

## CONTRACTION OF HOROSPHERE-CONVEX HYPERSURFACES BY POWERS OF THE MEAN CURVATURE IN THE HYPERBOLIC SPACE

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ABSTRACT. This paper concerns the evolution of a closed hypersurface of the hyperbolic space, convex by horospheres, in direction of its inner unit normal vector, where the speed equals a positive power  $\beta$  of the positive mean curvature. It is shown that the flow exists on a finite maximal interval, convexity by horospheres is preserved and the hypersurfaces shrink down to a single point as the final time is approached.

### 1. Introduction and main result

Let  $M^n$  be a smooth, compact oriented manifold of dimension  $n \geq 2$  without boundary,  $(N^{n+1}, \bar{g})$  be an  $(n+1)$ -dimensional complete Riemannian manifold, and  $X_0 : M^n \rightarrow N^{n+1}$  a smooth immersion. Consider a one-parameter family of smooth immersions:  $X_t : M^n \rightarrow N^{n+1}$ . The hypersurfaces  $M_t = X_t(M^n)$  are said to move by powers of the mean curvature, if  $X_t = X(\cdot, t)$  satisfies the evolution equation

$$(1.1) \quad \begin{cases} \frac{\partial}{\partial t} X(p, t) = -H^\beta(p, t) \cdot \nu(p, t), & p \in M^n, \\ X(\cdot, 0) = X_0(\cdot), \end{cases}$$

where  $\beta > 0$ ,  $\nu(p, t)$  is the outer unit normal to  $M_t$  at  $X(p, t)$  in the tangent space  $TN^{n+1}$ , and  $H(p, t)$  is the trace of the Weingarten map  $\mathcal{W}_{-\nu}(p, t) = -\mathcal{W}_\nu(p, t)$  on the tangent space  $TM^n$  induced by  $X_t$ . Throughout the paper, we will call such a flow  $H^\beta$ -flow.

For  $\beta = 1$ , this flow is the well-known mean curvature flow, Huisken [14] showed that, when  $N^{n+1}$  is the Euclidean space  $\mathbb{R}^{n+1}$ , any closed convex hypersurface  $M_0$  evolving by mean curvature flow contracts to a point in finite

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Received December 28, 2012; Revised July 21, 2013.

2010 *Mathematics Subject Classification.* Primary 53C44, 35K55, 58J35, 35B40.

*Key words and phrases.*  $H^\beta$ -curvature flow, horosphere, convex hypersurface, hyperbolic space.

The research is partially supported by NSFC (No.11171096), RFDP (No.20104208110002), Funds for Disciplines Leaders of Wuhan (No.Z201051730002) and Natural Science Foundation of the Fujian Province, China (No.2013J01030).

time, becoming spherical in shape as the limit is approached. In [15], he extended this result to compact hypersurfaces in general Riemannian manifolds with suitable bounds on curvature. In fact, the speed of the mean curvature flow can be viewed as a symmetric function of the principal curvature with homogeneous degree one, the results of [14] and [15] have been generalized to a class of fully nonlinear parabolic equations of degree one in the Euclidean space (or some Riemannian manifolds), see [1], [2], [8], [9], and [19]. If one considers the flows for which the speed has other positive degrees of homogeneity in the principal curvature it is more difficult to prove corresponding results for the flows. In some case it is known that if the initial hypersurface has an appropriate pinching condition on the principal curvature (unless the case the dimension of the hypersurface is two, see [3], [5] and [23]), then the evolving hypersurfaces converge to a single point (see [4], [8] and [26]).

The present flow (1.1) has been considered by Schulze in [24] when  $N^{n+1}$  is the Euclidean space for  $M_0$  of strictly positive mean curvature hypersurface, he proved that (1.1) has a unique, smooth solution on a finite time interval  $[0, T)$  and  $M_t$  converges to a point as  $t \rightarrow T$  if  $M_0$  is strictly convex for  $0 < \beta < 1$  or  $M_0$  is weakly convex for  $\beta \geq 1$ . Here “weakly convex” and “strictly convex”, resp., are defined as all the eigenvalues of Weingarten map being positive and nonnegative, resp.. But some counterexamples show that in general the evolving hypersurfaces may not become spherical in shape as the limit is approached.

However, the result of [24] does not closely relate to the ambient space, we face the challenges of extending the above result to hypersurface to more general ambient spaces. But not every Riemannian manifold is well suited to deal with the situation analogous to the setting in the Euclidean space. The present paper wants to consider the case that the ambient space is a simply connected Riemannian manifold of constant sectional curvature  $\kappa (< 0)$  whose flow behaves quite differently compared to the Euclidean space to a certain extent.

Set  $a = \sqrt{|\kappa|}$  and  $N_\kappa^{n+1}$  be isometric to the hyperbolic space  $\mathbb{H}_\kappa^{n+1}$  of radius  $1/a$ :

$$\mathbb{H}_\kappa^{n+1} := \{p \in L^{n+2} : \langle p, p \rangle = -\frac{1}{a^2}\}.$$

Here  $(L^{n+2}, \langle \cdot, \cdot \rangle)$  denotes the  $(n+2)$ -dimensional Lorentz-Minkowski space. To consider the flow (1.1) in  $N_\kappa^{n+1}$  is then equivalent to considering the flow (1.1) in  $\mathbb{H}_\kappa^{n+1}$ . Indeed, in order to formulate the main result of this work, it is necessary to provide some definitions as in [6, 7] as follows.

**Definition 1.1.** A horosphere  $\mathcal{H}$  of  $\mathbb{H}_\kappa^{n+1}$  is the limit of a geodesic sphere of  $\mathbb{H}_\kappa^{n+1}$  as its center goes to the infinity along a fixed geodesic ray.

**Definition 1.2.** An horoball  $\mathcal{H}$  is the convex domain whose boundary is a horosphere.

**Definition 1.3.** A hypersurface  $M$  of  $\mathbb{H}_\kappa^{n+1}$  is said to be convex by horospheres ( $h$ -convex for short) if it bounds a domain  $\Omega$  satisfying that for every  $p \in M =$

$\partial\Omega$ , there is a horosphere  $\mathcal{H}$  of  $\mathbb{H}_\kappa^{n+1}$  through  $p$  such that  $\Omega$  is contained in  $\mathcal{H}$  of  $\mathbb{H}_\kappa^{n+1}$  bounded by  $\mathcal{H}$ .

*Remark 1.4.* In fact, Currier in [10] showed that  $h$ -convex immersions of smooth compact hypersurfaces are embedded spheres, and Borisenko and Miquel in [6] showed that horosphere  $\mathcal{H}$  of  $\mathbb{H}_\kappa^{n+1}$  is weakly (strictly)  $h$ -convex if and only if all its principal curvatures are (strictly) bounded from below by  $a$  at each point.

Now our main result which is an analogue of that on the flow (1.1) of convex hypersurface of the Euclidean space in [24] can be stated by the following theorem.

**Theorem 1.5.** *Let  $X_0 : M^n \rightarrow \mathbb{H}_\kappa^{n+1}$  be a smooth immersion with the mean curvature strictly bounded from below by  $na$ , that is  $H(M_0) > na$ . Then there exists a unique, smooth solution to the flow (1.1) on a finite maximal time interval  $[0, T)$  and  $T$  is between  $\frac{1}{\beta+1}(H_{\max}(M_0))^{-(\beta+1)}$  and  $\frac{n}{\beta+1}(H_{\min}(M_0) - na)^{-(\beta+1)}$ . In the case that*

- i)  $M_0$  is strictly  $h$ -convex for  $0 < \beta < 1$ ,
- ii)  $M_0$  is weakly  $h$ -convex for  $\beta \geq 1$ ,

*then the hypersurfaces  $M_t$  are strictly  $h$ -convex for all  $t > 0$  and they contract to a point in  $\mathbb{H}_\kappa^{n+1}$  as  $t$  approaches  $T$ .*

*Remark 1.6.* The hypothesis of the mean curvature on the initial hypersurface  $M_0$  in  $\mathbb{H}_\kappa^{n+1}$  is essential to ensure short-time existence like the Euclidean case in [24], and the  $h$ -convexity we work with for  $M_0$  is the same as that of [7] in order to ensure that the initial hypersurface is sufficiently positively curved to overcome the obstructions from the negative curvature imposed by the ambient spaces.

About techniques used to prove the above theorem, this paper perturbs the second fundamental form by adding a suitable multiple of the induced metric and follows the ideas introduced in the Euclidean case [24]. The organization of the paper is as follows: Section 2 introduces the notation for the paper and summarizes preliminary results employed in the rest of the paper. Section 3 gives the proof of short-time existence and uniqueness of solutions, and derives the induced evolution equations for some geometric quantities and the corresponding turbulent quantities. Using these, Section 4 deduces that solutions of the flow (1.1) remain  $h$ -convex as long as it exists. Section 5 shows the lower and upper bounds on the maximal time, and establishes the long time existence for solutions of the flow (1.1). Section 6 proves that these hypersurfaces shrink down to a single point in  $\mathbb{H}_\kappa^{n+1}$  as the final time is approached.

## 2. Notation and preliminary results

From now on, we use the same notation as in [7, 14, 24] in local coordinates  $\{x^i\}$ ,  $1 \leq i \leq n$ , near  $p \in M^n$  and  $\{y^\alpha\}$ ,  $0 \leq \alpha, \beta \leq n$ , near  $F(p) \in \mathbb{H}_\kappa^{n+1}$ .

Denote all quantities on  $\mathbb{H}_\kappa^{n+1}$  by a bar, for example by  $\bar{g} = \{\bar{g}_{\alpha\beta}\}$  the metric, by  $\bar{g}^{-1} = \{\bar{g}^{\alpha\beta}\}$  the inverse of the metric, by  $\bar{\nabla}$  the covariant derivative, by  $\bar{\Delta}$  the rough Laplacian, and by  $\bar{R} = \{\bar{R}_{\alpha\beta\gamma\delta}\}$  the Riemann curvature tensor. Components are sometimes taken with respect to the tangent vector fields  $\partial_\alpha (= \frac{\partial}{\partial y^\alpha})$  associated with a local coordinate  $\{y^\alpha\}$  and sometimes with respect to a moving orthonormal frame  $e_\alpha$ , where  $\bar{g}(e_\alpha, e_\beta) = \delta_{\alpha\beta}$ . The corresponding geometric quantities on  $M^n$  will be denoted by  $g$  the induced metric, by  $g^{-1}$  the inverse of  $g$ ,  $\nabla$ ,  $\Delta$ ,  $R$ ,  $\partial_i$  and  $e_i$  the covariant derivative, the rough Laplacian, the curvature tensor, the natural frame fields and a moving orthonormal frame field, respectively. Then further important quantities are the second fundamental form  $A(p) = \{h_{ij}\}$  and the Weingarten map  $\mathcal{W} = \{g^{ik}h_{kj}\} = \{h_j^i\}$  as a symmetric operator and a self-adjoint operator respectively. The eigenvalues  $\lambda_1(p) \leq \dots \leq \lambda_n(p)$  of  $\mathcal{W}$  are called the principal curvatures of  $X(M^n)$  at  $X(p)$ . The mean curvature is given by

$$H := \text{tr}_g \mathcal{W} = h_i^i = \sum_{i=1}^n \lambda_i,$$

the squared norm of the second fundamental form by

$$|A|^2 := \text{tr}_g(\mathcal{W}^t \mathcal{W}) = h_j^i h_i^j = h^{ij} h_{ij} = \sum_{i=1}^n \lambda_i^2,$$

and the Gauß-Kronecker curvature by

$$K := \det(\mathcal{W}) = \det\{h_j^i\} = \frac{\det\{h_{ij}\}}{\det\{g_{ij}\}} = \prod_{i=1}^n \lambda_i.$$

More generally, the mixed mean curvatures  $E_r, 1 \leq r \leq n$ , are given by the elementary symmetric functions of the  $\lambda_i$

$$E_r(\lambda) = \sum_{1 \leq i_1 < \dots < i_r \leq n} \lambda_{i_1} \dots \lambda_{i_r} = \frac{1}{r!} \sum_{i_1, \dots, i_r} \lambda_{i_1} \dots \lambda_{i_r} \text{ for } \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n,$$

and their quotients are

$$Q_r(\lambda) = \frac{E_r(\lambda)}{E_{r-1}(\lambda)} \text{ for } \lambda \in \Gamma_{r-1},$$

where  $E_0 \equiv 1$ , and  $E_l \equiv 0$ , if  $r > n$ ,  $\Gamma_r := \{\lambda \in \mathbb{R}^n \mid E_i > 0, i = 1, \dots, r\}$ . Denote the sum of all terms in  $E_r(\lambda)$  not containing the factor  $\lambda_i$  by  $E_{r;i}(\lambda)$ . Then the following identities for  $E_r$  and the properties on the quotients  $Q_r$  were proved by Huisken and Sinestrari in [16].

**Lemma 2.1.** *For any  $r \in \{1, \dots, n\}$ ,  $i \in \{1, \dots, n\}$ , and  $\lambda \in \mathbb{R}^n$ ,*

$$(2.1) \quad \frac{\partial E_{r+1}}{\partial \lambda_i}(\lambda) = E_{r;i}(\lambda),$$

$$(2.2) \quad E_{r+1}(\lambda) = E_{r+1;i}(\lambda) + \lambda_i E_{r;i}(\lambda),$$

$$(2.3) \quad \sum_{i=1}^n E_{r;i}(\lambda) = (n-r)E_r(\lambda),$$

$$(2.4) \quad \sum_{i=1}^n \lambda_i E_{r;i}(\lambda) = (r+1)E_{r+1}(\lambda),$$

$$(2.5) \quad \sum_{i=1}^n \lambda_i^2 E_{r;i}(\lambda) = E_1(\lambda)E_{r+1}(\lambda) - (r+2)E_{r+2}(\lambda).$$

**Lemma 2.2.** i)  $Q_{r+1}$  is concave on  $\Gamma_r$  for  $r \in \{0, \dots, n-1\}$ .  
 ii)  $\frac{\partial Q_r}{\partial \lambda_i}(\lambda) > 0$  on  $\Gamma_r$  for  $i \in \{1, \dots, n-1\}$  and  $r \in \{2, \dots, n-1\}$ .

Consider the functions:

$$s_\kappa(x) = \frac{\sinh(\sqrt{|\kappa|x})}{\sqrt{|\kappa|}} = \frac{\sinh(ax)}{a}, \quad c_\kappa(r) = s'_\kappa(x),$$

$$ta_\kappa(x) = \frac{s_\kappa(x)}{c_\kappa(x)}, \quad co_\kappa(x) = \frac{1}{ta_\kappa(x)}.$$

Denote  $r_p$  the function “distance to  $p$ ” in  $\mathbb{H}_\kappa^{n+1}$  and use the notation  $\partial_{r_p} = \bar{\nabla} r_p$ . and denote the component of  $\partial_{r_p}$  by  $\partial_{r_p}^\top$  tangent to  $M_t$ , which satisfies  $\partial_{r_p} = \nabla(r_p|_{M^n})$ .

Hadamard’s theorem in the hyperbolic space implies that a hypersurface bounds a strictly convex body makes it possible to represent it as a graph over a geodesic sphere. In our case,  $M^n$  is a strictly convex hypersurface in  $\mathbb{H}_\kappa^{n+1}$ , consider geodesic polar coordinates centered at  $p$ . Then the metric takes the following representation:

$$(2.6) \quad \bar{g} = dr^2 + s_\kappa^2(r)\sigma_{ij}du^i du^j,$$

where  $\sigma := \sigma_{ij}du^i du^j$  is the canonical metric of the unit sphere  $\mathbb{S}^n$  in  $T_p\mathbb{H}_\kappa^{n+1}$ . Consider this embedding

$$X_t : \mathbb{S}^n \rightarrow M^n \hookrightarrow \mathbb{H}_\kappa^{n+1}, \quad t \in [0, T].$$

Let  $D$  be the Levi-Civita connection on  $\mathbb{S}^n$ . For each  $t$ , regard  $r_p$  as a function on  $\mathbb{S}^n$ . Then a local coordinate vector field of  $M_t$  has the following representation

$$(2.7) \quad X_{t*}\left(\frac{\partial}{\partial u^i}\right) = D_i r(u)\partial_{r_p} + s_\kappa(r(u))e_i, \quad 1 \leq i \leq n,$$

and the outward unit normal vector of  $M_t$  can be expressed as

$$(2.8) \quad \nu = \frac{1}{|\xi|} \left( s_\kappa(r_t)\partial_{r_p} - \sum_{i=1}^n D_i r_t e_i \right)$$

with

$$(2.9) \quad |\xi| = \sqrt{s_\kappa^2(r_t) + |Dr_t|^2}.$$

After a standard computation, the second fundamental form of  $M_t$  can be expressed as

$$(2.10) \quad h_{ij} = -\frac{1}{|\xi|} \left( s_\kappa(r_t) D_j D_i r_t - s_\kappa^2(r_t) c_\kappa(r_t) \sigma_{ij} - 2c_\kappa(r_t) D_i r_t D_j r_t \right),$$

and the metric  $g_{ij}$  induced from  $\mathbb{H}_\kappa^{n+1}$  is

$$(2.11) \quad g_{ij} = D_i r D_j r + s_\kappa^2(r) \sigma_{ij}.$$

From this, the inverse metric can be expressed as

$$(2.12) \quad g^{ij} = \frac{1}{s_\kappa^2(r)} \left( \sigma^{ij} - \frac{1}{|\xi|^2} D^i r D^j r \right),$$

where  $(\sigma^{ij}) = (\sigma_{ij})^{-1}$  and  $D^i r = \sigma^{ij} D_j r$ . Then equations (2.10) and (2.12) imply that

$$(2.13) \quad H = -\frac{1}{|\xi| s_\kappa(r)} \left( \Delta_{\mathbb{S}^n} r - \frac{1}{|\xi|^2} \nabla_{\mathbb{S}^n}^2 r(Dr, Dr) \right) + \frac{c_\kappa(r)}{|\xi|} \left( n + \frac{|Dr|^2}{|\xi|^2} \right).$$

Using (2.11) and (2.12) the Christoffel symbols have the expression:

$$(2.14) \quad \Gamma_{ij}^k = \frac{1}{s_\kappa^2(r)} \left[ D_i D_j r D_l r + s_\kappa(r) c_\kappa(r) (D_i r \sigma_{lj} + D_j r \sigma_{il} - D_l r \sigma_{ij}) \right] \cdot \left( \sigma^{kl} - \frac{1}{|\xi|^2} D^k r D^l r \right).$$

Finally, define the inner radius  $\rho_-$  by

$$\rho_-(t) = \sup \{ r : B_r(q) \text{ is enclosed by } M_t \text{ for some } q \in \mathbb{H}_\kappa^{n+1} \},$$

where  $B_r(q)$  is the geodesic ball of radius  $r$  with centered at  $q$ . The following well-known result in  $\mathbb{H}_\kappa^{n+1}$  will be applied in the later sections.

**Lemma 2.3.** *Let  $\Omega$  be a compact  $h$ -convex domain,  $o$  the center of an inball of  $\Omega$  (the largest ball enclosed by  $\Omega$ ),  $\rho_-$  its inner radius. Furthermore let  $\tau := \text{ta}_\kappa(\frac{a\rho_-}{2})$ , then*

- i) *the maximal distance  $\max d(o, \partial\Omega)$  between  $o$  and the points in  $\partial\Omega$  satisfies the inequality*

$$\max d(o, \partial\Omega) \leq \rho_- + a \frac{\ln(1 + \sqrt{\tau})}{1 + \tau} < \rho_- + a \ln 2.$$

- ii) *For any interior point  $p$  of  $\Omega$ ,  $\langle \nu, \partial_{r_p} \rangle \geq \text{ata}_\kappa(\text{dist}((p, \partial\Omega)))$ , where  $\text{dist}$  denotes the distance in the ambient space  $\mathbb{H}_\kappa^{n+1}$ .*

*Proof.* See ([6], Theorem 3.1) for the proof. □

### 3. Short time existence and evolution equations

This section first considers short time existence for the initial value problem (1.1).

**Theorem 3.1.** *Let  $X_0 : M^n \rightarrow \mathbb{H}_\kappa^{n+1}$  be a smooth closed immersion with the mean curvature strictly bounded from below by  $\nu$  everywhere. Then there exists a unique smooth solution  $X_t$  of problem (1.1), defined on some time interval  $[0, T)$ , with  $T > 0$ .*

*Proof.* In fact, if  $f$  is any symmetric function of the curvatures  $\lambda_i, i \in \{1, \dots, n\}$ , it is well-known (see e.g. Theorem 3.1 of [17]) that a flow of the form

$$(3.1) \quad \frac{\partial}{\partial t} X(p, t) = -f(p, t) \cdot \nu(p, t)$$

is parabolic on a given hypersurface with the condition  $\frac{\partial f}{\partial \lambda_i} > 0$  for all  $i$  holds everywhere. Then, given any initial immersion  $X_0$  satisfying the parabolicity assumption, standard techniques ensure the local existence and uniqueness of a solution to (1.1) with initial value  $X_0$ . In our case  $f = H^\beta$  and the condition reads

$$(3.2) \quad \frac{\partial H^\beta}{\partial \lambda_i} = \beta H^{\beta-1} \frac{\partial H}{\partial \lambda_i} = \beta H^{\beta-1} > 0,$$

which is satisfied the condition of Theorem 3.1 of [17].  $\square$

The rest of this section is devoted to compute the induced evolution equations of geometric quantities under the flow (1.1). The derivation of some induced evolution equations can follow the Theorem 3.15 in [1], for example, using Simon's identity for the rough Laplacian of the second fundamental form (see [25]), the evolution equation for the second fundamental form of evolving hypersurfaces by  $H^\beta$ -flow in an arbitrary background space can be written as a reaction-diffusion equation:

$$(3.3) \quad \begin{aligned} \partial_t h_{ij} &= \beta H^{\beta-1} \Delta h_{ij} + \beta(\beta-1) H^{\beta-2} \nabla_i H \nabla_j H - (\beta+1) H^\beta h_i^k h_{kj} \\ &+ \beta H^{\beta-1} |A|^2 h_{ij} + (1-\beta) H^\beta \bar{R}_{0i0j} + \beta H^{\beta-1} h_{ij} \bar{R}_{0k0}^k \\ &- \beta H^{\beta-1} h_{jk} \bar{R}_{li}^{kl} - \beta H^{\beta-1} h_{ik} \bar{R}_{lj}^{kl} + 2\beta H^{\beta-1} h_{kl} \bar{R}_{ij}^{kl} \\ &- \beta H^{\beta-1} \bar{\nabla}_j \bar{R}_{0ki}^k - \beta H^{\beta-1} \bar{\nabla}_k \bar{R}_{0ij}^k, \end{aligned}$$

where  $\nu$  is arranged to be  $e_0$ . Also note that in our case where the background space is a hyperbolic space, the ambient space is locally symmetric ( $\bar{\nabla} \bar{R} = 0$ ) and the Riemann curvature tensor takes the form

$$(3.4) \quad \bar{R}_{\alpha\beta\gamma\delta} = -a^2 (\bar{g}_{\alpha\gamma} \bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta} \bar{g}_{\beta\gamma}).$$

Then, compared with the Euclidean case [24], extra terms of following equations involving the second fundamental form are now due to the background curvature.

**Theorem 3.2.** *On any solution  $M_t$  of (1.1) the following hold:*

$$(3.5) \quad \partial_t g_{ij} = -2H^\beta h_{ij},$$

$$(3.6) \quad \partial_t \nu = \beta H^{\beta-1} \nabla H,$$

$$(3.7) \quad \partial_t (d\mu_t) = -H^{\beta+1} d\mu_t,$$

$$(3.8) \quad \begin{aligned} \partial_t h_{ij} &= \beta H^{\beta-1} \Delta h_{ij} + \beta(\beta-1) H^{\beta-2} \nabla_i H \nabla_j H - (\beta+1) H^\beta h_i^k h_{kj} \\ &\quad + \beta \left( |A|^2 + na^2 \right) H^{\beta-1} h_{ij} - a^2 (\beta+1) H^\beta g_{ij}, \end{aligned}$$

$$(3.9) \quad \begin{aligned} \partial_t h_i^j &= \beta H^{\beta-1} \Delta h_i^j + \beta(\beta-1) H^{\beta-2} \nabla_i H \nabla^j H - (\beta-1) H^\beta h_i^k h_k^j \\ &\quad + \beta \left( |A|^2 + na^2 \right) H^{\beta-1} h_i^j - a^2 (\beta+1) H^\beta \delta_i^j, \end{aligned}$$

$$(3.10) \quad \partial_t H = \beta H^{\beta-1} \Delta H + \beta(\beta-1) H^{\beta-2} |\nabla H|^2 + \left( |A|^2 - na^2 \right) H^\beta,$$

$$(3.11) \quad \begin{aligned} \partial_t H^l &= \beta H^{\beta-1} \Delta H^l + l\beta(\beta-l) H^{\beta+l-3} |\nabla H|^2 \\ &\quad + l \left( |A|^2 - na^2 \right) H^{\beta+l-1}, \quad l \in \mathbb{R}. \end{aligned}$$

For the proof of the main theorem, as mentioned in the introduction, it is convenient for us to define some suitable perturbations of the second fundamental form. Define the turbulent second fundamental form

$$(3.12) \quad \tilde{h}_{ij} = h_{ij} - ag_{ij}.$$

Denote  $\tilde{A}$  (resp.  $\tilde{\mathcal{W}}$ ) the matrix whose entries are  $\tilde{h}_{ij}$  (resp.  $\tilde{h}_i^j$ ). Then  $\tilde{\lambda}_i$  given by

$$(3.13) \quad \tilde{\lambda}_i = \lambda_i - a, \quad i \in 1, \dots, n,$$

are the eigenvalues of  $\tilde{\mathcal{W}}$ . Denote the elementary symmetric functions of the  $\tilde{\lambda}_i$  by  $\tilde{E}_r, 1 \leq r \leq n$ . From the definition it follows that

$$\begin{aligned} \tilde{H} &= \text{tr}_g \tilde{\mathcal{W}} = \tilde{E}_1 = \sum_{i=1}^n \tilde{\lambda}_i = H - na, \\ |\tilde{A}|^2 &= \text{tr}_g (\tilde{\mathcal{W}}^t \tilde{\mathcal{W}}) = \sum_{i=1}^n \tilde{\lambda}_i^2 = |A|^2 + na^2 - 2Ha, \end{aligned}$$

and their quotients

$$\tilde{Q}_r(\tilde{\lambda}) = \frac{\tilde{E}_r(\tilde{\lambda})}{\tilde{E}_{r-1}(\tilde{\lambda})} \quad \text{for } \tilde{\lambda} \in \tilde{\Gamma}_{r-1},$$

where  $\tilde{\Gamma}_r := \{\tilde{\lambda} \in \mathbb{R}^n \mid \tilde{E}_i > 0, i = 1, \dots, r\}$ , if  $\tilde{E}_r$  is considered to be a function of  $\tilde{\lambda}$ . It is easy to check that

$$\nabla_k \tilde{h}_{ij} = \nabla_k h_{ij},$$

and therefore the Codazzi equations hold for  $\nabla_k \tilde{h}_{ij}$ .

The following theorem is easily obtained from Theorem 3.2.



**Theorem 3.3.** *On any solution  $M_t$  of (1.1) the following hold:*

$$(3.14) \quad \partial_t \tilde{h}_{ij} = \beta H^{\beta-1} \Delta \tilde{h}_{ij} + \beta(\beta-1) H^{\beta-2} \nabla_i \tilde{H} \nabla_j \tilde{H} - (\beta+1) H^\beta h_i^k \tilde{h}_{kj} \\ + \beta H^{\beta-1} |\tilde{A}|^2 h_{ij} + a(\beta+1) H^\beta \tilde{h}_{ij},$$

$$(3.15) \quad \partial_t \tilde{h}_i^j = \beta H^{\beta-1} \Delta \tilde{h}_i^j + \beta(\beta-1) H^{\beta-2} \nabla_i \tilde{H} \nabla^j \tilde{H} - (\beta-1) H^\beta h_i^k \tilde{h}_k^j \\ + \beta |\tilde{A}|^2 H^{\beta-1} h_i^j + a(\beta+1) H^\beta \tilde{h}_i^j,$$

$$(3.16) \quad \partial_t \tilde{H} = \beta H^{\beta-1} \Delta \tilde{H} + \beta(\beta-1) H^{\beta-2} |\nabla \tilde{H}|^2 + H^\beta |\tilde{A}|^2 + 2a H^\beta \tilde{H},$$

$$(3.17) \quad \partial_t \tilde{H}^l = \beta H^{\beta-1} \Delta \tilde{H}^l + l\beta \left[ (\beta-1) \tilde{H} - (l-1) H \right] \tilde{H}^{l-2} H^{\beta-2} |\nabla H|^2 \\ + l H^\beta \tilde{H}^{l-1} |\tilde{A}|^2 + 2al H^\beta \tilde{H}^l, \quad l \in \mathbb{R}.$$

Furthermore, the quotients  $\tilde{Q}_r(\tilde{\lambda})$  satisfy the following evolution equation which is an extension of ([24] Lemma 2.4) to hypersurfaces of (1.1) in  $\mathbb{H}_\kappa^{n+1}$  :

**Lemma 3.4.** *For  $\beta \geq 1$  let  $X : M^n \times [0, T) \rightarrow \mathbb{H}_\kappa^{n+1}$  be an  $H^\beta$ -flow with*

$$\tilde{E}_{r-1}(p, t) > 0, \quad \tilde{E}_{r+1}(p, t) \geq 0 \quad \text{for all } (p, t) \in M^n \times [0, T).$$

Then

$$(3.18) \quad \partial_t \tilde{Q}_r \geq \beta H^{\beta-1} \Delta \tilde{Q}_r + H^{\beta-1} \left[ \beta |\tilde{A}|^2 - r(\beta-1) H \tilde{Q}_r + 2aH \right] \tilde{Q}_r.$$

*Proof.* As in [24], from the evolving equation (3.15) of  $\tilde{h}_i^j$ , using

$$\partial_t \tilde{Q}_r = \frac{\partial \tilde{Q}_r}{\partial \tilde{h}_i^j} \left( \partial_t \tilde{h}_i^j \right) \quad \text{and} \quad \Delta \tilde{Q}_r = \frac{\partial \tilde{Q}_r}{\partial \tilde{h}_i^j} \Delta \tilde{h}_i^j + \frac{\partial^2 \tilde{Q}_r}{\partial \tilde{h}_i^j \partial \tilde{h}_p^q} \nabla^k \tilde{h}_i^j \nabla_k \tilde{h}_p^q$$

it is easy to calculate the derivative of  $\tilde{Q}_r$ :

$$\partial_t \tilde{Q}_r = \beta H^{\beta-1} \Delta \tilde{Q}_r - \beta H^{\beta-1} \frac{\partial^2 \tilde{Q}_r}{\partial \tilde{h}_i^j \partial \tilde{h}_p^q} \nabla^k \tilde{h}_i^j \nabla_k \tilde{h}_p^q \\ + \beta(\beta-1) H^{\beta-2} \frac{\partial \tilde{Q}_r}{\partial \tilde{h}_i^j} \nabla_i \tilde{H} \nabla^j \tilde{H} - (\beta-1) H^\beta \frac{\partial \tilde{Q}_r}{\partial \tilde{h}_i^j} h_i^k \tilde{h}_k^j \\ + a(\beta+1) H^\beta \frac{\partial \tilde{Q}_r}{\partial \tilde{h}_i^j} \tilde{h}_i^j + \beta H^{\beta-1} |\tilde{A}|^2 \frac{\partial \tilde{Q}_r}{\partial \tilde{h}_i^j} h_i^j.$$

Choosing a frame  $\{e_i\}$  which diagonalises  $\tilde{\mathcal{W}}$ , the fifth and the sixth term appearing here can be simplified using the following simple calculation with the aid of Lemma 2.1:

$$\frac{\partial \tilde{Q}_r}{\partial \tilde{h}_i^j} h_i^k \tilde{h}_k^j = \frac{\partial \tilde{Q}_r}{\partial \tilde{h}_i^j} \left( \tilde{h}_i^k + a\delta_i^k \right) \tilde{h}_k^j \\ = \sum_{i=1}^n \frac{\partial \tilde{Q}_r}{\partial \tilde{\lambda}_i} \tilde{\lambda}_i^2 + a \frac{\partial \tilde{Q}_r}{\partial \tilde{h}_i^j} \tilde{h}_i^j$$

$$\begin{aligned}
 &= \frac{1}{\tilde{E}_{r-1}^2} \left( \tilde{E}_{r-1} \sum_{i=1}^n \tilde{E}_{r-1,i} \tilde{\lambda}_i^2 - \tilde{E}_r \sum_{i=1}^n \tilde{E}_{r-2,i} \tilde{\lambda}_i^2 \right) + a\tilde{Q}_r \\
 &= -(r+1) \frac{\tilde{E}_{r+1}}{\tilde{E}_{r-1}} + r\tilde{Q}_r + a\tilde{Q}_r
 \end{aligned}$$

and

$$\begin{aligned}
 \frac{\partial \tilde{Q}_r}{\partial \tilde{h}_i^j} \tilde{h}_i^j &= \sum_{i=1}^n \frac{\partial \tilde{Q}_r}{\partial \tilde{\lambda}_i} \tilde{\lambda}_i \\
 &= \frac{1}{\tilde{E}_{r-1}^2} \left( \tilde{E}_{r-1} \sum_{i=1}^n \tilde{E}_{r-1,i} \tilde{\lambda}_i - \tilde{E}_r \sum_{i=1}^n \tilde{E}_{r-2,i} \tilde{\lambda}_i \right) \\
 &= \tilde{Q}_r.
 \end{aligned}$$

In view of the Lemma 2.2, the second, the third and the last term in the right hand side (RHS for short) of the evolution equation of  $Q_r$  are positive by monotonicity and concavity of the  $\tilde{Q}_r$ . So the desired inequality can be obtained with the hypotheses.  $\square$

If the hypersurfaces  $M_t$  are strictly  $h$ -convex, consider the inverse  $\tilde{\mathcal{W}}_p^{-1}$  of  $\tilde{\mathcal{W}}_p$  at a given point  $p \in M^n$ , set  $\tilde{\mathcal{W}}_p^{-1} = \{\tilde{b}_i^j\}$ , where  $\tilde{b}_i^j$  is given by  $\tilde{b}_i^k \tilde{h}_k^j = \delta_i^j$ . The evolution equation of  $\tilde{b}_i^j$  is similar to the Euclidean case:

**Lemma 3.5.** *For  $\beta > 0$ , let  $M_t$  be an  $H^\beta$ -flow of strictly  $h$ -convex hypersurfaces in  $\mathbb{H}_\kappa^{n+1}$ . Then*

$$\begin{aligned}
 \partial_t \tilde{b}_i^j &= \beta H^{\beta-1} \Delta \tilde{b}_i^j - 2\beta H^{\beta-1} \left( \nabla^k \tilde{b}_i^p \right) \tilde{h}_p^q \left( \nabla_k \tilde{b}_q^j \right) \\
 &\quad - \beta(\beta-1) H^{\beta-2} \left( \tilde{b}_i^p \nabla_p \tilde{H} \right) \left( \nabla^q \tilde{H} \tilde{b}_q^j \right) \\
 &\quad + (\beta-1) H^\beta \delta_i^j - \beta H^{\beta-1} |\tilde{A}|^2 \tilde{b}_i^j - 2a H^\beta \tilde{b}_i^j - a\beta H^{\beta-1} |\tilde{A}|^2 \tilde{b}_i^p \tilde{b}_p^j \\
 &\leq \beta H^{\beta-1} \Delta \tilde{b}_i^j + (\beta-1) H^\beta \delta_i^j - \beta H^{\beta-1} |\tilde{A}|^2 \tilde{b}_i^j \\
 &\quad - 2a H^\beta \tilde{b}_i^j - a\beta H^{\beta-1} |\tilde{A}|^2 \tilde{b}_i^p \tilde{b}_p^j.
 \end{aligned}$$

*Proof.* Compute from  $\tilde{b}_i^k \tilde{h}_k^j = \delta_i^j$ ,

$$\partial_t \tilde{b}_i^j = -\tilde{b}_i^p \left( \partial_t \tilde{h}_p^q \right) \tilde{b}_q^j$$

and

$$\nabla_k \tilde{b}_i^j = -\tilde{b}_i^p \left( \nabla_k \tilde{h}_p^q \right) \tilde{b}_q^j$$

which implies

$$\Delta \tilde{b}_i^j = -\tilde{b}_i^p \left( \Delta \tilde{h}_p^q \right) \tilde{b}_q^j + 2\nabla^k \tilde{b}_i^p \tilde{h}_p^q \nabla_k \tilde{b}_q^j.$$

Together with Theorem 3.3, this gives the equality. For  $\beta \geq 1$ , the inequality follows immediately. To show that also for  $0 < \beta < 1$ , the two gradient terms in the RHS of the equality in Lemma 3.5 have the desired sign, we have to work

a bit more. As in ([24], Lemma 2.5), note that  $\tilde{H}^\beta(\tilde{\lambda}) = (\tilde{Q}_n^\beta(\theta))^{-1}$ , where the  $\theta_i$  are the principle radii, i.e.,  $\theta_i = 1/\tilde{\lambda}_i$ . For general functions  $f, g$  satisfying  $f(\tilde{h}_i^j) = 1/g(\tilde{b}_i^j)$  one can compute that

$$(3.19) \quad \frac{\partial^2 f}{\partial \tilde{h}_i^j \partial \tilde{h}_p^q} = \frac{2}{f} \frac{\partial f}{\partial \tilde{h}_i^j} \frac{\partial f}{\partial \tilde{h}_p^q} - f^2 \frac{\partial^2 g}{\partial \tilde{b}_m^n \partial \tilde{b}_k^l} \tilde{b}^{ni} \tilde{b}_{mj} \tilde{b}^{kp} \tilde{b}_{lq} - \frac{\partial f}{\partial \tilde{h}^{jq}} \tilde{b}^{ip} - \frac{\partial f}{\partial \tilde{h}^{ip}} \tilde{b}^{jq}.$$

By the chain rule

$$(3.20) \quad \frac{\partial \tilde{H}^\beta}{\partial \tilde{h}_p^q}(\tilde{\lambda}) = \beta \tilde{H}^{\beta-1}(\tilde{\lambda}) \frac{\partial \tilde{H}}{\partial \tilde{h}_p^q}(\tilde{\lambda}) = \beta \tilde{H}^{\beta-1}(\tilde{\lambda}) \delta_q^p$$

and

$$(3.21) \quad \frac{\partial^2 \tilde{H}^\beta}{\partial \tilde{h}_m^n \partial \tilde{h}_p^q}(\tilde{\lambda}) = \beta(\beta - 1) \tilde{H}^{\beta-2}(\tilde{\lambda}) \delta_n^m \delta_q^p.$$

From (3.19) (with  $f = \tilde{H}^\beta$ ) and (3.20), it follows

$$(3.22) \quad \begin{aligned} \frac{\partial^2 \tilde{H}^\beta}{\partial \tilde{h}_m^n \partial \tilde{h}_p^q} &= 2\beta^2 \tilde{H}^{\beta-2} \delta_n^m \delta_q^p - \tilde{H}^{2\beta} \frac{\partial^2 \tilde{Q}_n^\beta}{\partial \tilde{b}_r^s \partial \tilde{b}_t^u} \tilde{b}^{sp} \tilde{b}_{rq} \tilde{b}^{um} \tilde{b}_{tn} \\ &\quad - \beta \tilde{H}^{\beta-1} \delta_{nq} \tilde{b}^{mp} - \beta \tilde{H}^{\beta-1} \delta^{mp} \tilde{b}_{nq}. \end{aligned}$$

Replacing (3.22) into (3.21), by multiplication with  $\nabla^v \tilde{h}_m^n \nabla_w \tilde{h}_p^q$  and summation

$$\begin{aligned} \beta(\beta - 1) \tilde{H}^{\beta-2} \nabla^v \tilde{H} \nabla_w \tilde{H} &= 2\beta^2 \tilde{H}^{\beta-2} \nabla^v \tilde{H} \nabla_w \tilde{H} - \tilde{H}^{2\beta} \frac{\partial^2 \tilde{Q}_n^\beta}{\partial \tilde{b}_r^s \partial \tilde{b}_t^u} \nabla^v \tilde{b}_r^s \nabla_w \tilde{b}_t^u \\ &\quad - 2\beta \tilde{H}^{\beta-1} \tilde{b}^{mq} \nabla^v \tilde{h}_m^p \nabla_w \tilde{h}_{pq}, \end{aligned}$$

which is

$$\beta(\beta + 1) \nabla^v \tilde{H} \nabla_w \tilde{H} - \tilde{H}^{\beta+2} \frac{\partial^2 \tilde{Q}_n^\beta}{\partial \tilde{b}_r^s \partial \tilde{b}_t^u} \nabla^v \tilde{b}_r^s \nabla_w \tilde{b}_t^u - 2\beta \tilde{H} \tilde{b}^{mq} \nabla^v \tilde{h}_m^p \nabla_w \tilde{h}_{pq} = 0.$$

Using the Codazzi equations for  $\nabla_k \tilde{h}_{ij}$  and the concavity of  $\tilde{Q}_n^\beta(\theta)$  for  $0 < \beta < 1$ , it follows that

$$\beta(\beta + 1) \nabla^v \tilde{H} \nabla_w \tilde{H} - 2\beta \tilde{H} \tilde{b}^{mq} \nabla^v \tilde{h}_m^p \nabla_w \tilde{h}_{pq} \leq 0.$$

Using  $\tilde{H} < H$ , this leads to

$$\beta(\beta + 1) H^{\beta-2} \left( \tilde{b}_i^v \nabla_v \tilde{H} \right) \left( \nabla^w \tilde{H} \tilde{b}_w^j \right) - 2\beta H^{\beta-1} \left( \nabla^k \tilde{b}_i^p \right) \tilde{h}_p^q \left( \nabla_k \tilde{b}_q^j \right) \leq 0.$$

which implies that the sum of the two gradient terms in the RHS of the equality in Lemma 3.5 is non-positive for  $0 < \beta < 1$ . Then this shows the desired inequality for  $0 < \beta < 1$ .  $\square$

### 4. Preserving $h$ -convexity

With the notation of Theorem 1.5, this section shall show that  $h$ -convex hypersurface remains so under the  $H^\beta$ -flow, while in the case  $\beta \geq 1$  they immediately become strictly  $h$ -convex.

As a first step the maximum principle applied to the evolution equation of  $\tilde{H}$  guarantee that the minimum  $H_{\min}$  of  $H$  is increasing under the flow (1.1).

**Proposition 4.1.** *Under the assumptions of Theorem 1.5,*

$$H_{\min}(t) \geq na + (H_{\min}(0) - na) \left( 1 - \frac{\beta + 1}{n} (H_{\min}(0) - na)^{\beta+1} t \right)^{-\frac{1}{\beta+1}}$$

which gives an upper bound on the maximal existence time  $T$ :

$$T \leq \frac{n}{\beta + 1} (H_{\min}(0) - na)^{-(\beta+1)}.$$

*Proof.* A direct calculation using  $|A|^2 \geq \frac{1}{n}H^2$  gives an estimate

$$|\tilde{A}|^2 = |A|^2 - 2aH + na^2 \geq \frac{1}{n}H^2 - 2aH + na^2 = \frac{1}{n}\tilde{H}^2,$$

which implies that from the evolution equation (3.16) of  $\tilde{H}$

$$\partial_t \tilde{H}_{\min} \geq \frac{1}{n}H_{\min}^\beta \tilde{H}_{\min}^2 + 2aH_{\min}^\beta \tilde{H}_{\min} \geq \frac{1}{n}\tilde{H}_{\min}^{\beta+2}.$$

Now let  $\phi$  be the solution of the ODE

$$\begin{cases} \frac{d\phi}{dt} = \frac{1}{n}\phi^{\beta+2}, \\ \phi(0) = \tilde{H}_{\min}(0), \end{cases}$$

then by the maximum principle

$$\tilde{H} \geq \phi \quad \text{on} \quad 0 \leq t \leq T.$$

On the other hand  $\phi$  is explicitly given by

$$\phi(t) = \phi(0) \left( 1 - \frac{\beta + 1}{n} (\phi(0))^{\beta+1} t \right)^{-\frac{1}{\beta+1}},$$

which implies

$$\tilde{H}_{\min}(t) \geq \tilde{H}_{\min}(0) \left( 1 - \frac{\beta + 1}{n} (\tilde{H}_{\min}(0))^{\beta+1} t \right)^{-\frac{1}{\beta+1}}.$$

Thus,

$$\tilde{H}_{\min}(t) \rightarrow \infty \quad \text{as} \quad 1 - \frac{\beta + 1}{n} (\tilde{H}_{\min}(0))^{\beta+1} t \rightarrow 0+,$$

which proves Proposition 4.1. □

To show that  $h$ -convexity of  $M_t$  is preserved, next consider the evolution of  $\lambda_{\min} := \min_{M_t} \lambda_i$  as in Chap. 3 of [13]. In order to do so, define a smooth approximation  $\mathcal{A}$  to  $\max(x_1, \dots, x_n)$  as follows: for  $\delta > 0$  let

$$(4.1) \quad \mathcal{A}_2(x_1, x_2) = \frac{x_1 + x_2}{2} + \sqrt{\left(\frac{x_1 - x_2}{2}\right)^2 + \delta^2},$$

$$\mathcal{A}_{n+1}(x_1, \dots, x_{n+1}) = \frac{1}{n+1} \sum_{i=1}^{n+1} \mathcal{A}_2(x_i, \mathcal{A}_n(x_1, \dots, \hat{x}_i, \dots, x_{n+1})), \quad n \geq 2.$$

The approximation has the following properties, for a proof see ([13], Lemma 3.3).

**Lemma 4.2.** *For  $n \geq 2$  and  $\delta > 0$ ,*

- i)  $\mathcal{A}_n(x_1, \dots, x_n)$  is smooth, symmetric, monotonically increasing and convex,
- ii)  $\max\{x_1, \dots, x_n\} \leq \mathcal{A}_n(x_1, \dots, x_n) \leq \max\{x_1, \dots, x_n\} + (n - 1)\delta$ ,
- iii)  $\frac{\partial \mathcal{A}_n(x_1, \dots, x_n)}{\partial x_i} \leq 1$ ,
- iv)  $\mathcal{A}_n(x_1, \dots, x_n) - (n - 1)\delta \leq \sum_{i=1}^n \frac{\partial \mathcal{A}_n(x_1, \dots, x_n)}{\partial x_i} x_i \leq \mathcal{A}_n(x_1, \dots, x_n)$ ,
- v)  $\sum_{i=1}^n \frac{\partial \mathcal{A}_n(x_1, \dots, x_n)}{\partial x_i} = 1$ .

Schulze in [24] proved that the minimal principal curvatures of the hypersurfaces under the  $H^\beta$ -flow is increasing by applying the properties of  $\mathcal{A}_n$ , which is also valid in the context of  $\mathbb{H}_\kappa^{n+1}$ .

**Lemma 4.3.** *Let  $M_t$  be a solution of the flow (1.1) in  $\mathbb{H}_\kappa^{n+1}$ . Suppose the initial hypersurface  $M_0$  is strictly  $h$ -convex. Then all  $M_t$  are also strictly  $h$ -convex and  $\lambda_{\min}(t)$  is monotonically increasing for  $t > 0$ .*

*Proof.* Note that the monotonicity of  $\tilde{\lambda}_{\min}(t)$  in time is the same as that of  $\lambda_{\min}(t)$ , so it is sufficient to prove that  $\tilde{\lambda}_{\min}(t)$  is monotonically increasing. Firstly, Proposition 4.1 ensures that  $H$  preserves positivity in time.

For  $\beta \geq 1$ , using a frame which diagonalises  $\tilde{\mathcal{W}}$ , consider the evolution of  $\tilde{\lambda}_{\min}(t)$  in the evolution equation (3.15) of  $\tilde{\mathcal{W}}$ . Then

$$(4.2) \quad \begin{aligned} \partial_t \tilde{\lambda}_{\min}(p, t) &\geq \beta H^{\beta-1} \Delta \tilde{\lambda}_{\min}(p, t) - (\beta - 1) H^\beta \tilde{\lambda}_{\min}^2(p, t) \\ &\quad + 2aH^\beta \tilde{\lambda}_{\min}(p, t) + \beta |\tilde{A}|^2 H^{\beta-1} (a + \tilde{\lambda}_{\min}(p, t)) \\ &= \beta H^{\beta-1} \Delta \tilde{\lambda}_{\min}(p, t) + 2aH^\beta \tilde{\lambda}_{\min}(p, t) + H^\beta \tilde{\lambda}_{\min}^2(p, t) \\ &\quad + \beta H^{\beta-1} \left[ |\tilde{A}|^2 (a + \tilde{\lambda}_{\min}(p, t)) - H \tilde{\lambda}_{\min}^2(p, t) \right]. \end{aligned}$$

The part in the square brackets is nonnegative by the identity  $H = \tilde{H} + na$ , the estimates  $|\tilde{A}|^2 \geq \tilde{H} \tilde{\lambda}_{\min}^2$  and  $|\tilde{A}|^2 \geq n \tilde{\lambda}_{\min}^2$ . Then the maximum principle shows the desired result.

For  $0 < \beta < 1$ , observe that the gradient terms have the wrong sign, we have to work a little bit more as in [24]. For a fixed  $\delta > 0$  now choose a smooth

approximation  $\mathcal{A}(\tilde{b}_i^j) := \mathcal{A}_n(\theta_1, \dots, \theta_n)$  to  $\max(\theta_1, \dots, \theta_n)$ , as defined in (4.1), where the  $\theta_i$  are the eigenvalues of  $\tilde{b}_i^j$ , i.e.,  $\theta_i = 1/\tilde{\lambda}_i$ . By the chain rule

$$\partial_t \mathcal{A} = \frac{\partial \mathcal{A}}{\partial \tilde{b}_i^j} \frac{\partial \tilde{b}_i^j}{\partial t} \quad \text{and} \quad \Delta \mathcal{A} = \frac{\partial \mathcal{A}}{\partial \tilde{b}_i^j} \Delta \tilde{b}_i^j + \frac{\partial^2 \mathcal{A}}{\partial \tilde{b}_r^s \partial \tilde{b}_t^u} \nabla^v \tilde{b}_r^s \nabla_v \tilde{b}_t^u,$$

grouping the two identities and applying Lemma 3.5  $\mathcal{A}$  satisfies the following evolution inequality:

$$\begin{aligned} \partial_t \mathcal{A} \leq & \beta H^{\beta-1} \Delta \mathcal{A} - \beta H^{\beta-1} \frac{\partial^2 \mathcal{A}}{\partial \tilde{b}_r^s \partial \tilde{b}_t^u} \nabla^v \tilde{b}_r^s \nabla_v \tilde{b}_t^u + (\beta - 1) H^\beta \operatorname{tr} \left( \frac{\partial \mathcal{A}}{\partial \tilde{b}_i^j} \right) \\ & - \beta H^{\beta-1} |\tilde{A}|^2 \frac{\partial \mathcal{A}}{\partial \tilde{b}_i^j} \tilde{b}_i^j - 2a H^\beta \frac{\partial \mathcal{A}}{\partial \tilde{b}_i^j} \tilde{b}_i^j - a \beta H^{\beta-1} |\tilde{A}|^2 \frac{\partial \mathcal{A}}{\partial \tilde{b}_i^j} \tilde{b}_i^p \tilde{b}_p^j. \end{aligned}$$

The various terms on the RHS of this inequality can be easily estimated: First, in view of Lemma 4.2 i) convexity of  $\mathcal{A}$  implies convexity of  $\mathcal{A}$ , then the second term can be estimated by

$$-\beta H^{\beta-1} \frac{\partial^2 \mathcal{A}}{\partial \tilde{b}_r^s \partial \tilde{b}_t^u} \nabla^v \tilde{b}_r^s \nabla_v \tilde{b}_t^u \leq 0.$$

Using Lemma 4.2 v),  $0 < \beta < 1$ , and  $\tilde{H} \leq H$  the third term can be estimated by

$$(\beta - 1) H^{\beta-1} \tilde{H}.$$

Lemma 4.2 iv) implies that the next term can be estimated by

$$-\beta H^{\beta-1} |\tilde{A}|^2 (\mathcal{A} - (n - 1)\delta)$$

and the fifth term can be dropped. The last term also can be dropped since  $\frac{\partial \mathcal{A}}{\partial \tilde{b}_i^j} \tilde{b}_i^p \tilde{b}_p^j$  is positive. The following estimate is obtained:

$$\partial_t \mathcal{A} \leq \beta H^{\beta-1} \Delta \mathcal{A} + (\beta - 1) H^{\beta-1} \left( \frac{\tilde{H}}{\mathcal{A}} - |\tilde{A}|^2 \right) \mathcal{A} - H^{\beta-1} |\tilde{A}|^2 (\mathcal{A} - (n - 1)\beta\delta).$$

At a point  $(p, t)$  with  $\mathcal{A} - (n - 1)\beta\delta > 0$ , since

$$\frac{\tilde{H}}{\mathcal{A}} \leq \frac{\tilde{H}}{\theta_{\max}} = \tilde{H} \tilde{\lambda}_{\min} \leq |\tilde{A}|^2,$$

this gives an estimate of the form

$$\partial_t \mathcal{A} \leq \beta H^{\beta-1} \Delta \mathcal{A},$$

which gives a contradiction if  $\mathcal{A}$  attains a first maximum larger than  $(n - 1)\beta\delta$ . The limit as  $\delta$  is approached to 0 then implies the conclusion of Lemma 4.3.  $\square$

**Corollary 4.4.** *Let  $X : M^n \times [0, T) \rightarrow \mathbb{H}_\kappa^{n+1}$  be an  $H^\beta$ -flow of strictly  $h$ -convex hypersurfaces. Then*

$$|A|(p, t) \leq H(p, t) \leq (H_{\max}(0)^{-(\beta+1)} - (\beta + 1)t)^{-\frac{1}{(\beta+1)}}.$$

*Proof.* Lemma 4.3 implies that if  $M_0$  is strictly  $h$ -convex, under the flow (1.1),  $M_t$  is strictly  $h$ -convex as long as it exists, then  $|A| \leq H$ , which implies that from the evolution equation (3.10) of  $H$

$$\partial_t H_{\max} \leq H_{\max}^{\beta+2} - a^2 H_{\max}^{\beta} \leq H_{\max}^{\beta+2}.$$

Now let  $\phi$  be the solution of the ODE

$$\begin{cases} \frac{d\phi}{dt} = \phi^{\beta+2}, \\ \phi(0) = \tilde{H}_{\max}(0), \end{cases}$$

then by the maximum principle

$$H \leq \phi \quad \text{on} \quad 0 \leq t \leq T.$$

On the other hand  $\phi$  is explicitly given by

$$\phi(t) = (\phi(0)^{-(\beta+1)} - (\beta + 1)t)^{-\frac{1}{(\beta+1)}}.$$

Thus, this gives the desired estimate. □

**Corollary 4.5.** *Let  $X : M^n \times [0, T] \rightarrow \mathbb{H}_\kappa^{n+1}$  be an  $H^\beta$ -flow of weakly  $h$ -convex hypersurfaces. Then  $M_t$  is weakly  $h$ -convex for all  $t \in [0, T]$ .*

*Proof.* The initial surface  $M_0$  can be smoothly approximated by strictly  $h$ -convex hypersurfaces  $M_0^i$ . Let these hypersurfaces move by  $H^\beta$ -flow, which by Lemma 4.3 remain strictly  $h$ -convex. For any  $t \in [0, T]$ , Corollary 4.4 implies the uniform  $C^2$ -estimates for these hypersurfaces. For  $\alpha > 0$  the uniform  $C^{2,\alpha}$ -estimates can be obtained for these hypersurfaces as follows: For  $0 < \beta \leq 1$  the speed  $H^\beta$  is concave in  $h_i^j$  and in this case with the uniform  $C^{2,\alpha}$ -bounds are known in general for concave operators (see [20], Theorem 2, Chapter 5.5, or also see [18]). For  $\beta > 1$ ,  $M_t^i$  can be locally reparameterized as graphs over the unit sphere  $\mathbb{S}^n$  with center  $p$  in  $T_p \mathbb{H}_\kappa^{n+1}$ . From (1.1) and (2.8), a short computation yields that the distance function on  $\mathbb{S}^n$  satisfies the following parabolic PDE

$$(4.3) \quad \partial_t r = -s_\kappa^{-1}(r)H^\beta |\xi|,$$

where the mean curvature  $H$  and the outward normal vector length  $|\xi|$  are given by the expressions (2.13) and (2.9), respectively. The function  $H^\beta$  in the coordinate system under consideration is a function of  $D^2r$  and  $Dr$ . Since  $H(\cdot, t)$  is larger than  $na$  and bounded above by Corollary 4.4 this implies that  $H^{\beta-1}$  are also uniformly Hölder continuous functions. Then this ensures that (4.3) is a linear, strictly parabolic partial differential equation

$$(4.4) \quad \partial_t r = a^{ij} D_i D_j r + b^{ij} D_i r D_j r + c^{ij} \sigma_{ij},$$

with coefficients given by

$$a^{ij} = g^{ij} H^{\beta-1}, \quad b^{ij} = -g^{ij} H^{\beta-1} \text{co}_\kappa(r) \quad \text{and} \quad c^{ij} = -g^{ij} H^{\beta-1} s_\kappa(r) c_\kappa(r)$$

in space and time. The interior Schauder estimates by the general theory of Krylov and Safonov [18], [20] lead to  $C^{2,\alpha}$ -estimates. In both cases, i.e.,  $0 < \beta \leq 1$  and  $\beta > 1$ , such a property implies all the higher order estimates by using standard linearization and bootstrap techniques (see [18], [20]). By extracting a convergent subsequence of strictly  $h$ -convex flows it follows that the original flow also had to be  $h$ -convex.  $\square$

For  $\beta \geq 1$ , the following Proposition shows that weakly  $h$ -convex hypersurfaces immediately become strictly  $h$ -convex under the  $H^\beta$ -flow in  $\mathbb{H}_\kappa^{n+1}$ .

**Proposition 4.6.** *For  $\beta \geq 1$ , let  $M_t$  be a solution of the  $H^\beta$ -flow in  $\mathbb{H}_\kappa^{n+1}$ . Suppose the initial hypersurface  $M_0$  is a weakly  $h$ -convex hypersurface with  $H(0) > na$ . Then  $M_t$  is strictly  $h$ -convex for all  $[0, T)$ .*

*Proof.* It is sufficient to prove that  $\tilde{\lambda}_i > 0$ . Since  $H(t) > na$ , i.e.,  $\tilde{H}(t) > 0$ , for all  $[0, T)$ ,  $\tilde{Q}_2$  is well-defined and Corollary 4.5 implies that  $M_t$  is weakly  $h$ -convex. Then an immediate consequence is

$$\tilde{Q}_2 = \frac{|\tilde{H}|^2 - |\tilde{A}|^2}{2\tilde{H}} \geq 0.$$

For  $t \in [0, \varepsilon]$ ,  $\varepsilon < T$ , the bounds on  $|A|^2$  imply the bounds on  $|\tilde{A}|^2$ ,  $|H|$ , and  $\tilde{Q}_2$  which imply

$$H^{\beta-1} \left[ \beta|\tilde{A}|^2 - r(\beta-1)H\tilde{Q}_2 + 2aH \right] \leq C$$

on this interval. An application of Lemma 3.4 for  $\omega := e^{Ct}\tilde{Q}_2$  shows the following estimate:

$$\partial_t \omega \geq \beta H^{\beta-1} \Delta \omega.$$

Suppose that there exists  $(p_0, t_0) \in M^n \times (0, \varepsilon)$  with  $\tilde{Q}_2(p_0, t_0) = 0$ , then also  $\omega(p_0, t_0) = 0$ . The Harnack's inequality in the parabolic case (see i.e., [20]) applied to the above equation shows that  $\omega \equiv 0$  for all  $t \in (0, t_0)$ , i.e.,  $\tilde{Q}_2 \equiv 0$ , which is in contradiction to the existence of strictly convex points on  $M_t$ , and so  $\tilde{Q}_2 > 0$  on  $M^n \times (0, T)$ . An iterative application of this yields that  $\tilde{Q}_r > 0$  on  $M^n \times (0, T)$ . This concludes the proposition.  $\square$

### 5. The long time existence

The third section has shown that the equation (1.1) has a (unique) smooth solution on a short time if the initial hypersurface in  $\mathbb{H}_\kappa^{n+1}$  is  $h$ -convex. This section considers the long time behavior of (1.1) and establishes the existence of a solution on a finite maximal interval.

**Theorem 5.1.** *Let  $[0, T)$  be the maximal existence interval of the flow (1.1)  $M_t$  with  $H(\cdot, 0) > na$  in  $\mathbb{H}_\kappa^{n+1}$ . Then*

$$\frac{1}{\beta+1} (H_{\max}(0))^{-(\beta+1)} \leq T \leq \frac{n}{\beta+1} (H_{\min}(0) - na)^{-(\beta+1)}.$$



Moreover,  $\max_{M_t} |A|^2 \rightarrow +\infty$  as  $t \rightarrow T$ .

*Proof.* The estimates on the maximal time  $T$  of existence can be easily derived from Proposition 4.1, Corollary 4.4 and the proof of Corollary 4.5. To complete the proof of the theorem, assume that  $|A|^2$  remains bounded on the interval  $[0, T)$ , and derive a contradiction. Then the evolution equation (1.1) implies that

$$|X(p, \sigma) - X(p, \tau)| \leq \int_{\tau}^{\sigma} H(p, t) dt$$

for  $0 \leq \tau \leq \sigma < T$ . Since  $H$  is bounded,  $X(\cdot, t)$  tends to a unique continuous limit  $X(\cdot, T)$  as  $t \rightarrow T$ . In order to conclude that  $X(\cdot, T)$  represents a hypersurface  $M_T$ , next under this assumption and in view of the evolution equation (3.5) the induced metric  $g$  remains comparable to a fix smooth metric  $\tilde{g}$  on  $M^n$ :

$$\left| \frac{\partial}{\partial t} \left( \frac{g(u, u)}{\tilde{g}(u, u)} \right) \right| = \left| \frac{\partial_t g(u, u) g(u, u)}{g(u, u) \tilde{g}(u, u)} \right| \leq 2|H||h|_g \frac{g(u, u)}{\tilde{g}(u, u)},$$

for any non-zero vector  $u \in TM^n$ , so that ratio of lengths is controlled above and below by exponential functions of time, and hence since the time interval is bounded, there exists a positive constant  $C$  such that

$$\frac{1}{C} \tilde{g} \leq g \leq C \tilde{g}.$$

Then the metrics  $g(t)$  for all different times are equivalent, and they converge as  $t \rightarrow T$  uniformly to a positive definite metric tensor  $g(T)$  which is continuous and also equivalent by following Hamilton’s ideas in [12]. Using the uniform  $C^{2,\alpha}$ -estimates from the proof of Corollary 4.5 it is enough to imply bounds on all derivatives of  $X$ . Therefore the hypersurfaces  $M_t$  converge to a smooth limit hypersurface  $M_T$ . Finally, applying the local existence result with initial data  $X(\cdot, t)$ , the solution can be continued to a later times, contradicting the maximality of  $T$ . This completes the proof of Theorem 5.1.  $\square$

**Example 5.2.** For the evolution of a geodesic sphere  $\mathcal{S}_0$  with radius  $R_0$  and the origin point of  $\mathbb{H}_\kappa^{n+1}$  its center under the flow (1.1), we get

$$\begin{cases} \frac{dR(t)}{dt} = -(nc\text{co}_\kappa(R(t)))^\beta, \\ R(0) = R_0. \end{cases}$$

A straightforward analysis for the existence of solution of the above ODE implies that the evolving geodesic spheres  $\mathcal{S}_t$  with radii  $R(t)$  contract to the center of the  $\mathcal{S}_0$  on a finite maximal existence time  $T$  satisfying

$$\frac{1}{(an)^\beta} (R_0 + f_\beta(R_0)) \leq T \leq \frac{1}{(an)^\beta} \left( \frac{1}{a} \ln(\cosh(aR_0)) + f_\beta(R_0) \right),$$

where the function  $f_\beta(x)$  on  $[0, +\infty)$  is given by

$$f_\beta(x) = \begin{cases} \frac{1}{a} \sum_{m=0}^{\lfloor \frac{\beta}{2} \rfloor} \frac{1}{\beta + 1 - 2m} (\tanh(ax))^{\beta+1-2m}, & \text{if } [\beta] \text{ is even,} \\ \frac{1}{a} \sum_{m=0}^{\lfloor \frac{\beta-1}{2} \rfloor} \frac{1}{\beta + 1 - 2m} (\tanh(ax))^{\beta+1-2m}, & \text{if } [\beta] \text{ is odd.} \end{cases}$$

In the following, we want to show that the flow exists as long as it bounds a non-vanishing volume. In order to achieve this, using a trick of Tso [26] for the Gauß curvature flow, see also [1], [7] and [21], we study the evolution under (1.1) of the function

$$(5.1) \quad Z_t = \frac{H^\beta}{\Phi - \epsilon}.$$

Here  $\Phi = s_\kappa(r_p)\langle \nu, \partial_{r_p} \rangle$ , which could be seen as “support function” of  $M^n$  in  $\mathbb{H}_\kappa^{n+1}$ , and  $\epsilon$  is a constant to be chosen later.

**Corollary 5.3.** *For  $t \in [0, T)$  and any constant  $\epsilon$ ,*

$$\begin{aligned} \partial_t Z &= \beta H^{\beta-1} \Delta Z + \frac{2\beta H^{\beta-1}}{\Phi - \epsilon} \langle \nabla Z, \nabla \Phi \rangle + (\beta + 1)c_\kappa(r)Z^2 \\ &\quad - \epsilon \beta \frac{|A|^2}{H} Z^2 - n\beta a^2 H^{\beta-1} Z. \end{aligned}$$

*Proof.* For every  $X, Y$  tangent to  $M_t$ , the following formulas is well-known (see [22] page 46 or [11]):

$$(5.2) \quad \bar{\nabla}_X \partial_{r_p} = \text{co}_\kappa(r_p) (X - \langle \partial_{r_p}, X \rangle \partial_{r_p}).$$

This implies that

$$(5.3) \quad \bar{\Delta} r_p = n \text{co}_\kappa(r_p).$$

On the other hand,

$$\begin{aligned} \bar{\nabla}^2 r_p(X, Y) &= \langle \bar{\nabla}_X \partial_{r_p}, Y \rangle \\ (5.4) \quad &= \langle \nabla_X \partial_{r_p}^\top, Y \rangle - \langle \partial_{r_p}, \bar{\nabla}_X Y \rangle \\ &= \nabla^2 r_p(X, Y) + h(X, Y) \langle \partial_{r_p}, \nu \rangle. \end{aligned}$$

This implies that

$$(5.5) \quad \Delta r_p = \text{co}_\kappa(r_p) \left( n - \left| \partial_{r_p}^\top \right|^2 \right) - H \langle \partial_{r_p}, \nu \rangle.$$

Using (1.1) and (5.2) a direct calculation gives

$$(5.6) \quad \bar{\nabla}_t (s_\kappa(r_p) \partial_{r_p}) = -c_\kappa(r_p) H^\beta \nu,$$

which implies that

$$(5.7) \quad \partial_t \Phi = \beta H^{\beta-1} s_\kappa(r_p) \langle \partial_{r_p}, \nabla H \rangle - c_\kappa(r_p) H^\beta.$$

On the other hand, straightforward computations having into account (5.2) and (5.5) as in (Section 4 of [7]) give

$$(5.8) \quad \begin{aligned} \Delta\Phi &= \langle \nu, \partial_{r_p} \rangle \Delta s_\kappa(r_p) + 2 \langle \nabla s_\kappa(r_p), \nabla \langle \nu, \partial_{r_p} \rangle \rangle + s_\kappa(r_p) \Delta \langle \nu, \partial_{r_p} \rangle \\ &= c_\kappa(r_p)H + s_\kappa(r_p) \langle \partial_{r_p}, \nabla H \rangle - \Phi |A|^2. \end{aligned}$$

Combining this with (5.7) yields

$$(5.9) \quad \partial_t \Phi = \beta H^{\beta-1} \Delta \Phi - (\beta + 1) c_\kappa(r_p) H^\beta + \beta H^{\beta-1} \Phi |A|^2.$$

From (5.1), (3.11) with  $l = \beta$  and (5.9), it follows

$$(5.10) \quad \begin{aligned} \partial_t Z &= \frac{1}{\Phi - \epsilon} \left( \beta H^{\beta-1} \Delta H^\beta + \beta H^{2\beta-1} \left( |A|^2 - na^2 \right) \right) \\ &\quad - \frac{H^\beta}{(\Phi - \epsilon)^2} \left( \beta H^{\beta-1} \Delta \Phi - (\beta + 1) c_\kappa(r_p) H^\beta + \beta H^{\beta-1} \Phi |A|^2 \right). \end{aligned}$$

Another computation leads to

$$(5.11) \quad \beta H^{\beta-1} \Delta Z = \frac{\beta H^{\beta-1} \Delta H^\beta}{\Phi - \epsilon} - \frac{\beta H^{2\beta-1} \Delta \Phi}{(\Phi - \epsilon)^2} - 2 \frac{\beta H^{\beta-1}}{\Phi - \epsilon} \langle \nabla Z, \nabla \Phi \rangle$$

by combining (5.11) with (5.10), then the desired equation follows easily.  $\square$

To get an upper upper bound for  $Z$  which is finite and independenat of  $t$ , applying the maximum principle, first it is necessary to get an upper bound for  $r_p$ .

**Lemma 5.4.** *Let  $A(M_t)$  be the total area of  $M_t$ ,  $A_0 = A(M_0)$ ,  $A(\mathbb{S}^n)$  the total area of the unit sphere  $\mathbb{S}^n$  in  $\mathbb{R}^{n+1}$ ,  $\varphi$  the inverse of the function  $s \mapsto A(\mathbb{S}^n) s_\kappa^n(s)$ , and  $\rho_-(t)$  the inner radius of  $\Omega_t$  whose boundary is  $M_t$ . Then*

$$\rho_-(t) \leq \varphi(A_0)$$

for all  $t \in [0, T)$ .

*Proof.* Since the total area  $A(M_t)$  under the flow (1.1) is decreasing from the evolution equation (3.7), then  $A(M_t) \leq A_0$  for all  $t \in [0, T)$ . In view of Lemma 2.3 ii)  $h$ -convexity of the initial hypersurface in  $\mathbb{H}_\kappa^{n+1}$  implies that  $\langle \partial_{r_p}, \nu \rangle > 0$  for all  $t \in [0, T)$ . Now choosing geodesic polar coordinates in  $\mathbb{H}_\kappa^{n+1}$  around a center  $p_t$  of an inball of  $\Omega_t$  as mentioned in Section 2, the total area of  $M_t$  is given by

$$A(M_t) = \int_{\mathbb{S}^n} \frac{s_\kappa^n(s(u))}{\langle \partial_{r_p}, \nu \rangle} du,$$

where  $du$  is the volume form of the unit sphere  $\mathbb{S}^n$  in  $\mathbb{R}^{n+1}$ . But the facts  $\rho_-(t) \leq s(u)$  and  $A(M_t) \leq A_0$ , thus the conclusion of the lemma follows having into account that  $0 < \langle \partial_{r_p}, \nu \rangle \leq 1$ ,  $s_\kappa^n$  and  $\varphi^{-1}$  are increasing function.  $\square$

A direct consequence of the above lemma and Lemma 2.3 i) is:

**Corollary 5.5.** *For any  $t \in [0, T)$ , if  $p, q \in \Omega_t$ , then*

$$\text{dist}(p, q) \leq 2(\varphi(A_0) + a \ln 2).$$

**Theorem 5.6.** *Let  $M_t$  be a solution of the  $H^\beta$ -flow in  $\mathbb{H}_\kappa^{n+1}$ . Suppose the initial hypersurface  $M_0$  is a  $h$ -convex hypersurface,  $\delta > 0$ ,  $q_0 \in \mathbb{H}_\kappa^{n+1}$  and  $B_\delta(q_0) \subset \Omega_t$  for all  $t \in [0, \tau)$ , whose boundary is  $M_t$ . Then*

$$H(p, t) \leq C(M_0, \delta, \beta, n, a) \quad \text{for all } (p, t) \in M^n \times [0, \tau).$$

*Proof.* Since it is proved previously that  $M_t$  is  $h$ -convex along the flow (1.1), Lemma 2.3 ii) gives

$$\Phi = s_\kappa(r_{q_0}) \langle \nu, \partial_{r_{q_0}} \rangle \geq as_\kappa(\delta) \text{ta}_\kappa(\delta).$$

Then taking the constant  $\epsilon$  in the definition (5.1) of  $Z$  as

$$\epsilon = \frac{a}{2} s_\kappa(\delta) \text{ta}_\kappa(\delta)$$

implies

$$\Phi - \epsilon \geq \epsilon > 0.$$

Combining this,  $h$ -convexity of  $M_t$  implies that

$$(5.12) \quad Z \geq 0 \quad \text{and} \quad |A|^2 \geq \frac{1}{n} H^2.$$

On the other hand, by taking  $D := 2(\varphi(A_0) + a \ln 2)$  a direct consequence of Corollary 5.5 is that  $r_{q_0}$  on  $M_t$  satisfies:

$$(5.13) \quad r_{q_0} \leq D.$$

From Corollary 5.3, (5.12) and (5.13) the following inequality can be obtained:

$$\partial_t Z \leq \beta H^{\beta-1} \Delta Z + \frac{2\beta}{H} Z \langle \nabla Z, \nabla \Phi \rangle + \left( (\beta + 1) c_\kappa(D) - \frac{\epsilon \beta}{n} H \right) Z^2.$$

Assume that in  $(p_0, t_0)$ ,  $Z$  attains a big maximum  $C \gg 0$  for the first time. Then

$$H^\beta(p_0, t_0) \geq C(\Phi - \epsilon)(p_0, t_0) \geq \epsilon C,$$

which gives a contradiction if

$$C \geq \max_{p \in M^n} \left\{ Z(p, 0), \frac{1}{\epsilon} \left( \frac{n(\beta + 1) c_\kappa(D)}{\epsilon \beta} \right)^\beta \right\}. \quad \square$$

### 6. Contraction to a point

Now proceeding exactly as in [24], this section shows that  $M_t$  shrink down to a single point as the final time is approached.

*Proof of Theorem 1.5.* Theorem 5.1 and Theorem 5.6 ensure that the flow exists as long as it bounds a non-vanishing domain. Lemma 4.3 and Proposition 4.6 show that all hypersurfaces are strictly  $h$ -convex for  $t \in [0, \tau)$ , thus  $\lim_{t \rightarrow T} \lambda_{\min}(t) \geq a + \delta > a$ . Suppose for all  $t \in [0, \tau)$  there exist two distinct points  $q_1, q_2$  in  $\Omega_t \subset \mathbb{H}_\kappa^{n+1}$ , whose boundary is  $M_t$ . Let  $P$  be any 2-dimensional plane through  $q_1$  and  $q_2$ , then  $P$  intersects  $M_t$  transversally in regular curves  $\gamma_t^P$ . Since  $\lambda_{\min}(t) > a + \frac{\delta}{2}$ , then the curvature of the curves  $\gamma_t^P$  has a lower bound  $a + \delta' > a$ . Let  $I_t := P \cap \Omega_t$ . The fact the  $n$ -dimensional Hausdorff measure  $\mathfrak{H}^n(\Omega_t) \rightarrow 0$  implies that there is a  $P$  such that  $\mathfrak{H}^2(I_t) \rightarrow 0$ , contradicting that  $q_1, q_2 \in I_t$  and that the curvature of the curves  $\gamma_t^P$  is uniformly bounded from below by  $a$ .  $\square$

**Acknowledgments.** We would like to express special thanks to the anonymous referees for their careful reading of this paper and many helpful suggestions.

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