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MAXIMALITY OF THE ANALYTIC SUBALGEBRAS OF C^*-ALGEBRAS WITH FLOWS

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Abstract. Given a faithful flow α on a C^*-algebra A, when A is α-simple we will show that the closed subalgebra of A consisting of elements with non-negative Arveson spectra is maximal if and only if the crossed product of A by α is simple. We will also show how the general case can be reduced to the α-simple case, which roughly says that any flow with the above maximality is an extension of a trivial flow by a flow of the above type in the α-simple case. We also propose a condition of essential maximality for such closed subalgebras.

1. Introduction

Let α be a flow on a C^*-algebra A, i.e., α is a one-parameter automorphism group of A such that $t \mapsto \alpha_t(x)$ is continuous for $x \in A$. We denote by $\mathrm{Sp}_\alpha(x)$ the Arveson spectrum of $x \in A$ and by $\mathrm{Sp}(\alpha)$ the Arveson spectrum of α; the latter being the closure of the union of all $\mathrm{Sp}_\alpha(x)$, $x \in A$. Note that $\mathrm{Sp}(\alpha)$ is a closed subset of $\mathbb{R}$ with $\mathrm{Sp}(\alpha) = -\mathrm{Sp}(\alpha)$ and $\mathrm{Sp}(\alpha) \ni 0$. We define the spectral subspaces $A^\alpha(\Omega)$ for closed or open subsets Ω of $\mathbb{R}$ (see [2] or Chapter 8 of [8]). If α is not trivial, i.e., $\mathrm{Sp}(\alpha) \neq \{0\}$, then $A^\alpha[0,\infty)$ is a proper closed subalgebra of A, called the *analytic subalgebra* for α. If B is a closed subalgebra of A such that $B \supset A^\alpha[0,\infty)$, then it is known that B is α-invariant (Corollary 6 of [9]). We would be interested, following [9], in the property that the analytic subalgebra $A^\alpha[0,\infty)$ is maximal, i.e., if B is a closed subalgebra of A with $B \supset A^\alpha[0,\infty)$, then either $B = A^\alpha[0,\infty)$ or $B = A$. We note that the subalgebras are also studied from different perspectives (e.g., [1], [3]).

When α is periodic this problem was completely solved by Peligrad and Zsidó (see Theorem 13 of [9]). When α is a faithful action of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ on a C^*-algebra A such that A is α-simple, i.e., has no α-invariant ideals except for $\{0\}$ and A, and $\mathrm{Sp}(\alpha)$ has more than three points, i.e., $\mathrm{Sp}(\alpha) \supsetneqq \{1,0,-1\}$, the maximality of $A^\alpha[0,\infty)$ is equivalent to the simplicity of the crossed product of $A \times_\alpha \mathbb{T}$, which is again equivalent to the fullness of the strong Connes spectrum

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of α (see [4]). Note that this generalizes a classical result for $A = C(\mathbb{T})$ with α induced by translations on the base space $\mathbb{T}$, due to Wermer [11]. In this case the analytic subalgebra $A^\alpha[0,\infty)$ is identified with the subalgebra of continuous functions on $\mathbb{D} = \{x \in \mathbb{C} \mid |z| \le 1\}$ which are analytic in the interior, where the restriction map to $C(\mathbb{T})$ with $\mathbb{T} = \partial\mathbb{D}$ is injective, and the crossed product $A \times_\alpha \mathbb{T}$ is isomorphic to the compact operators on $L^2(\mathbb{T})$.

Our result for non-periodic flows is completely parallel to their result (and also to the result [10] in the von Neumann algebra case): When α is a *faithful* action of $\mathbb{R}$ on a C^*-algebra A which is α-simple, the maximality of $A^\alpha[0,\infty)$ is equivalent to the simplicity of $A \times_\alpha \mathbb{R}$, which is again equivalent to the fullness of the strong Connes spectrum of α (In this case $\mathrm{Sp}(\alpha)$ automatically contains at least five points; so no additional assumption on $\mathrm{Sp}(\alpha)$ is necessary). In the proof we will use the definition of strong Connes spectrum $\tilde{\mathbb{R}}(\alpha)$, which is in general a closed subsemigroup of $\mathbb{R}$ containing 0: λ belongs to $\tilde{\mathbb{R}}(\alpha)$ if for any open interval J containing λ and any non-zero α-invariant hereditary C^*-subalgebra D the support projection of $D^\alpha(J)$ is 1_D, i.e., the hereditary C^*-subalgebra generated by $D^\alpha(J)^*D^\alpha(J)$ is D.

There are many examples of (A,α) which give simple crossed products. For example if α is a flow on the Cuntz algebra $\mathcal{O}_n$ generated by n isometries $s_1,\ldots,s_n$ such that $\alpha_t(s_i) = e^{ip_it}s_i$ for some $p_1,\ldots,p_n \in \mathbb{R}$ and if $p_1,\ldots,p_n,-p_i$ generate $\mathbb{R}$ as a closed subsemigroup for all $i = 1,\ldots,n$, then $\mathcal{O}_n \times_\alpha \mathbb{R}$ is simple [4]. Hence in this case $\mathcal{O}_n^\alpha[0,\infty)$ is maximal.

In the next section we will consider the maximality of $A^\alpha[0,\infty)$ without α-simplicity; the result we present can essentially be read from [9] and reduces the problem to the α-simple case; If $A^\alpha[0,\infty)$ is maximal with α faithful, then A has an α-simple ideal I such that $I \times_\alpha \mathbb{R}$ is simple and α induces a trivial flow on A/I (see 2.1 for details). In Section 3 we treat the α-simple case and prove the result quoted above (see 3.7). In Section 4 we propose the notion of essential maximality for $A^\alpha[0,\infty)$ in case A is not α-simple and $A^\alpha[0,\infty)$ does not include a non-zero ideal of A, i.e., $A^\alpha[0,\infty)$ is called *essentially maximal* if any closed subalgebra B strictly containing $A^\alpha[0,\infty)$ contains a non-zero ideal of A. We point out some relationship of this notion with the strong Connes spectrum. When α satisfies additional conditions including the *no energy gap condition* [7], we shall show that $A^\alpha[0,\infty)$ is essentially maximal if and only if there is an α-invariant essential ideal I of A such that the strong Connes spectrum of $\alpha|I$ is full (see 4.9).

2. General case

If α (resp. β) is a flow on a C^*-algebra A (resp. B) and ϕ is a homomorphism of A into B such that $\phi\alpha_t = \beta_t\phi$, then we have that $\mathrm{Sp}_\beta(\phi(x)) \subset \mathrm{Sp}_\alpha(x)$ for $x \in A$. Hence it follows that $\phi(A^\alpha[0,\infty)) \subset B^\beta[0,\infty)$. We do not know whether the equality holds when ϕ is onto.

Proposition 2.1. *Let α be a non-trivial flow on a C^*-algebra A. Then the following conditions are equivalent.*

(1) $A^\alpha[0,\infty)$ *is maximal.*

(2) *There is a minimal α-invariant closed ideal I of A such that the induced flow on the quotient A/I is trivial, the image of $A^\alpha[0,\infty)$ is dense in A/I under the quotient map from A onto A/I, and $I^{\alpha|I}[0,\infty)$ is maximal.*

Proof. Suppose that $A^\alpha[0,\infty)$ is maximal. If J is a non-zero α-invariant ideal of A such that $\alpha|J$ is non-trivial, then the induced flow $\beta = \dot{\alpha}$ on A/J must be trivial. Otherwise $B = Q^{-1}((A/J)^\beta[0,\infty))$ is a proper closed subalgebra strictly containing $A^\alpha[0,\infty)$ because $B \supset J$, where Q is the quotient map of A onto A/J. In this case we must have $\mathrm{Sp}(\alpha) = \mathrm{Sp}(\alpha|J)$. If (J_i) is a decreasing net of α-invariant ideals of A such that $\alpha|J_i$ is non-trivial, then $\bigcap_i J_i \neq 0$. This is shown as follows. If $\bigcap_i J_i = 0$ and $x \in A$, then $\alpha_t(x) - x \in J_i$ for all i and $t \in \mathbb{R}$, which implies that $\alpha_t(x) = x$ for all $t \in \mathbb{R}$, i.e., α is trivial, a contradiction. Thus we obtain a minimal α-invariant ideal I such that the induced flow on A/I is trivial and $\alpha|I$ is non-trivial. Suppose that $I \neq A$; otherwise there is nothing to prove.

First we assert that if 0 is not isolated in $\mathrm{Sp}(\alpha)$ the support and range projections of $A^\alpha(0,\infty) = I^{\alpha|I}(0,\infty)$ are 1_I (We may simply write α for $\alpha|I$ later). For example if the hereditary C^*-subalgebra generated by $I^\alpha(0,\infty)^* I^\alpha(0,\infty)$ is not I, there is a pure state ϕ of I such that $\phi(I^\alpha(0,\infty)^* I^\alpha(0,\infty)) = 0$. Then ϕ is an α-invariant state called a ceiling state. Regarding ϕ as a state of A let $(\pi_\phi, \mathcal{H}_\phi, \Omega_\phi)$ be the GNS representation triple associated with ϕ, where π_ϕ is an irreducible representation on a Hilbert space $\mathcal{H}_\phi$ and $\Omega_\phi \in \mathcal{H}_\phi$ is a cyclic vector for $\pi_\phi(A)$ such that $\phi(x) = \langle \Omega_\phi, \pi_\phi(x)\Omega_\phi \rangle$, $x \in A$. Define a unitary flow U on $\mathcal{H}_\phi$ by $U_t \pi_\phi(x)\Omega_\phi = \pi_\phi \alpha_t(x)\Omega_\phi$, $x \in A$. Then the self-adjoint generator H of U satisfies that $H \leq 0$ and $H\Omega_\phi = 0$. If E denotes the spectral measure of H choose $a < b < 0$ such that all $E(-\infty, a], E(a,b], E(b,0]$ are non-zero. Then the closure B of $AA^\alpha[-b,\infty) + A^\alpha[0,\infty)$ gives a non-trivial closed subalgebra containing $A^\alpha[0,\infty)$ as $\pi_\phi(B)E(b,0] = E(b,0]\pi_\phi(B)E(b,0]$ and $\pi_\phi(B)E(a,0] \neq E(a,0]\pi_\phi(B)E(a,0]$ and $\pi_\phi(A^\alpha[0,\infty))E(c,0] = E(c,0]\pi_\phi(A^\alpha[0,\infty))E(c,0]$ for any $c \leq 0$. This contradiction shows that if 0 is not isolated in $\mathrm{Sp}(\alpha)$ the support projection of $I^\alpha(0,\infty)$ is 1_I. Similarly one can show the statement for the range projection.

If $I^\alpha[0,\infty)$ is not maximal, then there is a proper closed subalgebra B of I such that B strictly contains $I^\alpha[0,\infty)$. Since B is α-invariant by [9], $\mathrm{Sp}(\alpha|B)$ contains some $p < 0$. For any $\delta \in (0, -p/4)$ there is an $x \in B$ such that $\mathrm{Sp}_\alpha(x) \subset (p-\delta, p+\delta)$. There are $b_1, b_2 \in I^\alpha[0,\delta)$ such that $b_1 x b_2 \neq 0$.

We will show this last claim. Let B_1 be the α-invariant hereditary C^*-subalgebra generated by $\alpha_t(xx^*)$, $t \in \mathbb{R}$. If 0 is isolated in $\mathrm{Sp}(\alpha|B_1)$ there is a non-zero $b_1 \in B_1$ such that $\mathrm{Sp}_\alpha(b_1) = \{0\}$. Since $b_1 \in B_1 \subset I$ and $b_1\alpha_t(x) \neq 0$ for some $t \in \mathbb{R}$, we conclude that b_1 satisfies that $b_1 x \neq 0$. If 0 is not isolated there is a non-zero $c_1 \in B_1$ such that $\mathrm{Sp}_\alpha(c_1) \subset (0,\delta)$. Since $c_1\alpha_t(x) \neq 0$ for some $t \in \mathbb{R}$ we may set $b_1 = \alpha_{-t}(c_1)$ which satisfies $b_1 x \neq 0$. Next we set B_2 to be the α-invariant hereditary C^*-subalgebra generated by $\alpha_t(x^* b_1^* b_1 x)$, $t \in \mathbb{R}$. Then one can construct as above an element $b_2 \in B_2^{\alpha|B_2}[0,\delta) \subset I$ such that $b_1 x b_2 \neq 0$.

Note that $\mathrm{Sp}_\alpha(b_1xb_2) \subset (p-\delta, p+3\delta)$ and $b_1xb_2 \in B \setminus I^{\alpha|I}[0,\infty)$. Let D denote the closed subalgebra generated by b_1xb_2 and $A^\alpha[0,\infty)$. Since

$$A^\alpha[0,\infty)b_1xb_2A^\alpha[0,\infty) \subset I^\alpha[0,\infty)xI^\alpha[0,\infty) \subset B$$

we derive that the subalgebra generated by b_1xb_2 and $A^\alpha[0,\infty)$ is contained in $B + A^\alpha[0,\infty)$. Hence it follows that D is contained in the closure X of $B + A^\alpha[0,\infty)$. We assert that $X \not\supset I$. If $X \supset I$, then for any $x \in I$ with $\mathrm{Sp}_\alpha(x) \subset (-\infty,0)$ there is a sequence $b_n + a_n$ with $b_n \in B$ and $a_n \in A^\alpha[0,\infty)$ such that $b_n + a_n \to x$. But since $\mathrm{Sp}_\alpha(x) \subset (-\infty,0)$ and $\mathrm{Sp}(a_n) \subset [0,\infty)$ it follows that $\alpha_f(b_n) \to x$, where $f \in L^1(\mathbb{R})$ is chosen to be such that $\alpha_f(x) = \int f(t)\alpha_t(x)dt = x$ and $\alpha_f = 0$ on $A^\alpha[0,\infty)$; Hence $x \in B$, or $B \supset I^\alpha(-\infty,0)$. Since $B \supset I^\alpha[0,\infty)$ if 0 is isolated in $\mathrm{Sp}(\alpha)$, then $B \supset I$, a contradiction. Suppose that 0 is not isolated in $\mathrm{Sp}(\alpha)$ and let $x \in I$. Then there is a sequence $b_n + a_n$ with $b_n \in B$ and $\mathrm{Sp}_\alpha(a_n) \subset [0,1/n)$ such that $b_n + a_n \to x$ because $A^\alpha(0,\infty) = I^\alpha(0,\infty) \subset B$. If $z_i \in I$ satisfies $\mathrm{Sp}_\alpha(z_i) \subset (-\infty,0)$ for $i = 1,2,\ldots,k$, we obtain that $z_i^* z_i(a_n + b_n) \in B$ for all large n. Hence $z_i^* z_i x \in B$. Since the elements of the form $(1 + \sum_i z_i^* z_i)^{-1} \sum_i z_i^* z_i$ with k arbitrary gives an approximate identity for I we conclude that $B \supset I$. Thus it follows that $X \not\supset I$; in particular D is a proper subalgebra.

Since D contains $A^\alpha[0,\infty)$ and $b_1xb_2 \notin A^\alpha[0,\infty)$, it follows that D is a closed subalgebra strictly bigger than $A^\alpha[0,\infty)$, which contradicts that $A^\alpha[0,\infty)$ is maximal. Thus one can conclude that $I^\alpha[0,\infty)$ is maximal.

Suppose that the closure B of $Q(A^\alpha[0,\infty))$ is a proper closed subalgebra of A/I. Then $Q^{-1}(B)$ is a proper closed subalgebra strictly containing $A^\alpha[0,\infty)$ since $B \supset I$; this contradiction shows $B = A/I$.

Suppose the second condition holds. If there is a closed subalgebra B of A strictly containing $A^\alpha[0,\infty)$, there is a non-zero $x \in B$ such that $\mathrm{Sp}_\alpha(x) \subset (-\infty,0)$. Since the induced flow on A/I is trivial we derive that $x \in I$. Since $I^\alpha[0,\infty)$ and x generates I as a closed subalgebra it follows that $B \supset I$. Since $B \supset A^\alpha[0,\infty)$, $Q(A^\alpha[0,\infty))$ is dense in A/I, and $Q(B)$ is closed as being the quotient of B by $I \subset B$, one can conclude that $B = A$. This contradiction shows that $A^\alpha[0,\infty)$ is maximal. □

3. α-simple case

When α is a flow on a C^*-algebra A we construct another C^*-algebra called the crossed product $A \times_\alpha \mathbb{R}$ of A by α. We recall the following result: $A \times_\alpha \mathbb{R}$ is simple if and only if A is α-simple and the strong Connes spectrum $\tilde{\mathbb{R}}(\alpha)$ is equal to $\mathbb{R}$ [4].

Lemma 3.1. *Let α be a flow on a C^*-algebra A such that $A \times_\alpha \mathbb{R}$ is simple. Let D be a non-zero α-invariant hereditary C^*-subalgebra of A. Then for any non-zero open interval I the closed linear span of $A^\alpha(I)^* D A^\alpha(I)$ is A.*

Proof. Let $I = (p - \delta, p + \delta)$ for some $p \in \mathbb{R}$ and $\delta > 0$. We choose a sequence (p_i) in $\mathbb{R}$ such that $\bigcup_i (p_i - \delta/2, p_i + \delta/2) = \mathbb{R}$. Since A is α-simple, i.e., the α-invariant ideals are $\{0\}$ and A only, it follows that the hereditary C^*-subalgebra generated by all $A^\alpha(I_i)^* D A^\alpha(I_i)$ with $I_i = (p_i - \delta/2, p_i + \delta/2)$ for $i = 1, 2, \ldots$ is A. Since $\tilde{\mathbb{R}}(\alpha) = \mathbb{R}$, the hereditary C^*-subalgebra generated by $D^\alpha(-I_i + p)^* D^\alpha(-I_i + p)$ is D. Since $D^\alpha(-I_i + p) A^\alpha(I_i) \subset A^\alpha(I) \cap DA$ which is in the closure of $DA^\alpha(I)$, it follows by replacing D in $A^\alpha(I_i)^* D A^\alpha(I_i)$ by $D^\alpha(-I_i + p)^* D^\alpha(-I_i + p)$ that the hereditary C^*-subalgebra generated by $A^\alpha(I)^* D A^\alpha(I)$ is A, which is the same as the closed linear span of $A^\alpha(I)^* D A^\alpha(I)$. □

Proposition 3.2. *Let α be a flow on a C^*-algebra A such that $A \times_\alpha \mathbb{R}$ is simple. Then $A^\alpha[0, \infty)$ is maximal.*

Proof. Suppose that there is a proper closed subalgebra B of A strictly containing $A^\alpha[0, \infty)$. Then B is α-invariant [9] and $\mathrm{Sp}(\alpha|B)$ is strictly bigger than $[0, \infty)$. Let $p \in \mathrm{Sp}(\alpha|B) \setminus [0, \infty)$. For $\delta > 0$ we choose $x \in B$ such that $\mathrm{Sp}_\alpha(x) \subset (p - \delta, p + \delta)$ and set D to be the α-invariant hereditary C^*-subalgebra generated by $\alpha_t(x^* x)$, $t \in \mathbb{R}$, which is the same as the hereditary C^*-subalgebra generated by $\alpha_t(x)^* A^\alpha(0, \delta)^* A^\alpha(0, \delta) \alpha_t(x)$, $t \in \mathbb{R}$. By the previous lemma the hereditary C^*-subalgebra generated by $A^\alpha(0, \delta)^* D A^\alpha(0, \delta)$ is A; that is, the support projection of the family $A^\alpha(0, \delta) \alpha_t(x) A^\alpha(0, \delta)$ with $t \in \mathbb{R}$ is 1. In the same way we can conclude that the range projection of the family $A^\alpha(0, \delta) \alpha_t(x) A^\alpha(0, \delta)$, $t \in \mathbb{R}$ is 1. Since $A^\alpha(0, \delta) \alpha_t(x) A^\alpha(0, \delta) \subset B^\alpha(p - \delta, p + 3\delta)$ and δ is arbitrary, we conclude that the range and support projections of $B^\alpha(p - \epsilon, p + \epsilon)$ are 1 for any $\epsilon > 0$. Then in particular we derive that $\mathrm{Sp}(\alpha|B) \ni np$ for any $n = 1, 2, \ldots$.

Let $x_i \in B^\alpha(p - \epsilon, p + \epsilon)$ for $i = 1, 2, \ldots, n$ with n arbitrary and $p \in \mathrm{Sp}(\alpha|B) \setminus [0, \infty)$. Then the elements of the form $(1 + \sum_i x_i x_i^*)^{-1} (\sum_i x_i x_i^*) \in B$ constitutes an approximate identity for A (see 1.4 of [8]).

Let $y \in A$ be such that $\mathrm{Sp}_\alpha(y)$ is compact. Then there is a $p \in \mathrm{Sp}(\alpha|B)$ such that $-p + \mathrm{Sp}_\alpha(y) \subset (\delta, \infty)$ for some $\delta > 0$. Then for any $\epsilon > 0$ there are $x_i \in B^\alpha(p - \delta, p + \delta)$ for $i = 1, 2, \ldots, n$ such that $\|zy - y\| < \epsilon$ for $z = (1 + \sum_i x_i x_i^*)^{-1} (\sum_i x_i x_i^*)$. Since $(\sum_i x_i x_i^*) y = \sum_i x_i (x_i^* y) \in B$ it follows that $zy \in B$. Since $\epsilon > 0$ is arbitrary it follows that $y \in B$. Hence we conclude that $B = A$, which shows that $A^\alpha[0, \infty)$ is maximal. □

Lemma 3.3. *Let α be a flow on a C^*-algebra A such that A is α-simple and $\mathrm{Sp}(\alpha)$ contains more than three points. If $A^\alpha[0, \infty)$ is maximal, then the support and range projections of $A^\alpha(p, q)$ are 1 for all open intervals (p, q) with $0 < p < q \le \infty$ and $\mathrm{Sp}(\alpha) \cap (p, q) \neq \emptyset$. In particular $\mathrm{Sp}(\alpha)$ is a group.*

Proof. Suppose that the hereditary C^*-subalgebra S_p generated by

$$A^\alpha(p, \infty)^* A^\alpha(p, \infty)$$

is not A for some $p \geq 0$. Let S be the set of positive functionals f on A satisfying $f(S_p) = 0$ and $\|f\| \leq 1$. Since S is a closed face of the compact unit ball of positive functionals on A, it has a non-zero extreme point which is a pure state ϕ. Since the α-spectrum of ϕ is contained in $[-p, p]$, ϕ is α-covariant. The GNS representation π_ϕ associated with ϕ is a faithful irreducible representation and there is a unitary flow U on the representation space $\mathcal{H}_\phi$ such that $U_t\pi_\phi(x)U_t^* = \pi_\phi\alpha_t(x)$, $x \in A$. If Ω_ϕ is the associated unit vector, i.e., Ω_ϕ is a cyclic vector satisfying $\phi(x) = \langle \Omega_\phi, \pi_\phi(x)\Omega_\phi \rangle$, $x \in A$, then $\pi_\phi(A^\alpha(p, \infty))\Omega_\phi = 0$. Since the U-spectrum of Ω_ϕ is compact we may assume by adjusting U by a character on $\mathbb{R}$ that the spectrum of U is contained in $(-\infty, 0]$ including 0, i.e., if H is the self-adjoint operator defined by $U_t = e^{itH}$, then $H \leq 0$ with 0 in the spectrum of H. Let E denote the spectral measure of H. By the assumption on $\mathrm{Sp}(\alpha)$ the support of E has more than two points in addition to 0. We choose $a < b < 0$ such that all $E(-\infty, a], E(a, b], E(b, 0]$ are non-zero.

Let B be the closed linear span of $AA^\alpha[-b, \infty) + A^\alpha[0, \infty)$. Then B is a closed subalgebra containing $A^\alpha[0, \infty)$. Since $\pi_\phi(A^\alpha[-b, \infty))E(b, 0] = 0$ and $\pi_\phi(A^\alpha[0, \infty))E(b, 0] = E(b, 0]\pi_\phi(A^\alpha[0, \infty))E(b, 0]$, it follows that

$$\pi_\phi(B)E(b, 0] = E(b, 0]\pi_\phi(B)E(b, 0].$$

Since $E(b, 0] \neq 0, 1$ and $\pi_\phi(A)$ is irreducible we conclude that $B \neq A$. Note that $E(a, 0] \neq 0, 1, E(b, 0]$ and that

$$\pi_\phi(A^\alpha[0, \infty))E(a, 0] = E(a, 0]\pi_\phi(A^\alpha[0, \infty))E(a, 0].$$

But since the range projection of $\pi_\phi(B)E(a, 0]$, which dominates the range projection of $\pi_\phi(A)E(a - b, 0]$, is 1, we derive that

$$\pi_\phi(B)E(a, 0] \neq E(a, 0]\pi_\phi(B)E(a, 0].$$

Thus B is bigger than $A^\alpha[0, \infty)$, which contradicts the maximality of $A^\alpha[0, \infty)$. Thus we conclude that the support projection of $A^\alpha(p, \infty)$ is 1 for all $p \geq 0$.

Suppose that the hereditary C^*-subalgebra S_r generated by

$$A^\alpha(p, p + r)^* A^\alpha(p, p + r)$$

is non-zero and not equal to A for some $r > p \geq 0$. Suppose that $\mathrm{Sp}(\alpha) \cap (0, r) \neq \emptyset$ and let $q \in \mathrm{Sp}(\alpha) \cap (0, r)$. Let $\delta \in (0, (r - q)/4)$ and let $x \in A$ be such that $\mathrm{Sp}_\alpha(x) \subset (q - \delta, q + \delta)$. Let V denote the closed linear span of $AS_r + A^\alpha(p + q + \delta, \infty)$ and define $B = \{x \in A \mid xV \subset V\}$. Note that B is a closed subalgebra and $B \supset A^\alpha[0, \infty)$.

Note that $A^\alpha(p+q+\delta, p+q+r-\delta)+A^\alpha(p+q+r-2\delta, \infty) = A^\alpha(p+r+\delta, \infty)$. Since $x^* A^\alpha(p+q+\delta, p+q+r-\delta) \subset A^\alpha(p, p+r)$ and $x^* A^\alpha(p+q+r-2\delta, \infty) \subset A^\alpha(p + r - 3\delta, \infty) \subset A^\alpha(p + q + \delta, \infty)$, it follows that $x^*V \subset V$ and so $x^* \in B$. Thus B strictly contains $A^\alpha[0, \infty)$. On the other hand there is an $s > p + q + \delta$ such that the hereditary C^*-subalgebra generated by $A^\alpha(s, s + p + q)^* A^\alpha(s, s + p + q)$ is not contained in S_r. Let $y \in A^\alpha(s, s + p + q)$ be such that $y^*y \notin S_r$. Then $y^*y \in A^\alpha(-p - q, p + q)$. Since $y \in V$ and $y^*y \notin V$ we derive that $y^* \notin B$. This contradicts the maximality of $A^\alpha[0, \infty)$. Hence if $\mathrm{Sp}(\alpha) \cap (p, p + r) \neq \emptyset$

and $\mathrm{Sp}(\alpha) \cap (0, r) \neq \emptyset$, then it follows that $S_r = A$. If 0 is not isolated in $\mathrm{Sp}(\alpha)$ this implies that whenever $\mathrm{Sp}(\alpha) \cap (p, p+r) \neq \emptyset$ the support projection of $A^\alpha(p, p+r)$ is 1.

Suppose that 0 is isolated in $\mathrm{Sp}(\alpha)$ and let $q = \min \mathrm{Sp}(\alpha) \cap (0, \infty) > 0$. Then by the reasoning above we derive that the support projection of $A^\alpha(0, q+\epsilon)$ is 1 for any $\epsilon > 0$. Since $A^\alpha(0, q+\epsilon) = A^\alpha[q, q+\epsilon)$, it follows that the support projection of $A^\alpha[q, q+\epsilon)$ is 1. If $\mathrm{Sp}(\alpha) \cap (q, 2q)$ is not empty, let $s \in \mathrm{Sp}(\alpha) \cap (q, 2q)$. For a sufficiently small $\epsilon > 0$ let $x \in A$ be such that $\mathrm{Sp}_\alpha(x) \subset (s-\epsilon, s+\epsilon)$. Then as the support projection of $A^\alpha[q, q+\epsilon)$ is 1 there is a $y \in A^\alpha[q, q+\epsilon)$ such that $xy^* \neq 0$. This is a contradiction because $\mathrm{Sp}_\alpha(xy^*) \subset (s-q-2\epsilon, s-q+\epsilon)$ which would imply that $\mathrm{Sp}(\alpha) \cap (0, q) \neq \emptyset$ as $0 < s-q < q$. Hence we deduce that $\mathrm{Sp}(\alpha) \cap (q, 2q) = \emptyset$. Since the support projection of $A^\alpha[2q, 2q+\epsilon) \supset A^\alpha[q, q+\epsilon/2)A^\alpha[q, q+\epsilon/2)$ is 1 for any $\epsilon > 0$ we can repeat this argument to obtain $\mathrm{Sp}(\alpha) \cap (2q, 3q) = \emptyset$. Thus we conclude by induction that $\mathrm{Sp}(\alpha) = q\mathbb{Z}$ and that the support projection of $A^\alpha(\{nq\})$ is 1 for all $n = 1, 2, \ldots$.

Suppose that the hereditary C^*-subalgebra R_p generated by

$$A^\alpha(p, \infty)A^\alpha(p, \infty)^*$$

is not A for some $p \geq 0$. Then as above there is a pure state ϕ such that $\phi(R_p) = 0$. We have a unitary flow U on $\mathcal{H}_\phi$ such that $U_t\pi_\phi(x)U_t^* = \pi_\phi\alpha_t(x)$, $x \in A$ and $H \geq 0$ and $\mathrm{Sp}(H) \ni 0$ for the self-adjoint operator H satisfying $U_t = e^{itH}$. We choose $0 < a < b$ such that $E[0, a), E[a, b), E[b, \infty)$ are all non-zero with E the spectral measure of H, and set B to be the closed linear span of $A^\alpha[a, \infty)A + A^\alpha[0, \infty)$. Then it follows that B is a closed subalgebra containing $A^\alpha[0, \infty)$ and $E[0, a)\pi_\phi(B) = E[0, a)\pi_\phi(B)E[0, a)$. Then we will reach a contradiction as before, which shows that $R_p = A$. We omit similar arguments for the range projections of non-zero $A^\alpha(p, q)$. □

If $\mathrm{Sp}(\alpha)$ contains more than three points, then $\mathrm{Sp}(\alpha)$ is a group; so either $\mathrm{Sp}(\alpha) = \lambda\mathbb{Z}$ for some $\lambda > 0$ or $\mathrm{Sp}(\alpha) = \mathbb{R}$. In the former case α is periodic. Since the periodic case is treated by Peligrad and Zsidó, we will concentrate on the case $\mathrm{Sp}(\alpha) = \mathbb{R}$ in the next lemma.

Lemma 3.4. *Let α be a flow on a C^*-algebra A such that A is α-simple, $A^\alpha[0, \infty)$ is maximal, and $\mathrm{Sp}(\alpha) = \mathbb{R}$. If D is a non-zero α-invariant hereditary C^*-subalgebra of A, then $A^\alpha(p, q)^*DA^\alpha(p, q)$ generates A as a hereditary C^*-subalgebra for all open intervals (p, q).*

Proof. First note that for any open interval (p, q) the support and range projections of $A^\alpha(p, q)$ are 1. This follows from the previous lemma if $0 \leq p < q$ or $p < q \leq 0$. If $p < 0 < q$, then this follows because the support and range projections of $A^\alpha(0, q)$ are already 1.

Suppose that the hereditary C^*-subalgebra D_1 generated by

$$A^\alpha(p, q)^*DA^\alpha(p, q)$$

is not equal to A for some open interval (p, q). Let V be the closed linear span of $AD_1 + A^\alpha(p, \infty)$ and $B = \{x \in A \mid xV \subset V\}$. Then B is a closed subalgebra containing $A^\alpha[0, \infty)$. Let $\delta = (q-p)/3$ and let d be a non-zero positive element of D such that $\mathrm{Sp}_\alpha(d) \subset (-\delta, \delta)$. Since the range projection of $A^\alpha(\delta, 2\delta - \epsilon)$ is 1, there is an $x \in A$ such that $\mathrm{Sp}_\alpha(x) \subset (\delta, 2\delta - \epsilon)$ for some small $\epsilon > 0$ and $dx \neq 0$. Since $\mathrm{Sp}_\alpha(dx) \subset (0, 3\delta - \epsilon)$ it follows that $x^* dA^\alpha(q-\epsilon, \infty) \subset A^\alpha(p, \infty)$. Combining the fact that $x^* dA^\alpha(p, q)$ is contained in the closed linear span of AD_1, we derive that $x^* dV \subset V$, i.e., $x^* d \in B$. Since $x^* d \notin A^\alpha[0, \infty)$, we conclude that B is bigger than $A^\alpha[0, \infty)$. Let $y \in A^\alpha(p, 2p)$ be such that $y^* y \notin D_1$. Since $\mathrm{Sp}_\alpha(y^* y) \subset (-p, p)$ we derive that $y^* y \notin V$. Since $y \in V$ this implies that $y^* \notin B$. This contradicts the maximality of $A^\alpha[0, \infty)$. Hence $D_1 = A$. □

Remark 3.5. In the above lemma if we assume that $\mathrm{Sp}(\alpha)$ is isomorphic to $\mathbb{Z}$ instead of $\mathrm{Sp}(\alpha) = \mathbb{R}$, then the statement goes as follows: If D is a non-zero α-invariant hereditary C^*-subalgebra, then $A^\alpha(\{p\})^* DA^\alpha(\{p\})$ generates A as a hereditary C^*-subalgebra for all $p \in \mathrm{Sp}(\alpha)$. To prove this assume that the hereditary C^*-subalgebra D_1 generated by $A^\alpha(\{p\})^* DA^\alpha(\{p\})$ is not equal to A and then define V to be the closed linear span of $AD_1 + A^\alpha[p, \infty)$. Then $B = \{x \in A \mid xV \subset V\}$ is a proper closed subalgebra strictly containing $A^\alpha[0, \infty)$, which is a contradiction.

Lemma 3.6. *Let α be a flow on a C^*-algebra A such that A is α-simple, $A^\alpha[0, \infty)$ is maximal, and* $\mathrm{Sp}(\alpha)$ *contains more than three points. If D is a non-zero α-invariant hereditary C^*-subalgebra of A and (p, q) is an open interval with* $\mathrm{Sp}(\alpha) \cap (p, q) \neq \emptyset$*, then the hereditary C^*-subalgebra generated by $D^\alpha(p, q)^* D^\alpha(p, q)$ is D. In other words* $\tilde{\mathbb{R}}(\alpha) = \mathrm{Sp}(\alpha)$.

Proof. We know by Lemma 3.3 that $\mathrm{Sp}(\alpha)$ is a closed group.

First we assume that $\mathrm{Sp}(\alpha) = \mathbb{R}$. Let D_1 be the hereditary C^*-subalgebra generated by $D^\alpha(p, q)^* D^\alpha(p, q)$. Let $\delta > 0$. Since the hereditary C^*-subalgebra generated by $A^\alpha(-\delta, \delta)^* D_1 A^\alpha(-\delta, \delta)$ is A by Lemma 3.4, one can conclude that the hereditary C^*-subalgebra generated by

$$D^\alpha(-\delta, \delta)^* A^\alpha(-\delta, \delta)^* D_1 A^\alpha(-\delta, \delta) D^\alpha(-\delta, \delta)$$

is D. This implies that the support projection of $D^\alpha(p, q) A^\alpha(-\delta, \delta) D^\alpha(-\delta, -\delta)$, which is a subset of $D^\alpha(p-2\delta, q+2\delta)$, is 1_D. Since $p < q$ and $\delta > 0$ are arbitrary, this shows that the support projection of $D^\alpha(p - \epsilon, p + \epsilon)$ is 1_D for any $p \in \mathbb{R}$ and $\epsilon > 0$, which implies $\tilde{\mathbb{R}}(\alpha) = \mathbb{R}$.

The case $\mathrm{Sp}(\alpha) \cong \mathbb{Z}$ follows in the same way by using Remark 3.5. □

The following result has an analogous version in the von Neumann algebra case due to Solel (Theorem 3.7 of [10]), where α-simplicity is replaced by α-ergodicity on the center and the strong Connes spectrum $\tilde{\mathbb{R}}(\alpha)$ is replaced by the (original) Connes spectrum.

Theorem 3.7. *Let α be a non-trivial flow on a C^*-algebra A. Suppose that A is α-simple. Then $A^\alpha[0,\infty)$ is maximal if and only if one of the following three conditions holds*:

(1) $\mathrm{Sp}(\alpha)=\{\lambda,0,\lambda\}$ *for some* $\lambda>0$; *in this case A is simple and there is a projection E in the multiplier algebra $M(A)$ such that $\alpha_t=\mathrm{Ad}\,e^{it\lambda E}$; moreover $A^\alpha[0,\infty)=EA+(1-E)A(1-E)$.*

(2) $\mathrm{Sp}(\alpha)=\tilde{\mathbb{R}}(\alpha)=\lambda\mathbb{Z}$ *for some* $\lambda>0$; *in this case α is periodic with period* $1/\lambda$.

(3) $\mathrm{Sp}(\alpha)=\tilde{\mathbb{R}}(\alpha)=\mathbb{R}$.

Proof. The first and second cases were treated in [9] at least under the assumption α is periodic. We will give a slightly more general statement concerning the first case below.

If one of the three conditions is satisfied, then it follows that $A^\alpha[0,\infty)$ is maximal ([9] for the first two cases and Lemma 3.2 for the third).

If $A^\alpha[0,\infty)$ is maximal it follows from 3.3 that either $\mathrm{Sp}(\alpha)=\{\lambda,0,-\lambda\}$ for some $\lambda>0$, $\mathrm{Sp}(\alpha)\cong\mathbb{Z}$, or $\mathrm{Sp}(\alpha)=\mathbb{R}$. Then the first two cases follow from [9] (or 3.8, 3.6) while the third case follow from Lemma 3.6. □

Proposition 3.8. *Let α be a non-trivial universally weakly inner flow on a C^*-algebra A. If $A^\alpha[0,\infty)$ is maximal, then $\mathrm{Sp}(\alpha)=\{\lambda,0,-\lambda\}$ for some $\lambda>0$ and there is a non-zero simple ideal I of A and a projection $E\in M(I)$ such that $\mathrm{Ad}\,e^{it\lambda E}\pi(x)=\pi\alpha_t(x)$ and $[E,\pi(x)]\in I$ for $x\in A$ and α is trivial on $\mathrm{Ker}(\pi)$, where π is the canonical map of A into $M(I)$.*

Proof. Note that all ideals of A are α-invariant. By Proposition 2.1 we have a minimal ideal I of A such that $\alpha|I$ is non-trivial and the induced flow on A/I is trivial, which implies $\mathrm{Sp}(\alpha)=\mathrm{Sp}(\alpha|I)$. Since I is simple and $\alpha|I$ is universally weakly inner, it follows that $\alpha|I$ is inner [5]. Since $I^{\alpha|I}[0,\infty)$ is maximal it follows that $\mathrm{Sp}(\alpha|I)$ consists of only three points, say $\{\lambda,0,-\lambda\}$ for some $\lambda>0$, otherwise we would have $\mathrm{Sp}(\alpha)=\tilde{\mathbb{R}}(\alpha|I)$, which means that $\alpha|I$ is not inner. Thus there is a projection E in the multiplier algebra $M(I)$ such that $\alpha_t|I=\mathrm{Ad}\,e^{it\lambda E}$. Then it follows that $\pi\alpha_t(x)=\mathrm{Ad}\,e^{it\lambda E}\pi(x)$ for $x\in A$. Since the induced flow on the quotient A/I is trivial it follows that α is trivial on $\mathrm{Ker}(\pi)$ and $\mathrm{Ad}\,e^{it\lambda E}\pi(x)-\pi(x)\in I$ for all $x\in A$. The latter condition is simply expressed by $[E,\pi(x)]\in I$ for $x\in A$. □

4. Essential maximality

To conclude this note we comment on the case where A is not α-simple. We might want to mitigate the condition that $A^\alpha[0,\infty)$ is maximal, e.g., if $A^\alpha[0,\infty)$ is a direct sum of maximal subalgebras we might call it essentially maximal. Formally we propose the following definitions.

Given a flow α on A we shall say that $A^\alpha[0,\infty)$ is *essentially non-self-adjoint* if $I\cap A^\alpha[0,\infty)\neq I$ or $\alpha|I$ is non-trivial for any non-zero α-invariant ideal I. For such a flow we say that $A^\alpha[0,\infty)$ is *essentially maximal* if any closed subalgebra B strictly containing $A^\alpha[0,\infty)$ there is a non-zero α-invariant ideal I with $I \subset B$. We obtain the following easy results:

Proposition 4.1. *Let α be a flow on a C^*-algebra A. If $\tilde{\mathbb{R}}(\alpha) = \mathbb{R}$, then $A^\alpha[0,\infty)$ is essentially maximal.*

Proof. Let B be a closed subalgebra strictly containing $A^\alpha[0,\infty)$. Let $p \in \mathrm{Sp}(\alpha|B)$ with $p < 0$ and let x be a non-zero element of B such that $\mathrm{Sp}_\alpha(x) \subset (p-\delta, p+\delta)$ for $\delta = -p/4$. Let I be the ideal generated by $\alpha_t(x)$, $t \in \mathbb{R}$. Then the support and range projections of $A^\alpha(0,\delta)\alpha_t(x)A^\alpha(0,\delta)$ are 1_I as in the proof of Proposition 3.2. Since $A^\alpha(0,\delta)\alpha_t(x)A^\alpha(0,\delta) \subset B^\alpha(p-\delta, p+3\delta) \cap I$, it follows that the support and range projections of $B^\alpha(-5\delta,-\delta) \cap I$ are 1_I, which implies that the support and range projections of $B^\alpha(-5n\delta,-n\delta) \cap I$ are also 1_I for all $n = 1,2,\ldots$. Then for any $n \in \mathbb{N}$ we have an approximate identity for I of elements of the form $z = (1+\sum_i x_i x_i^*)^{-1}(\sum_i x_i x_i^*)$ with $x_i \in B^\alpha(-5n\delta,-n\delta) \cap I$ for $i = 1,\ldots,k$ with k arbitrary. Then as in the proof of Proposition 3.2 one can conclude that $B \supset I$. □

Proposition 4.2. *Let α be a flow on a C^*-algebra A. If $\tilde{\mathbb{R}}(\alpha) = \mathrm{Sp}(\alpha) = \lambda\mathbb{Z}$ for some $\lambda > 0$, then $A^\alpha[0,\infty)$ is essentially maximal.*

Proof. We denote by $A^\alpha(n\lambda)$ the spectral subspace $A^\alpha(\{n\lambda\})$ for all $n \in \mathbb{Z}$. Note that $A^\alpha[0,\infty)$ is the closed linear span of $A^\alpha(n\lambda)$ with $n = 0,1,2,\ldots$.

Let B be a closed subalgebra strictly bigger than $A^\alpha[0,\infty)$. Then there is a negative $p \in \mathbb{Z}$ such that $B^\alpha(p\lambda) \neq \{0\}$. Let x be a non-zero element of $B^\alpha(p\lambda)$. Let I be the ideal generated by x, which is α-invariant. We prove as in the proof of the previous lemma that the range and support projections of $A^\alpha(0)xA^\alpha(0)$ or $B^\alpha(p\lambda)$ are 1_I. Then one can show that $B \supset I$. □

But there would be much more examples. Let α be a flow on a C^*-algebra A such that A is α-simple and $\tilde{\mathbb{R}}(\alpha) = \mathbb{R}$ and define a flow β on $B = C[-1,1] \otimes A$ by $\beta_t(f)(s) = \alpha_{st}(f(s))$. Then $B^\beta[0,\infty)$ is the subalgebra consisting of $f \in B$ such that $f(s) \in A^\alpha[0,\infty)$ for $s > 0$ and $f(s) \in A^\alpha(-\infty,0]$ for $s < 0$ (and $f(0) \in A^\alpha(\{0\})$). One can see that $B^\beta[0,\infty)$ is essentially maximal and that $\tilde{\mathbb{R}}(\beta) = \{0\}$. But if we define an ideal I of B consisting of functions vanishing at 0, then $\tilde{\mathbb{R}}(\beta|I) = \mathbb{R}$. If α satisfies $\tilde{\mathbb{R}}(\alpha) = \mathbb{Z} = \mathrm{Sp}(\alpha)$ instead one can see that $B^\beta[0,\infty)$ is still essentially maximal and that $\tilde{\mathbb{R}}(\beta|J) = \{0\}$ for any non-zero β-invariant ideal J. Note that $\mathrm{Sp}(\beta) = \mathbb{R}$.

Proposition 4.3. *Let α be a flow on a C^*-algebra A such that $A^\alpha[0,\infty)$ is essentially non-self-adjoint and I be an essential α-invariant ideal. Then $A^\alpha[0,\infty)$ is essentially maximal if and only if $I^{\alpha|I}[0,\infty)$ is essentially maximal. In general if $A^\alpha[0,\infty)$ is essentially maximal, then $J^{\alpha|J}[0,\infty)$ is essentially maximal for any non-zero α-invariant ideal J of A.*

Proof. Suppose that $A^\alpha[0,\infty)$ is essentially maximal and let J be a non-zero α-invariant ideal of A. If $J^\alpha[0,\infty)$ is not essentially maximal, then there is a closed subalgebra B of J bigger than $J^\alpha[0,\infty)$ such that B contains no non-zero ideals. Let $y \in B \setminus J^\alpha[0,\infty)$ be a non-zero element of the form b_1xb_2 with $b_1, b_2 \in J^\alpha[0,\infty)$ and $x \in B$ (see the proof of Proposition 2.1 for the existence of such y). Let B_1 denote the closed subalgebra generated by b_1xb_2 and $A^\alpha[0,\infty)$. Since $A^\alpha[0,\infty)b_1xb_2A^\alpha[0,\infty) \subset J^\alpha[0,\infty)xJ^\alpha[0,\infty) \subset B$ it follows that B_1 is contained in the closure of $B + A^\alpha[0,\infty)$. If B_1 contains a non-zero ideal K of A, then it contains the non-zero ideal $P = K \cap J$ (If $K \cap J = 0$, then $K \subset B_1 \cap J^\perp = A^\alpha[0,\infty) \cap J^\perp$, contradicting $A^\alpha[0,\infty)$ is essentially non-self-adjoint). Since B_1 is contained in the closure of $B + A^\alpha[0,\infty)$, there are, for any $x \in P$, sequences (b_n) in B and (a_n) in $A^\alpha[0,\infty)$ such that $b_n + a_n \to x$. For any $y \in P^\alpha[0,\infty) \subset J^\alpha[0,\infty) \subset B$ it follows that $(b_n + a_n)y \in B$ as $b_ny \in B$ and $a_ny \in P^\alpha[0,\infty) \subset B$, which implies that $xy \in B$. Similarly $x_1yx_2 \in B$ for all $x_1, x_2 \in P$. If P_1 denotes the ideal generated by $P^\alpha[0,\infty)$, this implies that $P_1 \subset B$. Since $P_1 \neq 0$, this is a contradiction. Thus we conclude that $J^\alpha[0,\infty)$ is essentially maximal.

It remains to show that if J is essential and $J^\alpha[0,\infty)$ is essentially maximal, then $A^\alpha[0,\infty)$ is essentially maximal. This part is easy. □

Let α be a flow on a C^*-algebra A. We say α satisfies the *no energy gap condition* if for any non-zero α-invariant hereditary C^*-algebra B and for any $\lambda > 0$ the C^*-subalgebra generated by $B^\alpha(-\lambda,\lambda)$ is B (see [7]). We note the following:

Proposition 4.4. *Let α be a flow on a C^*-algebra A. Then the following conditions are equivalent for any $\lambda > 0$:*

(1) *$A^\alpha(-\lambda,\lambda)$ generates A as a closed subalgebra.*
(2) *$A^\alpha(0,\lambda)$ generates $A^\alpha(0,\infty)$ as a closed subalgebra.*
(3) *$A^\alpha[0,\lambda)$ generates $A^\alpha[0,\infty)$ as a closed subalgebra.*

Proof. (1)⇒(2): From the proof of Lemma 2.2 of [7] the linear combinations of monomials $y_1y_2\cdots y_n$ with all $y_i \in A^\alpha(0,\lambda)$ or all $y_i \in A^\alpha(-\lambda,0)$ and $z \in A^\alpha(-\lambda/2,\lambda/2)$ are dense in A. If $x \in A$ is such that $\mathrm{Sp}_\alpha(x)$ is a compact subset of $(0,\infty)$ there is, for any $\epsilon > 0$, a linear combination z_+ (resp. z_-) of monomials of the form $y_1\cdots y_n$ with $y_i \in A^\alpha(0,\lambda)$ (resp. $y_i \in A^\alpha(-\lambda,0)$) and $z_0 \in A^\alpha(-\lambda/2,\lambda/2)$ such that $\|x - z_+ - z_- - z_0\| < \epsilon$. Let f be in $L^1(\mathbb{R})$ such that $\hat{f}$ has compact support in $(0,\infty)$ and $\hat{f} = 1$ on $\mathrm{Sp}_\alpha(x)$. Then $\alpha_f(x) = \int f(t)\alpha_t(x)dt = x$ and $\|x - \alpha_f(z_+) - \alpha_f(z_0)\| < \|f\|_1\epsilon$. Since $\alpha_f(z_0) \in A^\alpha(0,\lambda/2)$ and $\alpha_f(z_+)$ can be approximated by a Riemann sum it follows that the closed subalgebra generated by $A^\alpha(0,\lambda)$ includes x.

(2)⇒(3): Since $A^\alpha[0,\infty)$ is the linear span of $A^\alpha[0,\lambda)$ and $A^\alpha(0,\infty)$ this is obvious.

(3)⇒(1): The closed subalgebra generated by $A^\alpha(-\lambda,\lambda)$ includes $A^\alpha(-\infty,0] \cup A^\alpha[0,\infty)$. Since A is the closed linear span of $A^\alpha(-\infty,0) \cup A^\alpha(-\lambda,\lambda) \cup A^\alpha(0,\infty)$ the conclusion follows. □

The following is Proposition 1.1 of [7].

Lemma 4.5. *Let α be a flow on a C^*-algebra. If $\tilde{\mathbb{R}}(\alpha)=\mathbb{R}$, then α satisfies the no energy gap condition.*

There are more flows with the no energy gap condition. Let (λ_n) be a sequence in $\mathbb{R}$ such that $\lim_n \lambda_n = 0$ and $\sum_n \lambda_n^2 = \infty$ and define a flow on the UHF algebra A of type 2^∞ by

$$\alpha_t = \bigotimes_n \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & e^{it\lambda_n} \end{pmatrix}.$$

Then for any flow β on a C^*-algebra B the flow $t \mapsto \alpha_t \otimes \beta_t$ on $A \otimes B$ satisfies the no energy gap condition. See Proposition 3.1 of [7].

Lemma 4.6. *Let α be a universally weakly inner flow on a C^*-algebra A such that α satisfies the no energy gap condition. Then α is trivial.*

Proof. Suppose that α is a universally weakly inner flow on A. By [6] there is a non-zero α-invariant hereditary C^*-subalgebra B of A such that $\alpha|B$ is uniformly continuous. Suppose that there is an α-invariant hereditary C^*-subalgebra B such that $\alpha|B$ is non-trivial and uniformly continuous. Let π be an irreducible representation of B such that $t \mapsto \pi\alpha_t$ is non-trivial. There is a self-adjoint operator h on $\mathcal{H}_\pi$ such that $h \geq 0$, $\mathrm{Sp}(h) \ni 0$, and $\pi\alpha_t = \mathrm{Ad}\, e^{ith}\pi$. Let $\delta = \|h\|/4 > 0$ and let D be the hereditary C^*-subalgebra of A generated by $B^\alpha(3\delta,\infty)^* B^\alpha(3\delta,\infty) + B^\alpha(3\delta,\infty)B^\alpha(3\delta,\infty)^*$. Then it follows that the C^*-subalgebra D_δ generated by $D^\alpha(-\delta,\delta)$ is not equal to D as $\pi(D_\delta)$ is not irreducible on the closure of $\pi(D)\mathcal{H}_\pi$. This contradicts the no energy gap condition of α.

Hence we conclude that if $\alpha|B$ is uniformly continuous, then $\alpha|B$ is trivial for all α-invariant hereditary C^*-subalgebras B. One can show that there is a maximal α-invariant hereditary C^*-subalgebra B_0 on which α is trivial. If there is an $x \in AB_0$ such that $\mathrm{Sp}_\alpha(x)$ is compact and $\mathrm{Sp}_\alpha(x) \not\ni 0$ and if B_1 denotes the hereditary C^*-subalgebra generated by $\alpha_s(x)B_0\alpha_t(x)^*$, $s,t \in \mathbb{R}$, then $B_0B_1 = 0$ and α is uniformly continuous, and so trivial, on the hereditary C^*-subalgebra generated by B_0 and B_1 as B_0AB_1 is a subset of the closed linear span of $B_0\alpha_t(x)^*$, $t \in \mathbb{R}$. Since this contradicts the maximality of B_0, we must have that α is trivial on AB_0. Hence we obtain that $AB_0A \subset B_0$, i.e., B_0 is an ideal. Then B_0 must be essential because otherwise we could apply the same argument to $B_0^\perp$, contradicting the maximality of B_0. Then for any $x \in A$ and $b \in B_0$ we have that $\alpha_t(x)b = \alpha_t(xb) = xb$, i.e., $\alpha_t(x) = x$. Hence $B_0 = A$. □

Lemma 4.7. *Let α be a flow on a C^*-algebra A. If $A^\alpha(-\lambda,\lambda)$ generates A for any $\lambda > 0$, then either* $\mathrm{Sp}(\alpha) = \{0\}$ *or* $\mathrm{Sp}(\alpha) = \mathbb{R}$.

Proof. If $\mathrm{Sp}(\alpha)$ is bounded, then the previous lemma implies that $\mathrm{Sp}(\alpha) = \{0\}$. If $\mathrm{Sp}(\alpha)$ is unbounded and has a gap, say $\mathrm{Sp}(\alpha) \cap (\mu, \mu+\epsilon) = \emptyset$ for some $\mu \geq 0$ and $\epsilon > 0$, then the closed subalgebra generated by $A^\alpha(-\epsilon,\epsilon)$ is contained in

$A^\alpha[-\mu,\mu]$, which implies that α does not satisfy the no energy gap condition. Hence if $\mathrm{Sp}(\alpha)$ is unbounded, then $\mathrm{Sp}(\alpha)=\mathbb{R}$. □

Proposition 4.8. *Let α be a flow on a C^*-algebra A such that α satisfies the no energy gap condition. Then there is a minimal α-invariant ideal I of A such that the induced flow on the quotient A/I is trivial. Moreover I is the ideal generated by $A^\alpha(0,\infty)$ and the Connes spectrum $\mathbb{R}(\alpha|I)$ of α is full.*

Proof. Let I be the ideal generated by $A^\alpha(0,\infty)$. Let $B\in H^\alpha(I)$. If $\mathrm{Sp}(\alpha|B)=\{0\}$, then $\mathrm{Sp}(\alpha|J)=\{0\}$ for the ideal generated by B. Since $J\subset I$ we have that $0\neq JA^\alpha(0,\infty)\subset J$, which contradicts that $\alpha|J$ is trivial. Hence $\mathrm{Sp}(\alpha|B)=\mathbb{R}$. Thus one can conclude that $\mathbb{R}(\alpha|I)=\mathbb{R}$. If there is an α-invariant ideal P of A such that the induced flow on A/P is trivial, then $P\supset A^\alpha(0,\infty)$, which implies that $P\supset I$. □

In the situation of the above proposition let R_0 denote the set of covariant irreducible representations π of A satisfying $\pi\alpha_t=\pi$, $t\in\mathbb{R}$. Then the above I is also defined by $I=\bigcap_{\pi\in R_0}\mathrm{Ker}(\pi)$.

Proposition 4.9. *Let α be a flow on a separable C^*-algebra A such that $A^\alpha[0,\infty)$ is essentially non-self-adjoint. Suppose that α satisfies the no energy gap condition and that all the α-invariant primitive ideals of A are maximal among the α-invariant ideals. Then $A^\alpha[0,\infty)$ is essentially maximal if and only if there is an essential ideal I of A such that $\tilde{\mathbb{R}}(\alpha|I)=\mathbb{R}$.*

Proof. The *if* part follows from Propositions 4.1 and 4.3.

Suppose that $A^\alpha[0,\infty)$ is essentially maximal. By the previous proposition $A^\alpha(0,\infty)$ generates an ideal I such that the induced flow on A/I is trivial. Since $A^\alpha[0,\infty)$ is essentially non-self-adjoint, I must be essential. Hence we may assume that $A^\alpha(0,\infty)$ generates A.

Let (π_i,U_i) be all covariant irreducible representations of (A,α) such that the kernel of $\pi_i\times U_i$ is not generated by $\mathrm{Ker}(\pi)$ (or equivalently $\mathrm{Ker}(\pi_i\times U_i)$ is not invariant under the dual flow $\hat{\alpha}$). We denote by $\mathcal{H}_i$ the representation space for π_i and by H_i the self-adjoint generator of U_i, i.e., $U_{i,t}=e^{itH_i}$ on $\mathcal{H}_i$. Then by the no energy gap condition the spectrum of $\mathrm{Sp}(H_i|[\pi_i(B)\mathcal{H}_i])$ is connected for all $B\in H^\alpha(A)$, where $H^\alpha(A)$ is the set of non-zero α-invariant hereditary C^*-subalgebras of A and $[\pi_i(B)\mathcal{H}_i]$ is the closed linear span (or the closure) of $\pi_i(B)\mathcal{H}_i$ (This follows from the definition; see page 455 of [7]). Since $\mathrm{Ker}(\pi_i\times U_i)$ is not $\hat{\alpha}$-invariant there is no $B\in H^\alpha(A)$ such that $\pi_i\times U_i$ is faithful on $B\times_\alpha\mathbb{R}$. And there is a $B\in H^\alpha(A)$ such that $\pi_i(B)\neq 0$ and $\mathrm{Sp}(H_i|[\pi_i(B)\mathcal{H}_i])\neq\mathbb{R}$. Hence $\mathrm{Sp}(H_i|[\pi_i(B)\mathcal{H}_i])=[p,\infty)$ or $(-\infty,p]$ (If $\mathrm{Sp}(H_i|[\pi_i(B)\mathcal{H}_i])$ is bounded, it must be a singleton, which is excluded from the beginning). If $\mathrm{Sp}(H_i|[\pi_i(B)\mathcal{H}_i])=[p,\infty)$ for some $B\in H^\alpha(A)$, then $\mathrm{Sp}(H_i|[\pi_i(D_\lambda)\mathcal{H}_i])$ is of the form $[q,\infty)$ with $q\in\mathbb{R}$ for the hereditary C^*-subalgebra D_λ generated by $A^\alpha(-\lambda,\lambda)BA^\alpha(-\lambda,\lambda)$ for any $\lambda>0$. In this case we call U_i *almost bounded below.* The same is true for the case $(-\infty,p]$, which will be called *almost bounded above.*

We denote by R_b (resp. R_a) the set of covariant irreducible representations (π, U) with U almost bounded below (resp. almost bounded above). We will also regard R_b etc. as a set of irreducible representations of A.

Let $J = \bigcap_{\pi\in R_a} \mathrm{Ker}(\pi)$. We shall show that J is an essential ideal. Suppose that J is not essential and let $I = J^\perp = \{x \in A \mid xJ = 0\}$. We regard $\pi \in R_a$ as a representation of I.

Since I is separable there is a sequence (π_i, U_i) in R_a such that $\bigcap_i \mathrm{Ker}(\pi_i) = \{0\}$. The closure of $\bigcup_{p\in\mathbb{R}} \hat{\alpha}_p(\mathrm{Ker}(\pi_i \times U_i))$ is given as $I(\pi_i) \times_\alpha \mathbb{R}$ for some α-invariant ideal $I(\pi_i)$ of I. Since $I(\pi_i)$ is bigger than the α-invariant primitive ideal $\mathrm{Ker}(\pi_i)$ of I the assumption implies that $I(\pi_i) = I$ (When we regard π_i as a representation of A, $\bigcup_p \hat{\alpha}_p(\mathrm{Ker}(\pi_i \times U_i))$ is dense in $A \times_\alpha \mathbb{R}$; take the intersection with $I \times_\alpha \mathbb{R}$). Let (x_i) be a dense sequence in the unit sphere of $I \times_\alpha \mathbb{R}$. Since $(\pi_i \times U_i)\hat{\alpha}_p = \pi_i \times \chi_p U_i$ where χ_p is the character of $\mathbb{R}$ defined by $\chi_p(t) = e^{ipt}$ and $\mathrm{Ker}(\pi_i \times \chi_p U_i)$ is increasing as p is decreasing, we replace U_i by $\chi_p U_i$ with some $p \in \mathbb{R}$ such that $\|(\pi_i \times U_i)(x_j)\| < 1/i$ for $j = 1, 2, \ldots, i$. Then it follows that $\sup_{i\geq j} \|(\pi_i \times U_i)(x_j)\| < 1/j$ for all j.

Let $\rho = \bigoplus_i \pi_i$ and $V = \bigoplus_i U_i$ on the representation space $\mathcal{H} = \bigoplus \mathcal{H}_i$. Since $\inf_p \|(\rho \times \chi_p V)(x_j)\| < 1/j$, we obtain that $\bigcup_p \hat{\alpha}_p(\mathrm{Ker}(\rho \times V))$ is dense in $I \times_\alpha \mathbb{R}$. Hence there is a $B \in H^\alpha(I)$ such that $\mathrm{Sp}(H|[\rho(B)\mathcal{H}]) = (-\infty, p]$ for some $p \in \mathbb{R}$, where H is the self-adjoint generator of V. We choose a maximal family $\{B_k\}$ in $H^\alpha(I)$ such that $\mathrm{Sp}(H|[\rho(B_k)\mathcal{H}]) = (-\infty, p_k]$ for some $p_k \in \mathbb{R}$ and $B_k I B_\ell = 0$ for $k \neq \ell$. Let B be the closed linear span of all B_k, which is in $H^\alpha(I)$ and generates an essential ideal I_0 of I.

Let E be the closure of $IBI^\alpha[1,\infty) + I^\alpha[0,\infty)$, which is a closed subalgebra of I containing $I^\alpha[0,\infty)$. Suppose that E contains a non-zero ideal P of I. Since $P \cap B \neq 0$, there is B_k such that $P \cap B_k \neq 0$. Since $\rho(P \cap B_k) \neq 0$, there is (π_i, U_i) such that $\pi_i(P \cap B_k) \neq 0$. Then there is a $p \in \mathbb{R}$ such that $\mathrm{Sp}(H_i|[\pi_i(B_k)\mathcal{H}_i]) = (-\infty, p]$, where H_i is the self-adjoint generator of U_i. Let ξ be a unit vector in $F_i(p-1, p]\mathcal{H}_\pi$, where F_i is the spectral measure of H_i. Then it follows that $\pi_i(IBI^\alpha[1,\infty))\xi = \pi_i(IB_kI^\alpha[1,\infty))\xi = 0$, which implies that $\pi_i(E)\xi = \pi_i(I^\alpha[0,\infty))\xi \in F_i(p-1,\infty)\mathcal{H}_i$. Since $\pi_i(P)$ is irreducible it follows that $\pi_i(E) \not\supseteq \pi_i(P)$. This contradiction shows that E does not contain a non-zero ideal.

For each B_k there is (π_i, U_i) such that $\pi_i(B_k) \neq 0$. Note that

$$\mathrm{Sp}(H_U|[\pi_i(B_k)\mathcal{H}_i]) = (-\infty, p]$$

for some $p \in \mathbb{R}$ and $\pi_i(I^\alpha(-2,-1]B_k) \neq 0$. Let $\xi \in F_i(p-\lambda, p]\mathcal{H}_i \cap [\pi_i(B_k)\mathcal{H}_i]$ and $x \in I^\alpha[1,2)$ be such that $\pi_i(x)^*\xi \neq 0$ for some $\lambda > 0$. Then

$$\pi_i(IBI^\alpha[1,\infty))\pi_i(x)^*\xi \supset \pi_i(IB_k)\pi_i(xx^*)\xi$$

is the whole space $\mathcal{H}_i$ while $\pi_i(I^\alpha[0,\infty))\pi_i(x^*)\xi \subset F_i(p-\lambda-2,\infty)\mathcal{H}_i$. Thus we can conclude that $E \supsetneqq I^\alpha[0,\infty)$. Hence $I^\alpha[0,\infty)$ is not essentially maximal, which is a contradiction. Thus $J = \bigcap_{\pi\in R_a} \mathrm{Ker}(\pi)$ must be essential.

In a similar way one concludes that $\bigcap_{\pi\in R_b}\mathrm{Ker}(\pi)$ is essential. Thus

$$J=\bigcap_{\pi\in R_a\cup R_b}\mathrm{Ker}(\pi)$$

is an essential ideal of A and we obtain the following property for $(J,\alpha|J)$: $\mathrm{Ker}(\pi\times U)$ is $\hat{\alpha}$-invariant or equivalently is generated by $\mathrm{Ker}(\pi)$ for any covariant irreducible representation (π,U) of $(J,\alpha|J)$. We will assert that $\tilde{\mathbb{R}}(\alpha|J)=\mathbb{R}$.

Suppose that $\tilde{\mathbb{R}}(\alpha|J)\neq\mathbb{R}$. Then there is a primitive ideal P of $J\times_\alpha\mathbb{R}$ such that P is not $\hat{\alpha}$-invariant, by the characterization of $\tilde{\mathbb{R}}(\alpha)$ in terms of the behavior of $\hat{\alpha}$ on the ideals [4]. Let ρ be an irreducible representation of $J\times_\alpha\mathbb{R}$ such that $\mathrm{Ker}(\rho)=P$. We express ρ as $\pi\times U$. Since $\pi(A)'\cap\{U_t\mid t\in\mathbb{R}\}'=\mathbb{C}1$ the flow $\beta_t=\mathrm{Ad}\,U_t$ on the von Neumann algebra $\pi(A)'$ is ergodic, which implies that $\mathrm{Sp}(\beta)$ is a group. Since $\mathrm{Sp}(\beta)$ is closed there are three cases: $\mathrm{Sp}(\beta)=\mathbb{R}$, $\mathrm{Sp}(\beta)=\lambda\mathbb{Z}$ for some $\lambda>0$, and $\mathrm{Sp}(\beta)=\{0\}$.

For each $B\in H^\alpha(A)$ with $\pi(B)\neq 0$ it follows that $\mathrm{Sp}(U|[\pi(B)\mathcal{H}_\pi])+\mathrm{Sp}(\beta)=\mathrm{Sp}(U|\pi(B)\mathcal{H}_\pi])$. If $\mathrm{Sp}(\beta)=\mathbb{R}$, then $\mathrm{Sp}(U|[\pi(B)\mathcal{H}_\pi])=\mathbb{R}$ for all $B\in H^\alpha(A)$ with $\pi(B)\neq 0$, which implies that $P=\mathrm{Ker}(\rho)$ is left invariant under $\hat{\alpha}$. This contradiction shows that the case $\mathrm{Sp}(\beta)=\mathbb{R}$ cannot arise. If $\mathrm{Sp}(\beta)=\lambda\mathbb{Z}$ for some $\lambda>0$, then it follows that $\mathrm{Sp}(U|[\pi(B)\mathcal{H}_\pi)$ is left invariant under the addition of λ for any $B\in H^\alpha(A)$ with $\pi(B)\neq 0$. Since the no energy gap condition implies that $\mathrm{Sp}(U|[\pi(B)\mathcal{H}_\pi)$ is connected this implies that $\mathrm{Sp}(U|[\pi(B)\mathcal{H}_\pi])=\mathbb{R}$. This shows again that P is $\hat{\alpha}$-invariant, a contradiction. Hence we are left with the case $\mathrm{Sp}(\beta)=0$ or π is irreducible. Since the $\pi\times U$ is not faithful, U is almost bounded below or almost bounded above. Hence $(\pi,U)\in R_b\cup R_a$. But then $\pi(J)=0$ by the definition of J, a contradiction. Hence we must have that $\tilde{\mathbb{R}}(\alpha|J)=\mathbb{R}$. □

We imposed a technical assumption on the α-invariant primitive ideals in the above proposition. What we needed was the property that $\bigcap_i I(\pi_i)$ is essential for the ideals $I(\pi_i)$ of I defined through $\bigcup_p\mathrm{Ker}(\pi\times\chi_pU_i)$, which may hold in general.

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