$.

A *subincline* of an incline algebra K is a non-empty subset M of K which is closed under the addition and multiplication. A subincline M is called an *ideal* if $x \in M$ and $y \leq x$ then $y \in M$. An element "0" in an incline algebra K is a *zero element* if $x + 0 = x = 0 + x$ and $x * 0 = 0 = 0 * x$ for any $x \in K$. An non-zero element "1" is called a *multiplicative identity* if $x * 1 = 1 * x = x$ for any $x \in K$. A non-zero element $a \in K$ is said to be a *left* (resp. *right*) *zero divisor* if there exists a non-zero $b \in K$ such hat $a * b = 0$ (resp. $b * a = 0$) A zero divisor is an element of K which is both a left zero divisor and a right zero divisor. An incline algebra K with multiplicative identity 1 and zero element 0 is called an *integral incline* if it has no zero divisors. By a homomorphism of inclines, we mean a mapping f from an incline algebra K into an incline algebra L such that $f(x + y) = f(x) + f(y)$ and $f(x * y) = f(x) * f(y)$ for all $x, y \in K$.

3. Right derivations of incline algebras

In what follows, let K denote an incline algebra with a zero-element unless otherwise specified.

Definition 3.1. Let K be an incline algebra. By a *right derivation* of K, we mean a self map d of K satisfying the identities

$$d(x+y) = d(x) + d(y) \text{ and } d(x*y) = (d(x)*y) + (d(y)*x)$$

for all $x, y \in K$.

Example 3.2. Let $K = \{0, a, b, 1\}$ be a set in which "+" and "$*$" is defined by

$+$	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

$*$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

Then it is easy to check that $(K, +, *)$ is an incline algebra. Define a map $d : K \to K$ by

$$d(x) = \begin{cases} a & \text{if } x = a, b, 1 \\ 0 & \text{if } x = 0 \end{cases}$$

Then we can see that d is a right derivation of the incline algebra K.

Proposition 3.3. *Let K be a commutative incline algebra. Then for a fixed $a \in K$, the mapping $d_a : K \to K$ given by $d_a(x) = x * a$, for every $x \in K$, is a right derivation of K.*

Proof. Let K be a commutative incline algebra. Then for a fixed $a \in K$, we have

$$\begin{aligned} d_a(x*y) &= (x*y)*a = ((x*y)*a) + ((x*y)*a) \\ &= ((x*a)*y) + ((y*a)*x) = d_a(x)*y + d_a(y)*x \end{aligned}$$

for all $x, y \in K$. This completes the proof. □

Proposition 3.4. *Let K be a commutative incline algebra. If K is a distributive lattice, $d_a(a) = a$ for each $a \in K$.*

Proof. Since K is a distributive lattice, we have $x * x = x$ for all $x \in K$. Hence $d_a(a) = a * a = a$. □

Proposition 3.5. *Let K be a commutative incline algebra and $a, b \in K$. Then $d_{a+b} = d_a + d_b$.*

Proof. Let K be a commutative incline algebra and $a, b \in K$. Then for all $c \in K$, we have

$$d_{a+b}(c) = c*(a+b) = (c*a) + (c*b) = d_a(c) + d_b(c) = (d_a + d_b)(c).$$

□

Proposition 3.6. *Let d be a right derivation of an incline algebra K. Then we have $d(0) = 0$.*

Proof. Let d be a right derivation of an incline algebra. Then we have

$$d(0) = d(0 * 0) = d(0) * 0 + d(0) * 0 = 0 + 0 = 0.$$

□

Proposition 3.7. *Let d be a right derivation of an incline algebra K. If K is a distributive lattice, then $d(x) \leq x$ for all $x \in K$.*

Proof. Let d be a right derivation of K and let K be a distributive lattice. Then

$$\begin{aligned} d(x) &= d(x * x) = d(x) * x + d(x) * x \\ &= d(x) * x \leq x \end{aligned}$$

from (K9) for all $x \in K$. □

Proposition 3.8. *Let K be an incline algebra and let d be a right derivation of K. Then we have $d(x * y) \leq d(x + y)$ for all $x, y \in K$.*

Proof. Let $x, y \in K$. By using (K9), we get $d(x) * y \leq d(x)$ and $d(y) * x \leq d(y)$. Thus we get

$$d(x * y) = (d(x) * y) + (d(y) * x) \leq d(x) + d(y) = d(x + y).$$

□

Proposition 3.9. *Let K be an incline algebra and a distributive lattice. Define $d^2(x) = d(d(x))$ for all $x \in K$. If $d^2 = d$, then $d(x * d(x)) = d(x)$ for all $x \in K$.*

Proof. Let K be an incline algebra and $x \in K$. Then

$$\begin{aligned} d(x * d(x)) &= (d(x) * d(x)) + (d^2(x) * x) \\ &= d(x) + (d(x) * x) \\ &= d(x). \end{aligned}$$

□

Proposition 3.10. *Let K be an incline algebra and let d be a right derivation of K. Then for all $x, y \in K$, $d(x*y) \leq d(x)$ and $d(x*y) \leq d(y)$.*

Proof. Let $x, y \in K$. Then by using (K7), we obtain

$$d(x) = d(x + x * y) = d(x) + d(x * y).$$

Hence we get $d(x*y) \leq d(x)$. Also, $d(y) = d(y+(x*y)) = d(y)+d(x*y)$, and so $d(x * y) \leq d(y)$. □

Definition 3.11. Let K be an incline algebra. A mapping f is *isotone* if $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in K$.

Proposition 3.12. *Let d be a right derivation of an incline algebra K. Then d is isotone.*

Proof. Let $x, y \in K$ be such that $x \leq y$. Then $x + y = y$. Hence we have $d(y) = d(x + y) = d(x) + d(y)$, which implies $d(x) \leq d(y)$. This completes the proof. □

Proposition 3.13. *A sum of two right derivations of an incline algebra K is again a right derivation of K.*

Proof. Let d_1 and d_2 be two right derivations of K respectively. Then we have for all $a, b \in K$,

$$\begin{aligned}(d_1 + d_2)(a * b) &= d_1(a * b) + d_2(a * b)\\ &= d_1(a) * b + d_1(b) * a + d_2(a) * b + d_2(b) * a\\ &= d_1(a) * b + d_2(a) * b + d_1(b) * a + d_2(b) * a\\ &= (d_1 + d_2)(a) * b + (d_1 + d_2)(b) * a.\end{aligned}$$

Clearly, $(d_1 + d_2)(a + b) = (d_1 + d_2)(a) + (d_1 + d_2)(b)$ for all $a, b \in K$. This completes the proof. □

Theorem 3.14. *Let K be a commutative incline algebra and let d_1, d_2 be right derivations of K, respectively. Define $d_1d_2(x) = d_1(d_2(x))$ for all $x \in K$. If $d_1d_2 = 0$, then d_2d_1 is a right derivation of K.*

Proof. Let K be a commutative incline algebra and $x, y \in K$. Then we have

$$\begin{aligned}0 &= d_1d_2(x * y) = d_1(d_2(x) * y + d_2(y) * x)\\ &= d_1d_2(x) * y + d_1(y) * d_2(x) + d_1d_2(y) * x + d_1(x) * d_2(y)\\ &= d_1(y) * d_2(x) + d_1(x) * d_2(y) = d_2(x) * d_1(y) + d_2(y) * d_1(x).\end{aligned}$$

Then

$$\begin{aligned}d_2d_1(x * y) &= d_2(d_1(x) * y + d_1(y) * x)\\ &= d_2d_1(x) * y + d_2(y) * d_1(x) + d_2d_1(y) * x + d_2(x) * d_1(y)\\ &= d_2d_1(x) * y + d_2d_1(y) * x.\end{aligned}$$

Finally, for all $x, y \in K$, we get

$$d_2d_1(x + y) = d_2(d_1(x) + d_1(y)) = d_2d_1(x) + d_2d_1(y).$$

This implies that d_2d_1 is a right derivation of a commutative incline algebra K. □

LEMMA 3.15. *Let K be an incline algebra. If every element a commutes with its right derivation $d(a)$, then we have*

$$d(a^n) = na^{n-1}d(a).$$

Proof. By using induction on n, we have the result. □

DEFINITION 3.16. Let K be an incline algebra and let d be a non-trivial right derivation of K. An ideal I of K is called a *d-ideal* if $d(I) = I$.

Since $d(0) = 0$, it can be easily observed that the zero ideal $\{0\}$ is a d-ideal of K. If d is onto, then $d(K) = K$, which implies K is a d-ideal of K.

EXAMPLE 3.17. In Example 3.2, let $I = \{0, a\}$. Then I is an ideal of K. It can be verified that $d(I) = I$. Therefore, I is an d-ideal of K.

LEMMA 3.18. *Let d be a right derivation of K and let I, J be any two d-ideals of K. Then we have $I \subseteq J$ implies $d(I) \subseteq d(J)$.*

Proof. Let $I \subseteq J$ and $x \in d(I)$. Then we have $x = d(y)$ for some $y \in I \subseteq J$. Hence we get $x = d(y) \in d(J)$. Therefore, $d(I) \subseteq d(J)$. □

PROPOSITION 3.19. *Let K be an incline algebra. Then, a sum of any two d-ideals is also a d-ideal of K.*

Proof. Let I and J be d-ideals of K. Then $I + J = d(I) + d(J) = d(I + J)$. Hence $I + J$ is a d-ideal of K. □

Let d be a right derivation of K. Define a set $Kerd$ by

$$Kerd := \{x \in K \mid d(x) = 0\}$$

for all $x \in K$.

PROPOSITION 3.20. *Let d be a right derivation of an incline algebra K. Then $Kerd$ is a subincline of K.*

Proof. Let $x, y \in Kerd$. Then $d(x) = 0, d(y) = 0$ and

$$\begin{aligned} d(x * y) &= (d(x) * y) + (d(y) * x) \\ &= (0 * y) + (0 * x) \\ &= 0 + 0 = 0, \end{aligned}$$

and

$$\begin{aligned} d(x + y) &= d(x) + d(y) \\ &= 0 + 0 = 0. \end{aligned}$$

Therefore, $x * y, x + y \in Kerd$. This completes the proof. □

Proposition 3.21. *Let d be a right derivation of an integral incline algebra K. Then $Kerd$ is an ideal of K.*

Proof. By Proposition 3.20, $Kerd$ is a subincline of K. Now let $x \in K$ and $y \in Kerd$ such that $x \leq y$. Then $d(y) = 0$ and

$$0 = d(y) = d(y + x * y) = d(y) + d(x * y) = 0 + d(x * y),$$

which $d(x * y) = 0$. Hence we have

$$0 = d(x * y) = (d(x) * y) + (d(y) * x) = d(x) * y.$$

Since K has no zero divisors, either $d(x) = 0$ or $y = 0$. If $d(x) = 0$, then $x \in Kerd$. If $y = 0$, then $x \leq y = 0$, i.e., $x = 0$, which implies $x \in Kerd$. □

Let d be a right derivation of K. Define a set $Fix_d(K)$ by

$$Fix_d(K) := \{x \in K \mid d(x) = x\}$$

for all $x \in K$.

Proposition 3.22. *Let K be a commutative incline algebra and let d be a right derivation. Then $Fix_d(K)$ is a subincline of K.*

Proof. Let $x, y \in Fix_d(K)$. Then we have $d(x) = x$ and $d(y) = y$, and so

$$\begin{aligned} d(x * y) &= d(x) * y + d(y) * x = x * y + y * x \\ &= x * y + x * y = x * y. \end{aligned}$$

Now

$$d(x + y) = d(x) + d(y) = x + y,$$

which implies $x + y, x * y \in Fix_d(K)$. This completes the proof. □

Definition 3.23. Let K be an incline algebra. An element $a \in K$ is said to be *additively left cancellative* if for all $a, b \in K$, $a + b = a + c \Rightarrow b = c$. An element $a \in K$ is said to be *additively right cancellative* if for all $a, b \in K$, $b+a = c+a \Rightarrow b = c$. It is said to be *additively cancellative* if it is both left and right cancellative. If every element of K is additively left cancellative, it is said to be *additively left cancellative.* If every element of K is additively right cancellative, it is said to be *additively right cancellative.*

Definition 3.24. A subincline I of an incline algebra K is called a *k-ideal* if $x + y \in I$ and $y \in I$, then $x \in I$.

Example 3.25. In Example 3.2, $I = \{0, a, b\}$ is an k-ideal of K.

Theorem 3.26. *Let K be a commutative incline algebra and additively right cancellative. If d is a right derivation of K, then $Fix_d(K)$ is a k-ideal of K.*

Proof. By Proposition 3.22, $Fix_d(K)$ is a subincline of K. Let $x + y, y \in Fix_d(K)$. Then $d(y) = y$ and $x + y = d(x + y)$. Hence $x + y = d(x + y) = d(x) + d(y) = d(x) + y$, which implies $x \in Fix_d(K)$. Hence $Fix_d(K)$ is a k-ideal of K. □

Proposition 3.27. *Let K be an incline algebra and let d be a right derivation of K. Then $Kerd$ is a k-ideal of K.*

Proof. From Proposition 3.20, $Kerd$ is a subincline of K. Let $x+y \in K$ and $y \in Kerd$. Then we have $d(x + y) = 0$ and $d(y) = 0$, and so

$$0 = d(x + y) = d(x) + d(y) = d(x) + 0 = d(x).$$

This implies $x \in Kerd$. □

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