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QUASI m-CAYLEY STRONGLY REGULAR GRAPHS

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Abstract. We introduce a new class of graphs, called quasi m-Cayley graphs, having good symmetry properties, in the sense that they admit a group of automorphisms G that fixes a vertex of the graph and acts semiregularly on the other vertices. We determine when these graphs are strongly regular, and this leads us to define a new algebro-combinatorial structure, called quasi-partial difference family, or QPDF for short. We give several infinite families and sporadic examples of QPDFs. We also study several properties of QPDFs and determine, under several conditions, the form of the parameters of QPDFs when the group G is cyclic.

1. Introduction

A regular graph $\mathcal{G}$ of valency k and order v is called *strongly regular* (in short SRG) with parameters (v, k, λ, μ) if any two adjacent vertices have λ common vertices and any two distinct non-adjacent vertices have μ common vertices. Strongly regular graphs have been extensively studied since their introduction by Bose [4]; in fact, they are one of the most basic association schemes, the ones with two classes. SRGs with certain symmetry properties have been an active topic of research. For example, one can ask when there exist a group of automorphisms G of the graph acting regularly on its vertices, that is, acting transitively on the vertex set with trivial vertex stabilizers (in other words, when a SRG $\mathcal{G}$ is a Cayley graph). It is known that this is the case if there exists a (v, k, λ, μ)-partial difference set D in G (see for example [8]); this is a

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subset D of G, with $|G| = v$ and $|D| = k$ such that $0 \notin D$, $D = -D$ and every element in D (respectively, every non-zero element in $G - D$) can be expressed in exactly λ ways (respectively, in exactly μ ways) as a difference of two distinct elements of D. The (v, k, λ, μ)-SRG generated by D is the graph with vertex set G where xy is an edge if $y - x \in D$. If we relax the condition that G acts transitively but keep the condition that all the stabilizers are trivial, then we are asking for SRGs $\mathcal{G}$ admitting a group of automorphisms G acting semiregularly on the vertex set of $\mathcal{G}$. This question was first studied by Marušič [12] and by de Resmini and Jungnickel [13]. In particular, they studied SRGs admitting a semiregular group of automorphisms with two orbits on the vertex set, and proved that the existence of such graphs is equivalent to the existence of certain algebraic structures that de Resmini and Jungnickel termed *partial difference triples* and that were specially studied by Leung and Ma [7]. More results about partial difference triples were given by Malnič, Marušič and Šparl [9]. Then SRGs admitting semiregular groups of automorphisms with three orbits were studied by Kutnar, Marušič, Miklavič and Šparl [6]. Recently, Martínez and Araluze [11] studied this question for an arbitrary number of orbits and for directed SRGs, introduced by Duval in [5]. They translated the problem into the language of the so-called *partial sum families* (see also [1, 2]).

In this paper we impose a different kind of symmetry in graphs. We will consider graphs admitting a group of automorphisms such that there is a unique vertex x, which we call the *vertex at infinity* ∞, such that the point stabilizer of x is G and for all other vertices the point stabilizer is trivial. When the number of orbits on the set of vertices different from the vertex at infinity under G is m we call it *a quasi m-Cayley graph*. The properties of these graphs will be studied in Section 2. We analyze the conditions for a quasi m-Cayley graph to be strongly regular, that gives rise to a new combinatorial structure that we call *quasi-partial difference family* (or QPDF for short). Finally, in Section 3 we consider the case when the group G is cyclic and obtain some structural properties of QPDFs (Theorem 3.2). Also, we find, in certain particular cases, the form of the parameters of QPDFs (Theorem 3.4), and present an infinite family of QPDFs and some sporadic examples of QPDFs.

Let us review at this point some definitions and notations relative to graphs with semiregular groups of automorphisms. Given integers $m \geq 1$ and $n \geq 2$, an automorphism group of a graph is called (m, n)-*semiregular* if it has m orbits of length n and no other orbit, and the action is regular on each orbit. A m-Cayley digraph $\mathcal{G}$ is a digraph admitting an (m, n)-semiregular group of automorphisms G. When G is abelian, we say that $\mathcal{G}$ is m-*Abelian*. If G is generated by an automorphism ρ (that is, when G is a cyclic group) and $m = 1$ (respectively, $m = 2$) we say that $\mathcal{G}$ is n-*circulant* (respectively, n-*bicirculant*). Every m-Cayley digraph $\mathcal{G}$ can be represented, following the terminology from [10], by an $m \times m$ array of subsets of G in the following way. Let $U_0, \ldots, U_{m-1}$ be the m orbits of G, and for each i let $u_i \in U_i$. For each i and j, let $S_{i,j}$ be defined by $S_{i,j} = \{\rho \in G \mid u_i \to \rho(u_j)\}$. Then the family $(S_{i,j})$ is called

the *symbol* of $\mathcal{G}$ relative to $(G; u_0, \ldots, u_{m-1})$. Of course, for the graph to be simple, we have to impose on the family the extra conditions that $0 \notin S_{i,i}$ for every i and that $S_{j,i} = -S_{i,j}$ for every i, j.

Lastly, let us present here some notation that will be used later on: Given a (v, k, λ, μ)-strongly regular graph, we will define $\beta = \lambda - \mu$, $\gamma = k - \mu$ and $\Delta = \sqrt{\beta^2 + 4\gamma}$. Also, if G is a group, A is a subset of G and $\mathbb{Z}[G]$ is the corresponding group ring of G over the integers, then we will use the same symbol A to denote the sum in $\mathbb{Z}[G]$ of the elements of A, and we will denote by $A^{(-1)}$ the sum of the inverses in G of the elements of A.

2. Quasi m-Cayley graphs

In this section we will give definitions and preliminaries.

Definition 2.1. A group G acts quasi-semiregularly on a set X if there exists an element ∞ in X such that the stabilizer G_∞ of the element ∞ in G is equal to G, and the stabilizer G_x of any element $x \in X - \{\infty\}$ in G is trivial. The element ∞ is called the point at infinity.

Definition 2.2. A graph $\mathcal{G}$ is a quasi m-Cayley graph on a group of automorphisms G if the group G acts quasi-semiregularly on $V(\mathcal{G})$ with m orbits on $V(\mathcal{G}) - \{\infty\}$. If G is cyclic, then the quasi m-Cayley graph $\mathcal{G}$ is said to be quasi (m, n)-circulant (and, more specifically, quasi bicirculant when $m = 2$).

If a graph $\mathcal{G}$ of order v is a quasi m-Cayley graph on a group G, then $m = (v-1)/n$ where $n = |G|$. Hence, a necessary condition for a graph to be a quasi m-Cayley graph is that n divides $v - 1$. Also, if $\mathcal{G}$ is a regular quasi m-Cayley graph, then it is of valency sn for some $s \geq 1$.

If $\mathcal{G}$ is a quasi m-Cayley graph on a group G, then evidently the complement of $\mathcal{G}$ is also a quasi m-Cayley graph on the same group G.

In this paper we are specially interested in the study of quasi m-Cayley graphs that are also strongly regular. An important source of examples of such graphs comes from cyclotomy of finite fields: If $\mathbb{F}_q$ is a finite field of order q and $q - 1 = ef$ with e, f positive integers, then the multiplicative group $G = \{x^e \mid x \in F_q^\times\}$ formed by the e-th powers of the non-zero elements of $\mathbb{F}_q$ acts by multiplication on the additive group of $\mathbb{F}_q$. The orbits of the mentioned action, which determines the cyclotomy on $\mathbb{F}_q$, are $\{0\}, C_0, \ldots, C_{e-1}$, where $C_i = \theta^i G$ and θ is a primitive root of $\mathbb{F}_q$. In many cases, a partial difference set D can be found by taking an appropriate union of orbits (see [8]), and in this case the strongly regular graph generated by D is obviously a quasi m-Cayley graph, where the point at infinity is the zero element of the field.

Similarly as for m-Cayley graphs we can define the *symbol* of a quasi m-Cayley graph in the following way: Let $\mathcal{G}$ be a quasi m-Cayley graph on a group G and let $\{U_0, \ldots, U_{m-1}\}$ be the set of m orbits of G on $V(\mathcal{G}) - \{\infty\}$. Let $u_i \in U_i$, $i \in \mathbb{Z}_m$, let $S_{i,j}$, $i, j \in \mathbb{Z}_m$ be defined by $S_{i,j} = \{\rho \in G \mid u_i \to \rho(u_j)\}$,

and let $S_\infty \subseteq \mathbb{Z}_m$ be defined by

$$S_\infty = \{i \in \mathbb{Z}_m \mid U_i \subseteq N(\infty)\}.$$

Then the family $(S_{i,j})$ together with S_∞ is called the *symbol* of $\mathcal{G}$ relative to $(G; u_0, \ldots, u_{m-1}, S_\infty)$, and determines the adjacencies in the graph. We can assume by renumbering the orbits if necessary that $S_\infty = \{0, \ldots, s-1\}$ for some s.

It is specially interesting the case when $|S_\infty| = 1$. In this situation we have that $S_\infty = \{0\}$ and we can omit S_∞ in the symbol; also, the group G can be identified with the neighborhood of the point at infinity.

Examples 2.3. If $n = mk + 1 \geq 2$, then the complete graph K_n is a quasi m-Cayley graph on a cyclic group $\mathbb{Z}_k$ with the symbol consisting of the sets:

$$S_{i,j} = \begin{cases} \mathbb{Z}_k - \{0\} & \text{if } i = j, \\ \mathbb{Z}_k & \text{if } i \neq j, \end{cases} \quad i, j \in \mathbb{Z}_m, \quad \text{and} \quad S_\infty = \mathbb{Z}_m.$$

Now we will characterize those quasi m-Cayley graphs that are strongly regular.

Definition 2.4. Let G be a group of order n and m, s positive integers with $s \leq m$. A family $\{S_{i,j}\}$ of subsets of G with $0 \leq i, j \leq m-1$, $i \neq j$ such that $0 \notin S_{i,i}$ and $S_{j,i} = -S_{i,j}$ is said to be a (m, n, s, λ, μ) quasi-partial difference family if

$$\sum_{j=1}^{m} |S_{i,j}| = \begin{cases} ns - 1 & \text{if } i \leq s \\ ns & \text{in other case,} \end{cases} \tag{1}$$

$$\sum_{j=1}^{s} |S_{i,j}| = \begin{cases} \lambda & \text{if } i \leq s \\ \mu & \text{in other case} \end{cases} \tag{2}$$

and if the following identities hold in the group ring $\mathbb{Z}[G]$:

$$\sum_{k=0}^{m-1} S_{i,k} S_{k,j} = \delta_{i,j} \gamma \{0\} + \beta S_{i,j} + \mu' G, \tag{3}$$

where $\delta_{i,j}$ is the Kronecker delta, $\gamma = ns - \mu$, $\beta = \lambda - \mu$ and

$$\mu' = \begin{cases} \mu - 1 & \text{if } i, j \leq s \\ \mu & \text{in other case.} \end{cases}$$

Observe that, in view of our assumptions about the $S_{i,j}$, in the previous definition it is sufficient to establish the identities (3) only for $i \leq j$.

Proposition 2.5. *The quasi m-Cayley graph defined by the symbol $(S_{i,j})$ with $S_\infty = \{0, \ldots, s-1\}$ is a $(mn+1, ns, \lambda, \mu)$-SRG if and on if it forms a (m, n, s, λ, μ) quasi-partial difference family.*

Proof. It is easy to prove that, with the notations introduced when the symbol of a quasi m-Cayley graph was defined, condition (3) in Definition 2.4 means that if $i, j \leq s$ and $x \in U_i$ and $y \in U_j$, then the number of paths $x - z - y$ of length 2 with $z \in V(\mathcal{G}) - \{\infty\}$ is $ns - 1$ if $x = y, \lambda - 1$ if $x \neq y$ and xy is an edge and $\mu - 1$ in other case and that, if $i > s$ or $j > s$ and $x \in U_i$ and $y \in U_j$, then the number of paths $x - z - y$ of length 2 with $z \in V(\mathcal{G}) - \{\infty\}$ is ns if $x = y, \lambda$ if $x \neq y$ and xy is an edge and μ in other case (the reason for subtracting one in the first situation is that the point at infinity is joined to x and y). Condition (2) means that the appropriate condition for the number of $x - z - y$ paths is satisfied when one of x and y is the point at infinity, and condition (1) guarantees the regularity of the graph. □

Observe that, in view of the proof of the previous proposition, condition (i) in Definition 2.4 is redundant and could be omitted; nonetheless, we keep it for the sake of clarity.

An example of a $(3, 3, 1, 0, 1)$ quasi-partial difference family on the cyclic group C_3 that generates the well known Petersen graph is the following one:

$$S_{0,0} = \emptyset,\ S_{0,1} = \{0\},\ S_{0,2} = \{0\},\ S_{1,1} = \emptyset,\ S_{1,2} = \{1, 2\},\ S_{2,2} = \emptyset.$$

3. (m, n)-Circulant quasi m-Cayley strongly regular graphs

In this section we study a specific case of quasi m-Cayley SRGs for which the group G is cyclic. A special emphasis is given to the case when $s = 1$, that is, when $|G| = k$, although Theorem 3.2 below holds for arbitrary s. Further the Payley type QPDFs in the Appendix also have $s > 1$.

Note that the case when $|G|$ is not a prime will also give examples of quasi m-Cayley graphs on cyclic groups with $s > 1$. In particular, if $\mathcal{G}$ is a quasi m-Cayley graph on a cyclic group C_{tn}, then $\mathcal{G}$ is also a quasi mt-Cayley graph on a cyclic group isomorphic to C_n with $s > 1$.

Under certain assumptions about the parameters of a QPDF over a cyclic group, we can obtain information about the structure of the family of the diagonal blocks $\{S_{i,i}\}$ of the QPDF. This is done in Theorem 3.2 below. More specifically, we show that all non-zero elements of the group appear the same number of times in the family, that is, the family is a 1-balanced incidence structure. The following lemma will be needed.

Lemma 3.1. *Let G be a cyclic group of order n, and let $A = \sum_{g \in G} a_g g$ be an element of the group ring $\mathbb{Z}[G]$. If there exists a set I of r integers in arithmetic progression with difference d such that $\chi(A) \in I$ for every non-trivial character χ of G, then the following hold*:

(i) *There exist integers c_m such that $A = \sum_{m|n} c_m U_m$, where U_m denotes the unique subgroup of G of order m.*

(ii) *If $m \neq 1, n$, then c_m has the form $c_m = dw_m/m$ with $w_m \in \mathbb{Z}$, and $w_m \neq 0$ only if m divides ud, where $u = l.c.m.\{2, 3, \ldots, r - 1\}$.*

(iii) *If $r = 3$ and the coefficients of A can take only the values 0, 1, and 2, then $c_m = 0$ unless m is a divisor of n in the set $\{1, n, 2d, d, d/2\}$.*

Proof. The case $r = 3$ is shown in [13, Lemma 3.5]. With the same proof you can also show it for general r, and hence the proof here is omitted. □

Note that for $r = 4$, parts (i) and (ii) of Lemma 3.1 reduce to [6, Lemma 5.1].

Theorem 3.2. *Let G be a cyclic group. If the family $\{S_{i,j}\}$ is an (m, n)-circulant (m, n, s, λ, μ) quasi-partial difference family and if one of the following three conditions is satisfied:*

(i) *n is a prime,*
(ii) *n is coprime to $(m!)\Delta$,*
(iii) *$m = 2$ and Δ does not divide $2n$.*

Then $\{S_{i,i}\}_{0 \le i \le m-1}$ covers all the elements of $G - \{0\}$ the same number of times. That is, $\sum_{i=0}^{m-1} S_{i,i} = \sigma(G - \{0\})$ in the group ring $\mathbb{Z}[G]$ for some natural number σ.

Proof. Let $U = \sum_{0 \le i \le m-1} S_{i,i}$ be the element in $\mathbb{Z}[G]$ that correspond with the multiset formed by the union of the $S_{i,i}$. Let us consider an arbitrary non-trivial character χ of the group G. Let A_χ be the square $m \times m$ matrix whose general term is $\chi(S_{i,j})$. It can be easily proved by using (3) that the matrix A_χ satisfies $A_\chi^2 = \gamma I + \beta A_\chi$, where I is the identity matrix. Observe that $\chi(U)$ is an integer, because it is the trace of A_χ and the polynomial $x^2 - \beta x - \gamma$ has integer roots (in fact, they are the eigenvalues of the DSRG generated by the QPDF distinct from k, whose difference is Δ). Thus, if we put $U = \sum_{g \in G} a_g g$, with a_g nonnegative integers, we can use Lemma 3.1. Now, if n is a prime, the result follows easily from the first part of the lemma, if is coprime to $(m!)\Delta$, it follows from the second part and, finally, if Δ does not divide $2n$, it follows from the third part. □

In fact, in all the (m, n)-circulant QPDFs that we have found, the conclusion of the theorem holds even if none of the conditions (i), (ii) and (iii) is satisfied. We wonder whether this is always true.

Definition 3.3. We will say that a $(m, n, 1, \lambda, \mu)$ quasi-partial difference family $\{S_{i,j}\}$ on a cyclic group C_n is uniform if the following three conditions hold:

(i) $\bigcup_{i=0}^{m-1} S_{i,i} = G - \{0\}$ (where the union is disjoint).
(ii) all the cardinalities $|S_{i,i}|$ with $i \ge 1$ are equal.
(iii) all the cardinalities $|S_{i,j}|$ with $i, j \ge 1$ and $i \ne j$ are equal.

We will denote by A the common value of the $|S_{i,i}|$ and by B the common value of the $|S_{i,j}|$ with $i \ne j$.

Theorem 3.4. *If $\{S_{i,j}\}$ is a $(m, n, 1, \lambda, \mu)$ uniform quasi-partial difference family on a cyclic group C_n and m is odd, then their parameters are of the form*

$$(m, d(md + 2), 1, d^2 - md + 3d - 1, d(d + 1))$$

with d an integer.

Proof. Let (v, k, λ, μ) be the parameters of the associated SRG $\mathcal{G}$. First observe that by (2) we have that $|S_{0,0}| = \lambda$ and, for every $i \geq 1$, $|S_{0,i}| = |S_{i,0}| = \mu$.

The matrix whose (i, j)-th entry is $|S_{i,j}|$ is

$$\mathfrak{C} = \begin{pmatrix} \lambda & \mu & \mu & \dots & \mu \\ \mu & A & B & \dots & B \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ \mu & B & B & \dots & A \end{pmatrix}.$$

Since by (1) the sum of the elements of the first row of $\mathfrak{C}$ is $k - 1$ we obtain

$$\lambda + (m-1)\mu = k - 1. \tag{4}$$

Now, we obtain from the fact that $G - \{0\}$ is the disjoint union of the $S_{i,i}$ that

$$\lambda + (m-1)A = \frac{v - m - 1}{m}, \tag{5}$$

and from the fact that there are m orbits for the action of G on $V(\mathcal{G}) - \infty$ we obtain

$$v - 1 = mk. \tag{6}$$

Now, by taking the trivial character in (3) with $i = 1, j = 1$ and with $i = 1, j = 2$, respectively, we have

$$\mu^2 + A^2 + (m-2)B^2 = \gamma + \beta A + \mu k, \tag{7}$$

$$\mu^2 + 2AB + (m-3)B^2 = \beta B + \mu k, \tag{8}$$

and substracting (8) from (7) we conclude that

$$(A - B)^2 = \gamma + \beta(A - B). \tag{9}$$

Solving in $A - B$ we obtain

$$A - B = \frac{\beta \pm \Delta}{2}. \tag{10}$$

Since by (1) the sum of the elements in the second row of $\mathfrak{C}$ is k we have

$$A + (m-2)B = \gamma. \tag{11}$$

We obtain now from (10) and (11) that

$$(m-1)A = \gamma + (m-2)\frac{\beta \pm \Delta}{2}. \tag{12}$$

From (5) and (12) we have

$$\lambda + \gamma + (m-2)\frac{\beta \pm \Delta}{2} = \frac{v - m - 1}{m}. \tag{13}$$

Now we conclude from (6) and (13) that

$$m\lambda - m\mu = -2 \mp (m-2)\Delta. \tag{14}$$

By solving the system obtained from (4),(6) and (14) we obtain

$$\lambda = \frac{m(k-1) \mp (m-1)(m-2)\Delta - 2(m-1)}{m^2}, \tag{15}$$

$$\mu = \frac{m(k-1) \pm (m-2)\Delta + 2}{m^2}, \tag{16}$$

$$v = mk + 1. \tag{17}$$

We have from (14) that $\beta = \frac{-2\mp(m-2)\Delta}{m}$, and substituting this expression of β in the definition of Δ and having into account that $\gamma = k - \mu$ we get

$$m^2k - m^2\mu = (m-1)\Delta^2 \mp (m-2)\Delta - 1. \tag{18}$$

By substituting (16) into (18) we obtain $k = \frac{\Delta^2-1}{m}$, and substituting this expression for k into (15), (16) and (17) we arrive to

$$\lambda = \frac{\Delta^2 \mp (m-1)(m-2)\Delta - 3m + 1}{m^2}, \tag{19}$$

$$\mu = \frac{\Delta^2 \pm (m-2)\Delta - m + 1}{m^2}, \tag{20}$$

$$v = \Delta^2. \tag{21}$$

We have from (14) that, if the upper sign holds, then $\Delta \equiv 1 \pmod m$ and, if the lower sign holds, then $\Delta \equiv -1 \pmod m$. Let us suppose that the upper sign holds. Then, by putting $\Delta = md+1$ and substituting it into (19), (20) and (21), respectively, we obtain parameters as in the statement of the proposition (with non-negative d, of course, since Δ is non-negative).

Let us suppose now that the lower sign holds. Now, by putting $\Delta = -md-1$ we obtain again parameters as in the statement of the proposition, this time with non-positive d. □

Observe that, by following the proof of the previous theorem, we can see that in fact $A = \mu$, and thus all the cardinalities $|S_{i,i}|$, $|S_{i,0}|$ and $|S_{0,i}|$ are equal.

Now we will prove that when $dm + 1$ is a prime power then, under certain assumptions, uniform QPDFs with parameters as in the theorem exist, but we will need first some definitions and results relative to cyclotomy of finite fields. If $q = ef + 1$ and if we consider the corresponding cyclotomy in $\mathbb{F}_q$ as defined in Section 2, then, for given i, j in $\{0, \ldots, e-1\}$, the cyclotomic number (i, j) is defined to be the number of solutions to the equation

$$1 + \theta^{re+i} = \theta^{se+j} \text{ with } 0 \le r, s \le f - 1.$$

Baumert, Mills and Ward introduced in [3] the concept of *uniform cyclotomy*. A cyclotomy is said to be uniform if there exist nonnegative integers A, B and C such that $(0,0) = A$, $(0,i) = (i,0) = (i,i) = B$ for every i in $\{1,\dots,e-1\}$ and $C = (i,j)$ for every i,j with $0 \neq i,j$ and $i \neq j$.

Proposition 3.5 ([3, Theorem 4])**.** *Let q be a power of the prime p, and let e be a divisor of $q-1$ such that $e \geq 3$. Then the cyclotomic numbers of order e over $GF(q)$ are uniform if and only if -1 is a power of p modulo e.*

Theorem 3.6. *If $dm+1 = p^r$ is a prime power and -1 is a power of p modulo m, where m is odd, then a uniform*

$$(m, d(md+2), 1, d^2 - md + 3d - 1, d(d+1))$$

QPDF exists.

Proof. Consider the cyclotomy with $e = m$ over the field $\mathbb{F}_{(dm+1)^2}$. We have, by Proposition 3.5, that the cyclotomy is uniform. Hence, in the group ring $\mathbb{Z}[G]$, where G is the additive group of the field, the identity $C_0C_0^{(-1)} = |C_0|\{0\} + AC_0 + B(C_1 + \cdots + C_{e-1})$ holds where $A = (0,0)$ and $B = (0,1)$, and therefore C_0 is a partial difference set (we have also used the trivial fact that in this field $C_0^{(-1)} = C_0$ is satisfied). The SRG generated by this partial difference set is obviously a quasi m-Cayley graph, and the QPDF associated to this graph is clearly uniform: condition (i) in the definition of uniform QPDF follows from the fact that $\mathbb{F}^{\times}_{(dm+1)^2}$ is the disjoint union of the cyclotomic orbits $C_0,\dots,C_{m-1}$, and conditions (ii) and (iii) hold trivially because the cyclotomy is uniform. Hence, the parameters of the PQDF are as stated in Theorem 3.4, and hence they also have the form indicated in the statement of this theorem, because the first two parameters determine the values of λ and μ. □

When we take parameters with the form described in Theorem 3.4 with $m = 2$ (although in that theorem it was assumed that m is odd), we obtain QPDFs that generates graphs with parameters of Paley type. Of course in this case, when $2d+1$ is a prime power, uniform QPDFs with these parameters exists: just observe that Payley graphs are generated by partial difference sets associated to a cyclotomy of order 2.

By [14] for $q \in \{13, 17\}$ Paley graphs $P(q)$ are (up to isomorphism) the only strongly regular quasi bicirculants of order q and valency $\frac{q-1}{2}$ (that is, the only strongly regular quasi bicirculants of order q apart from the complete graph K_q), and for $q = 21$ there is no such graph. Also, using program package Magma one can check that Paley graph $P(25)$ is the only strongly regular quasi bicirculant graph of order $q = 25$ (apart from the complete graph K_{25}).

By [14] there are 15 non-isomorphic strongly regular graphs with parameters $(25, 12, 5, 6)$. As mentioned above the Paley graph $P(25)$ is the only quasi bicirculant among these graphs. However, among the other 14 graphs in this family there are six more graphs that are quasi m-Cayley graphs on some cyclic group. In particular, two of the graphs are quasi m-Cayley graphs on a cyclic

group C_n for each $(m,n) \in \{(4,6),(6,4),(8,3)\}$, and four of the graphs are quasi 8-Cayley graphs on a cyclic group C_3 (but are neither quasi 4-Cayley graphs nor 6-Cayley graphs on a cyclic group). The symbols of the graphs that are distinct from the Paley graph $P(25)$ are listed in the appendix (only the symbols with respect to a quasi $(4,6)$-semiregular automorphism are given for the two first ones).

All (m,n)-circulant QPDFs with $s=1$ that we have found, except one, are uniform and hence have parameters in the form indicated in Theorem 3.4. The only exception is a $(7,7,1,0,1)$-QPDF whose symbol is listed in the Appendix.

4. Appendix

Quasi-partial difference families generating non-isomorphic (25, 12, 5, 6)-SRGs:

Family 1:

$S_{0,0} = \{\pm 2, 3\},\ S_{0,1} = \{0,3\},\ S_{0,2} = \{0,1,2,3\},\ S_{0,3} = \{0,2\},$
$S_{1,1} = \{\pm 2, 3\},\ S_{1,2} = \{2,4\},\ S_{1,3} = \{0,1,4,5\},$
$S_{2,2} = \{\pm 1\},\ S_{2,3} = \{0,2,3,5\},$
$S_{3,3} = \{\pm 1\}$

Family 2:

$S_{0,0} = \{\pm 2, 3\},\ S_{0,1} = \{0,3\},\ S_{0,2} = \{0,1,2,4\},\ S_{0,3} = \{0,5\},$
$S_{1,1} = \{\pm 2, 3\},\ S_{1,2} = \{0,5\},\ S_{1,3} = \{0,1,2,4\},$
$S_{2,2} = \{\pm 1\},\ S_{2,3} = \{1,2,4,5\},$
$S_{3,3} = \{\pm 1\}$

Family 3:

$S_{0,1} = \{0,1,2\},\ S_{0,2} = \{0\},\ S_{0,3} = \{0\},\ S_{0,4} = \{0\},\ S_{0,5} = \{0\},$
$S_{0,6} = \{0,1\},\ S_{0,7} = \{0,1\},$
$S_{1,2} = \{1\},\ S_{1,3} = \{2\},\ S_{1,4} = \{1\},\ S_{1,5} = \{2\},\ S_{1,6} = \{0,1\},\ S_{1,7} = \{0,1\},$
$S_{2,2} = \{\pm 1\},\ S_{2,3} = \{1\},\ S_{2,4} = \{2\},\ S_{2,5} = \{0,1,2\},\ S_{2,6} = \{1\},\ S_{2,7} = \{0\},$
$S_{3,3} = \{\pm 1\},\ S_{3,4} = \{0,1,2\},\ S_{3,5} = \{1\},\ S_{3,6} = \{0\},\ S_{3,7} = \{1\},$
$S_{4,5} = \{0,2\},\ S_{4,6} = \{0,2\},\ S_{4,7} = \{1,2\},$
$S_{5,6} = \{1,2\},\ S_{5,7} = \{0,2\},$
$S_{6,6} = \{\pm 1\},$
$S_{7,7} = \{\pm 1\}$

Family 4:

$S_{0,0} = \{\pm 1\},\ S_{0,1} = \{0\},\ S_{0,2} = \{2\},\ S_{0,3} = \{0\},\ S_{0,4} = \{0\},\ S_{0,5} = \{0,1,2\},$
$S_{0,6} = \{1\},\ S_{0,7} = \{0\},$

$S_{1,2} = \{1,2\},\ S_{1,3} = \{0,2\},\ S_{1,4} = \{0,1,2\},\ S_{1,5} = \{0\},\ S_{1,6} = \{2\},$
$S_{1,7} = \{1\},$
$S_{2,2} = \{\pm 1\},\ S_{2,4} = \{0\},\ S_{2,5} = \{2\},\ S_{2,6} = \{0,2\},\ S_{2,7} = \{0,1\},$
$S_{3,3} = \{\pm 1\},\ S_{3,4} = \{0\},\ S_{3,5} = \{2\},\ S_{3,6} = \{0,1\},\ S_{3,7} = \{1,2\},$
$S_{4,4} = \{\pm 1\},\ S_{4,5} = \{0,2\},\ S_{4,6} = \{0\},\ S_{4,7} = \{0\},$
$S_{5,6} = \{0,2\},\ S_{5,7} = \{0,2\},$
$S_{6,7} = \{0,1,2\}$

Family 5:

$S_{0,1} = \{0,1,2\},\ S_{0,2} = \{0\},\ S_{0,3} = \{0\},\ S_{0,4} = \{0\},\ S_{0,5} = \{0,2\},$
$S_{0,6} = \{0,1\},\ S_{0,7} = \{0\},$
$S_{1,2} = \{2\},\ S_{1,3} = \{1\},\ S_{1,4} = \{0\},\ S_{1,5} = \{1,2\},\ S_{1,6} = \{1,2\},\ S_{1,7} = \{0\},$
$S_{2,2} = \{\pm 1\},\ S_{2,3} = \{0\},\ S_{2,4} = \{0,1\},\ S_{2,5} = \{1,2\},\ S_{2,7} = \{1,2\},$
$S_{3,3} = \{\pm 1\},\ S_{3,4} = \{0,2\},\ S_{3,6} = \{1,2\},\ S_{3,7} = \{1,2\},$
$S_{4,4} = \{\pm 1\},\ S_{4,5} = \{0,1\},\ S_{4,6} = \{0,2\},$
$S_{5,6} = \{0,2\},\ S_{5,7} = \{0,1\},$
$S_{6,7} = \{0,2\},\ S_{7,7} = \{\pm 1\}$

Family 6:

$S_{0,0} = \{\pm 1\},\ S_{0,1} = \{0\},\ S_{0,2} = \{0,1\},\ S_{0,4} = \{0\},\ S_{0,5} = \{0,1\},$
$S_{0,6} = \{0\},\ S_{0,7} = \{0,2\},$
$S_{1,1} = \{\pm 1\},\ S_{1,2} = \{1\},\ S_{1,3} = \{2\},\ S_{1,4} = \{0,2\},\ S_{1,5} = \{0,1\},$
$S_{1,6} = \{1,2\},$
$S_{2,3} = \{1,2\},\ S_{2,4} = \{0,1\},\ S_{2,6} = \{0,2\},\ S_{2,7} = \{0,1\},$
$S_{3,3} = \{\pm 1\},\ S_{3,4} = \{0\},\ S_{3,5} = \{0,2\},\ S_{3,6} = \{0\},\ S_{3,7} = \{1,2\},$
$S_{4,5} = \{0\},\ S_{4,6} = \{0,1,2\},\ S_{4,7} = \{0,1\},$
$S_{5,5} = \{\pm 1\},\ S_{5,6} = \{1\},\ S_{5,7} = \{1,2\},$
$S_{6,7} = \{0,2\}$

An sporadic QPDF in the cyclic group C_7 with seven orbits:

$(v,k,\lambda,\mu) = (50,7,0,1),$
$S_{0,1} = \{3\},\ S_{0,2} = \{3\},\ S_{0,3} = \{0\},\ S_{0,4} = \{3\},\ S_{0,5} = \{5\},\ S_{0,6} = \{6\},$
$S_{1,1} = \{\pm 2\},\ S_{1,3} = \{3\},\ S_{1,4} = \{3,4\},\ S_{1,5} = \{3\},$
$S_{2,2} = \{\pm 3\},\ S_{2,3} = \{2\},\ S_{2,4} = \{1,6\},\ S_{2,5} = \{4\},$
$S_{3,5} = \{1,3,4\},\ S_{3,6} = \{3\},$

$$S_{4,6} = \{1, 5\},$$
$$S_{5,6} = \{4\},$$
$$S_{6,6} = \{\pm 1\}$$

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References

[1] A. Araluze, I. Kovács, K. Kutnar, L. Martínez, and D. Marušič, *Partial sum quadruples and bi-Abelian digraphs*, J. Combin. Theory Ser A **119** (2012), no. 8, 1811–1831.
[2] A. Araluze, K. Kutnar, L. Martínez, and D. Marušič, *Edge connectivity in difference graphs and some new constructions of partial sum families*, European J. Combin. **32** (2011), no. 3, 352–360.
[3] L. D. Baumert, W. H. Mills, and R. L. Ward, *Uniform cyclotomy*, J. Number Theory **14** (1982), no. 1, 67–82.
[4] R. C. Bose, *Strongly regular graphs, partial geometries and partially balanced designs*, Pacific J. Math. **13** (1963), 389–419.
[5] A. Duval, *A directed graph version of strongly regular graphs*, J. Combin. Theory Ser. A **47** (1988), no. 1, 71–100.
[6] K. Kutnar, D. Marušič, Š. Miklavič, and P. Šparl, *Strongly regular tri-Cayley graphs*, European J. Combin. **30** (2009), no. 4, 822–832.
[7] K. H. Leung and S. L. Ma, *Partial difference triples*, J. Algebraic Combin. **2** (1993), no. 4, 397–409.
[8] S. L. Ma, *A survey of partial difference sets*, Des. Codes Cryptogr. **4** (1994), no. 3, 221–261.
[9] A. Malnič, D. Marušič, and P. Šparl, *On strongly regular bicirculants*, European J. Combin. **28** (2007), no. 3, 891–900.
[10] A. Malnič, D. Marušič, P. Šparl, and B. Frelih, *Symmetry structure of bicirculants*, Discrete Math. **307** (2007), no. 3-5, 409–414.
[11] L. Martínez and A. Araluze, *New tools for the construction of directed strongly regular graphs*: *Difference digraphs and partial sum families*, J. Combin. Theory Ser. B **100** (2010), no. 6, 720–728.
[12] D. Marušič, *Strongly regular bicirculants and tricirculants*, Ars Combin. **25 C** (1988), 11–15.
[13] M. J. de Resmini and D. Jungnickel, *Strongly regular semi-Cayley graphs*, J. Algebraic Combin. **1** (1992), no. 2, 171–195.
[14] http://www.win.tue.nl/∼aeb/graphs/srg/srgtab1-50.html.

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