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TOTAL DOMINATIONS IN P_6-FREE GRAPHS

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Abstract. In this paper, we prove that the total domination number of a P_6-free graph of order $n \geq 3$ and minimum degree at least one which is not the cycle of length 6 is at most $\frac{n+1}{2}$, and the bound is sharp.

1. Introduction

A *total dominating set* of a graph G with no isolated vertex is a set S of vertices of G such that every vertex is adjacent to a vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [3]. For notation and graph theory terminology we in general follow [3]. Let $G = (V, E)$ be a graph with vertex set V of order n. The degree, neighborhood and closed neighborhood of a vertex v in the graph G are denoted by $d(v)$, $N(v)$ and $N[v] = N(v) \cup \{v\}$, respectively. The minimum degree and maximum degree of the graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For any $S \subseteq V$, $N(S) = \bigcup_{v \in S} N(v)$. Let $G[S]$ denote the graph induced by S. Let C_n, P_n and $K_{1,n-1}$ denote the cycle, the path and star of order n, respectively. A graph is P_n-free if it does not contain P_n as an induced subgraph.

Lemma 1 (Cockayne et al. [3]). *If G is a connected graph of order $n \geq 3$, then $\gamma_t(G) \leq \frac{2n}{3}$.*

A large family of graphs attaining the bound in Lemma 1 can be established using the following transformation of a graph. The 2-*corona* of a graph H is the graph of order $3|V(H)|$ obtained from H by attaching a path of length 2 to each vertex of H so that the resulting paths are vertex disjoint as illustrated in Figure 1. The 2-corona of a connected graph has total domination number two-thirds its order. The following characterization of connected graphs of order at least 3 with total domination number exactly two-thirds their order is obtained in [2].

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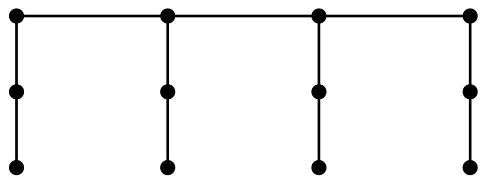

FIGURE 1. The 2-corona graph of a connected graph $H = P_4$.

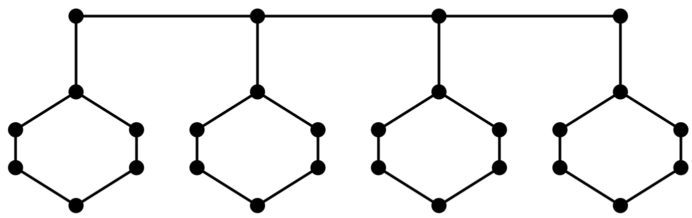

FIGURE 2. A graph in the collection $\Re$ with underlying tree $T \approx P_4$.

Lemma 2 (Brigham et al. [2]). *Let G be a connected graph of order $n \geq 3$. Then $\gamma_t(G) = \frac{2n}{3}$ if and only if G is C_3, C_6 or the 2-corona of some connected graph.*

If we restrict the minimum degree to be at least 2, then the upper bound in Lemma 1 can be improved.

Lemma 3 (Henning [7]). *If G is a connected graph of order n with $\delta(G) \geq 2$ and $G \notin \{C_3, C_5, C_6, C_{10}\}$, then $\gamma_t(G) \leq \frac{4n}{7}$.*

Let $\Re$ be the collection of graphs that can be obtained from a nontrivial tree T as follows. For each vertex v of T, add a 6-cycle C_v and join v to one vertex of C_v as shown in Figure 2. Let H_1 be the graph obtained from a 6-cycle by adding a new vertex and joining this vertex to two vertices at distance 2 apart on the cycle as depicted in Figure 3. The following characterization of those graphs of order n, which are edge-minimal with respect to satisfying G connected, $\delta(G) \geq 2$ and $\gamma_t(G) \geq \frac{4n}{7}$, that is, $\frac{4n}{7}$-minimal graphs, is obtained in [7].

Lemma 4 (Henning [7]). *A graph G is a $\frac{4}{7}$-minimal graph if and only if $G \in \Re \cup \{C_3, C_5, C_5, C_7, C_{10}, C_{14}, H_1\}$.*

Favaron et al. [6] conjectured that for any connected graph of order n with $\delta(G) \geq 3$, $\gamma_t(G) \leq \frac{n}{2}$. Archdeacon et al. [1] recently found an elegant one-page proof of this conjecture.

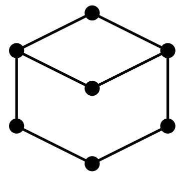

FIGURE 3. A graph H_1.

Lemma 5 ([1]). *If G is a connected graph of order n with $\delta(G) \geq 3$, then $\gamma_t(G) \leq \frac{n}{2}$.*

In 2008, Favaron et al. [5] gave an upper bound on total domination number in a claw-free graph.

Lemma 6 ([5]). *If G is a connected claw-free graph of order n and $\delta(G) \geq 2$, then $\gamma_t(G) \leq \frac{n+2}{2}$.*

Obviously, if G is a 2-corona graph or $G \in \Re$, then G contains an induced P_6. In this paper, we consider connected P_6-free graph. We show that every connected P_6-free graph G of order $n \geq 3$ with minimum degree at least one and $G \neq C_6$ satisfies $\gamma_t(G) \leq \frac{n+1}{2}$, and the bound is sharp.

2. Main results

Lemma 7. *Let G be a connected graph of order $n \geq 3$. If G is P_4-free, then $\gamma_t(G) = 2$.*

Proof. Since G is connected and P_4-free, it follows that its complement $\overline{G}$ is not connected. So, $\gamma_t(G) \leq 2$. Hence, $\gamma_t(G) = 2$. □

Theorem 1. *Let G be a connected graph of order $n \geq 3$. If G is P_5-free, then $\gamma_t(G) \leq \frac{n+1}{2}$.*

Proof. We will prove the inequality by induction on the order n of the graph. If G is P_4-free, by Lemma 7, we have $\gamma_t(G) = 2$. Since $2 \leq \frac{n+1}{2}$, the bound holds. This establishes the base cases for the graph contains no induced P_5. Suppose we now have a connected P_5-free graph G of order $n \geq 4$, and the desired result is true for any connected P_5-free graph of order less than n.

Case 1. G contains no induced subgraph C_5. Suppose that G contains an induced subgraph $P_4 : u_0, u_1, u_2, u_3$. Let $V_p = V(P_4)$, $A = N(V_p) \setminus V_p$, $B = V \setminus (A \cup V_p)$ and $C = N(B) \setminus B$. Then $C \subseteq A$. If $A = \emptyset$, then $G = P_4$. It is obvious that the result holds. So, we can assume that $A \neq \emptyset$. Since G is P_5-free and contains no induced subgraph C_5, it follows that $\{u_1, u_2\}$ dominates A. If $B = \emptyset$, then the result holds. So we can assume that $B \neq \emptyset$.

If $G[B \cup C]$ is not connected, then there is an induced P_5-path in the subgraph induced by the vertices of $B \cup C \cup V_p$, which is a contradiction. Hence, $G[B \cup C]$

is connected. Let $G' = G[B \cup C]$. If $|n(G')| = 2$, it is obvious that the result holds. If $|n(G')| \geq 3$, by induction, there exists a total dominating set S' of G' such that $|S'| \leq \frac{n(G')+1}{2} \leq \frac{n-3}{2}$. Then $S' \cup \{u_1, u_2\}$ is a total dominating set of G. So, $\gamma_t(G) \leq |S' \cup \{u_1, u_2\}| = |S'| + 2 \leq \frac{n-3}{2} + 2 = \frac{n+1}{2}$.

Case 2. G contains an induced subgraph C_5. Choose an induced subgraph $C_5 : u_0, u_1, u_2, u_3, u_4, u_0$. Let $V_c = V(C_5)$, $A = N(V_c) \setminus V_c$, $B = V \setminus (A \cup V_c)$ and $C = N(B) \setminus B$. If $A = \emptyset$, then $G = C_5$. It is obvious that the result holds. So, we can assume that $A \neq \emptyset$. Since G is P_5-free, it follows that $\{u_1, u_2, u_3\}$ dominates A. If $B = \emptyset$, then the result holds. So we can assume that $B \neq \emptyset$.

By a method similar to the proof of Case 1, we can see that $G[B \cup C]$ is connected. For any $0 \leq i \leq 4$, let $R_i = A \setminus N(\{u_{i\oplus 2}, u_{i\oplus 3}\})$, where $\oplus$ is the addition modulo 5. For any $x \in A$, $|N(x) \cap V_c| \geq 2$. Otherwise, if x is adjacent to exactly one vertex in V_c, say u_i, then $G[\{x, u_i, u_{i\oplus 1}, u_{i\oplus 2}, u_{i\oplus 3}\}] = P_5$, contradicting the assumption that G is P_5-free. For any $x \in R_i$, it is easy to prove that $x \in N(u_{i\oplus 1}) \cap N(u_{i\oplus 4})$, where $0 \leq i \leq 4$. Hence, $R_i \cap R_j = \emptyset$ for any $0 \leq i < j \leq 4$. For any i, $R_i \cap C = \emptyset$. Otherwise, say $x \in R_i \cap C$ and $y \in N(x) \cap B$, then $G[\{y, x, u_{i\oplus 1}, u_{i\oplus 2}, u_{i\oplus 3}\}] = P_5$, contradicting the assumption that G is P_5-free.

Case 2.1. For any i, $R_i \neq \emptyset$. Let $G' = G[B \cup C]$. Since all the R_i are not empty and disjoint, $n(G') \leq n - 10$. If $n(G') = 2$, it is obvious that the result holds. If $n(G') \geq 3$, let S' be a minimum total dominating set of G'. By induction, $|S'| \leq \frac{n(G')+1}{2}$. Since $S' \cup \{u_1, u_2, u_3\}$ is a total dominating set of G, it follows that $\gamma_t(G) \leq |S' \cup \{u_1, u_2, u_3\}| = |S'| + 3 \leq \frac{n-9}{2} + 3 \leq \frac{n+1}{2}$.

Case 2.2. There exists an i such that $R_i = \emptyset$, say $R_0 = \emptyset$. Then $\{u_2, u_3\}$ dominates A. Suppose that $u_0 \in N(C)$. Let $G' = G[B \cup C \cup \{u_0\}]$. Let S' be a γ_t-set of G'. By induction, $|S'| \leq \frac{n(G')+1}{2}$. Then $S' \cup \{u_2, u_3\}$ is a total dominating set of G. So, $\gamma_t(G) \leq |S' \cup \{u_2, u_3\}| = |S'| + 2 \leq \frac{n-4+1}{2} + 2 \leq \frac{n+1}{2}$.

Suppose that $u_0 \notin N(C)$. If $|A \setminus C| \geq 1$, let $G' = G[B \cup C]$. If $n(G') = 2$, it is obvious that the result holds. So we can assume that $n(G') \geq 3$. Let S' be a γ_t-set of G'. By induction, $|S'| \leq \frac{n(G')+1}{2} \leq \frac{n-5}{2}$. Since $S' \cup \{u_1, u_2, u_3\}$ is a total dominating set of G, $\gamma_t(G) \leq |S' \cup \{u_1, u_2, u_3\}| = |S'| + 3 \leq \frac{n+1}{2}$.

Suppose that $A = C$. Say $u_2 \in N(A)$. Let $G' = G[B \cup C \cup \{u_2\}]$. Let S' be a γ_t-set of G'. By induction, $|S'| \leq \frac{n-4+1}{2}$. Then $\gamma_t(G) \leq |S' \cup \{u_0, u_4\}| \leq \frac{n+1}{2}$. □

Theorem 2. *Let G be a connected P_6-free graph of order $n \geq 3$. If G is not C_6, then $\gamma_t(G) \leq \frac{n+1}{2}$, and this bound is sharp.*

Proof. We will prove the inequality by induction on the order n of the graph. If G is P_5-free, by Theorem 1, the result holds. This establishes the base cases for the graph contains no induced subgraph P_6. Suppose G is a connected P_6-free graph of order $n \geq 5$ and $G \neq C_6$, and the desired result is true for any connected P_6-free graph of order less than n, except C_6.

Case 1. G contains no induced subgraph C_6. Suppose that G contains an induced subgraph $P_5 : u_0, u_1, u_2, u_3, u_4$. Let $V_p = V(P_5)$, $A = N(V_p) \setminus V_p$, $B = V \setminus (A \cup V_p)$ and $C = N(B) \setminus B$. If $A = \emptyset$, then $G = P_5$. It is obvious that the result holds. So, we can assume that $A \neq \emptyset$. Since G is P_6-free and contains no induced subgraph C_6, it follows that $\{u_1, u_2, u_3\}$ dominates A. If $B = \emptyset$, then the result holds. So we can assume that $B \neq \emptyset$.

Case 1.1. $G[B \cup C]$ is not connected. Then C dominates B. Otherwise, there is an induced P_6 in the subgraph induced by the vertices $V_p \cup B \cup C$, which is a contradiction. Choose the minimum cardinality subset D of C such that D dominates B. Then $|D| \leq \frac{|C|+|B|}{2} \leq \frac{|A|+|B|}{2} \leq \frac{n-5}{2}$. Since $D \cup \{u_1, u_2, u_3\}$ is a total dominating set of G, it follows that $\gamma_t(G) \leq |D \cup \{u_1, u_2, u_3\}| = |D| + 3 \leq \frac{n-5}{2} + 3 = \frac{n+1}{2}$.

Case 1.2. $G[B \cup C]$ is connected. Suppose that $A \setminus C \neq \emptyset$. Let $G' = G[B \cup C]$. If $n(G') = 2$, the result holds. So, we can assume that $n(G') \geq 3$. Since G' is not C_6, by induction, there exists a total dominating set S' of G' such that $|S'| \leq \frac{n(G')+1}{2} \leq \frac{n-5}{2}$. Since $S' \cup \{u_1, u_2, u_3\}$ is a total dominating set of G, $\gamma_t(G) \leq |S' \cup \{u_1, u_2, u_3\}| = |S'| + 3 \leq \frac{n-5}{2} + 3 = \frac{n+1}{2}$.

Suppose that $A = C$. If $u_0 \in N(A)$, let $G' = G[A \cup B \cup \{u_0\}]$. Since G' is not C_6, by induction, there exists a total dominating set S' of G' such that $|S'| \leq \frac{n(G')+1}{2} \leq \frac{n-3}{2}$. Then $S' \cup \{u_2, u_3\}$ is a total dominating set of G. So, $\gamma_t(G) \leq |S' \cup \{u_2, u_3\}| = |S'| + 2 \leq \frac{n-3}{2} + 2 = \frac{n+1}{2}$. Let $u_0 \notin N(A)$. Similarly, we can assume that $u_4 \notin N(A)$. That is $d(u_0) = d(u_4) = 1$. If A dominates B, choose the minimum cardinality subset D of A such that D dominates B. Then $|D| \leq \frac{|A|+|B|}{2} \leq \frac{n-5}{2}$. Since $D \cup \{u_1, u_2, u_3\}$ is a total dominating set of G, it follows that $\gamma_t(G) \leq |D \cup \{u_1, u_2, u_3\}| = |D| + 3 \leq \frac{n-5}{2} + 3 = \frac{n+1}{2}$. If A does not dominate B, then $N(u_1) \cap A \neq \emptyset$ and $N(u_3) \cap A \neq \emptyset$. Let $G' = G[V \setminus \{u_2\}]$. Then G' is a connected P_6-free graph. By induction, there exists a total dominating set S' of G' such that $|S'| \leq \frac{n(G')+1}{2} \leq \frac{n}{2}$. Since S' is a total dominating set of G, $\gamma_t(G) \leq |S'| \leq \frac{n}{2}$.

Case 2. G contains an induced subgraph C_6. Let $u_0, u_1, \ldots, u_5, u_0$ be an induced subgraph C_6. Let $V_c = V(C_6)$, $A = N(V_c) \setminus V_c$, $B = V \setminus (A \cup V_c)$ and $C = N(B) \setminus B$. Since $G \neq C_6$, $A \neq \emptyset$. It is obvious that $\{u_1, u_2, u_3, u_4\}$ dominates A. If $B = \emptyset$, the result holds. So we can assume that $B \neq \emptyset$.

Case 2.1. $G[B \cup C]$ is not connected. Then C dominates B. Otherwise, there exists an induced P_6 in G, which is a contradiction. Choose the minimum cardinality subset D of C such that D dominates B. Then $|D| \leq \frac{|C|+|B|}{2} \leq \frac{|A|+|B|}{2}$. If $|D| < \frac{|A|+|B|}{2}$, since $D \cup \{u_1, u_2, u_3, u_4\}$ is a total dominating set of G, it follows that $\gamma_t(G) \leq |D \cup \{u_1, u_2, u_3, u_4\}| = |D| + 4 < \frac{n-6}{2} + 4 = \frac{n+2}{2}$. That is $\gamma_t(G) \leq \frac{n+1}{2}$. If $|D| = \frac{|A|+|B|}{2}$, $|A| = |C|$. Say $u_0 \in N(A)$. Then $D \cup \{u_2, u_3, u_4\}$ is a total dominating set of G. It follows that $\gamma_t(G) \leq |D \cup \{u_2, u_3, u_4\}| = |D| + 3 \leq \frac{n-5}{2} + 3 = \frac{n+1}{2}$.

Case 2.2. $G[B \cup C]$ is connected. For any $0 \leq i \leq 5$, we define the set $R_i = A \setminus N(\{u_{i\oplus 2}, u_{i\oplus 3}, u_{i\oplus 4}\})$, where $\oplus$ is the addition modulo 6. For any $x \in A$, $|N(x) \cap V_c| \geq 2$. For any $x \in R_i$, it is easy to prove that $x \in N(u_{i\oplus 1}) \cap N(u_{i\oplus 5})$ for $0 \leq i \leq 5$. Then $R_i \cap R_j = \emptyset$ for any $0 \leq i < j \leq 5$. For any i, $R_i \cap C = \emptyset$. Otherwise, say $x \in R_i \cap C$ and $y \in N(x) \cap B$, then $G[\{y, x, u_{i\oplus 1}, u_{i\oplus 2}, u_{i\oplus 3}, u_{i\oplus 4}\}] = P_6$, contradicting the assumption that G is P_6-free.

Case 2.2.1. For any i, $R_i \neq \emptyset$. Let $G' = G[B \cup C]$. Since all the R_i are not empty and disjoint, $n(G') \leq n - 12$. If $n(G') = 2$, the result holds. So, we can assume that $n(G') \geq 3$. Let S' be a minimum total dominating set of G'. Then $S' \cup \{u_1, u_2, u_3, u_4\}$ is a total dominating set of G. If G' is not C_6, by induction, we have $|S'| \leq \frac{n(G')+1}{2}$. So, $\gamma_t(G) \leq |S' \cup \{u_1, u_2, u_3, u_4\}| = |S'| + 4 \leq \frac{n-11}{2} + 4 \leq \frac{n+1}{2}$. If G' is C_6, $|S'| \leq 4$. Since $n \geq 18$, $\gamma_t(G) \leq |S' \cup \{u_1, u_2, u_3, u_4\}| \leq 8 \leq \frac{n+1}{2}$.

Case 2.2.2. There exists i such that $R_i = \emptyset$, say $R_0 = \emptyset$. Then $\{u_2, u_3, u_4\}$ dominates A. Suppose that $u_0 \in N(C)$. Assume $A \setminus C \neq \emptyset$. Let $G' = G[B \cup C \cup \{u_0\}]$. Then $n(G') \leq n - 6$ Let S' be a γ_t-set of G'. Then $S' \cup \{u_2, u_3, u_4\}$ is a total dominating set of G. If G' is not C_6, by induction, $|S'| \leq \frac{n(G')+1}{2}$. So, $\gamma_t(G) \leq |S' \cup \{u_2, u_3, u_4\}| = |S'| + 3 \leq \frac{n-6+1}{2} + 3 \leq \frac{n+1}{2}$. If G' is C_6, then $n \geq 12$. Let $G' = C_6 : u_0, v_1, \ldots, v_5, u_0$. Then $v_1, v_5 \in A$. Since v_5 is dominated by $\{u_2, u_3, u_4\}$, it follows that $\{u_2, u_3, u_4, v_1, v_2, v_3\}$ is a total dominating set of G. So, $\gamma_t(G) \leq 6 \leq \frac{n+1}{2}$. Suppose that $A = C$. Let $G' = G[B \cup C \cup \{u_0, u_5\}]$. Then $n(G') = n - 4$. Let S' be a γ_t-set of G'. Then $S' \cup \{u_2, u_3\}$ is a total dominating set of G. If G' is not C_6, by induction, $|S'| \leq \frac{n(G')+1}{2}$. So, $\gamma_t(G) \leq |S' \cup \{u_2, u_3\}| = |S'| + 2 \leq \frac{n-4+1}{2} + 2 \leq \frac{n+1}{2}$. If G' is C_6, then $n = 10$. Let $G' = C_6 : u_0, v_1, \ldots, v_4, u_5, u_0$. Then $\{v_1, v_2, u_2, u_3, u_4\}$ is a total dominating set of G. So, $\gamma_t(G) \leq 5 \leq \frac{n+1}{2}$.

Suppose that $u_0 \notin N(C)$. If $|A \setminus C| \geq 2$, let $G' = G[B \cup C]$. If $n(G') = 2$, the result holds. So we can assume that $n(G') \geq 3$. Let S' be a γ_t-set of G'. Then $S' \cup \{u_1, u_2, u_3, u_4\}$ is a total dominating set of G. If G' is not C_6, by induction, $|S'| \leq \frac{n(G')+1}{2} \leq \frac{n-7}{2}$. So, $\gamma_t(G) \leq |S' \cup \{u_1, u_2, u_3, u_4\}| = |S'| + 4 \leq \frac{n+1}{2}$. If G' is C_6, then $n \geq 14$. Let $G' = C_6 : v_1, \ldots, v_6, v_1$. Say $v_6 \in C$. Then $\{v_2, v_3, v_4, u_1, u_2, u_3, u_4\}$ is a total dominating set of G. So, $\gamma_t(G) \leq 7 \leq \frac{n+1}{2}$.

Suppose that $|A \setminus C| = 1$, say $v \in A \setminus C$. If $u_1 \in N(C)$, let $G' = G[B \cup C \cup \{u_0, u_1, u_5\}]$. Let S' be a γ_t-set of G'. If G' is not C_6, by induction, $|S'| \leq \frac{n-4+1}{2}$. Then $\gamma_t(G) \leq |S'| + 2 \leq \frac{n+1}{2}$. If G' is the graph C_6, then $n = 10$. It is easy to prove that $\gamma_t(G) \leq 5 \leq \frac{n+1}{2}$. Hence, we can assume that $u_1 \notin N(C)$. Then $u_2 \in N(C)$. Otherwise, $G[B \cup C \cup V_c]$ contains a P_6, which is a contradiction. By a similar way, if $u_2 \in N(C)$, the result holds.

Suppose that $A = C$. Without loss of generality, we can assume that $u_2 \in N(C)$. Let $G' = G[B \cup C \cup \{u_1, u_2\}]$. Let S' be a γ_t-set of G'. If G' is not C_6, by induction, $|S'| \leq \frac{n(G')+1}{2} \leq \frac{n-3}{2}$. So, $\gamma_t(G) \leq |S' \cup \{u_4, u_5\}| = |S'| + 2 \leq \frac{n+1}{2}$. If G' is C_6, then $n = 10$. It is easy to prove that $\gamma_t(G) \leq 5 \leq \frac{n+1}{2}$.

It remains to establish that the bound is sharp. Let G obtained from a star $K_{1,r}$ by subdividing each edge exactly one time. Then $n(G) = 2r + 1$. It is obvious that $\gamma_t(G) = r + 1 = \frac{n(G)+1}{2}$. □

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