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ON A RECURRENCE RELATION OF K-MITTAG-LEFFLER FUNCTION

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Abstract. The principal aim of this paper is to investigate a recurrence relation and an integral representation of k-Mittag-Leffler function introduced earlier by Dorrego and Cerutti [2] and several special cases have also been discussed.

1. Introduction and preliminaries

Fractional calculus deals with derivatives and integrals of arbitrary orders. During the last three decades fractional calculus has been applied to almost every field of mathematics like special functions, science, engineering and technology. Many applications of fractional calculus can be found in turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, non-linear control theory, image processing, non-linear biological systems and astrophysics. The Mittag-Leffler function has gained importance and popularity during the last one decade due mainly to its applications in the solution of fractional-order differential, integral and difference equations arising in certain problems of mathematical, physical, biological and engineering sciences. This function is introduced and studied by the Swedish mathematician Gosta Mittag-Leffler [9, 10] in terms of the power series

$$E_{\alpha}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + 1)} \quad (\alpha > 0). \tag{1.1}$$

In 1905, a generalization of this series in the following form

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \beta)} \quad (\alpha, \beta > 0) \tag{1.2}$$

has been studied by several authors notably by Mittag-Leffler [9, 10], Wiman [13], Agrawal [1], Humbert and Agrawal [8] and Dzrbashjan [3, 4, 5]. It is shown

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in [6] that the function defined by (1.1) and (1.2) are both entire functions. In 1971, Prabhakar [11] introduced the function which is defined by

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n + \beta) n!} \quad (\alpha, \beta, \gamma \in C, \mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\gamma) > 0), \tag{1.3}$$

where $(\gamma)_n$ is the Pochhammer's symbol (see e.g. [3]) defined ($\lambda \in C$) by

$$(\lambda)_n = \begin{cases} 1, & n = 0, \lambda \neq 0; \\ \lambda(\lambda+1)\cdots(\lambda - n + 1), & n \in \mathbb{N}, \lambda \in \mathbb{C}, \end{cases} \tag{1.4}$$

$$= \frac{\Gamma\lambda + n}{\Gamma\lambda}, \ n \in \mathbb{N} \cup \{0\}, \ \lambda \in \mathbb{C}/\mathbb{Z}, \tag{1.5}$$

$\mathbb{N}$ and $\mathbb{Z}$ being the set of positive integers and integers, respectively.

In 2012, G. A. Dorrego and R. A. Cerutti [2] introduced k-Mittag-Leffler function $E^{\gamma}_{k,\alpha,\beta}(z)$ defined by

$$E^{\gamma}_{k,\alpha,\beta}(z) = \sum\nolimits_{n=0}^{\infty} \frac{(\gamma)_{n,k} z^n}{\Gamma_k(\alpha n + \beta) n!}, \tag{1.6}$$

where $(\gamma)_{n,k}$ is the k-Pochhammer's symbol. k-Pochhammer's symbol and k-Gamma function are given below

$$(x)_{n,k} = x(x+k)(x+2k)\cdots(x+(n-1)k), \ \gamma \in \mathbb{C}, k \in \mathbb{R}, \text{ and } n \in \mathbb{N} \tag{1.7}$$

and $\Gamma_k(z) = \int_0^{\infty} e^{-\frac{t^k}{k}} t^{z-1}\, dt, \ k \in \mathbb{R}, z \in \mathbb{C}$ need more accurate convergence conditions. Particularly, $\Gamma_k(x) \to \Gamma(x)$ as $k \to 1$.

2. Recurrence relation

In this section we derive a recurrence relation of k-Mittag-Leffler function. The result is represented in the form of a theorem stated below:

Theorem 2.1. *For $R(\alpha + p) > 0, \ R(\beta + s) > 0, R(\gamma) > 0$,*

$$\begin{aligned} &E^{\gamma}_{k,\alpha+p,\beta+s+1}(z) - E^{\gamma}_{k,\alpha+p,\beta+s+2}(z) \\ &= (\beta+s)(\beta+s+2)E^{\gamma}_{k,\alpha+p,\beta+s+3}(z) + (\alpha+p)^2 z^2 \ddot{E}^{\gamma}_{k,\alpha+p,\beta+s+3}(z) \\ &\quad + (\alpha+p)\{\alpha+p+2(\beta+s+1)\} z \dot{E}^{\gamma}_{k,\alpha+p,\beta+s+3}(z), \end{aligned} \tag{2.1}$$

where $\dot{E}^{\gamma}_{k,\alpha,\beta}(z) = \frac{d}{dz} E^{\gamma}_{k,\alpha,\beta}(z)$ and $\ddot{E}^{\gamma}_{k,\alpha,\beta}(z) = \frac{d^2}{dz^2} E^{\gamma}_{k,\alpha,\beta}(z)$.

It is easy to obtain the following corollary by letting $\alpha + p = v$ and $\beta + s = m$.

Corollary 2.1. *We have, for $v, m \in \mathbb{N}$*

$$\begin{aligned} E^{\gamma}_{k,v,m+1}(z) = E^{\gamma}_{k,v,m+2}(z) &+ m(m+2)E^{\gamma}_{k,v,m+3}(z) + v^2 z^2 \ddot{E}^{\gamma}_{k,v,m+3}(z) \\ &+ vv + 2(m+1) z E^{\cdot\gamma}_{k,\alpha+p,\beta+s+3}(z). \end{aligned} \tag{2.2}$$

Proof of Theorem 2.1. By applying the fundamental relation of the Gamma function $\Gamma(z+1) = z\Gamma(z)$ to (1.4), we can write

$$E^{\gamma}_{k,\alpha+p,\beta+s+1}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} z^n}{\{(\alpha+p)n+\beta+s+1\}\Gamma_k((\alpha+p)n+\beta+s)n!}$$

and

$$\begin{aligned} &E^{\gamma}_{k,\alpha+p,\beta+s+2}(z) \\ &= \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} z^n}{((\alpha+p)n+\beta+s+1)((\alpha+p)n+\beta+s)\Gamma_k((\alpha+p)n+\beta+s)n!}, \end{aligned}$$

which we can write as follows:

(2.3)

$$\begin{aligned} &E^{\gamma}_{k,\alpha+p,\beta+s+2}(z) \\ &= \sum_{n=0}^{\infty} \left\{ \frac{1}{(\alpha+p)n+\beta+s} - \frac{1}{(\alpha+p)n+\beta+s+1} \right\} \frac{(\gamma)_{n,k} z^n}{((\alpha+p)n+\beta+s+1)\Gamma_k((\alpha+p)n+\beta+s)n!} \\ &= E^{\gamma}_{k,\alpha+p,\beta+s+1}(z) - \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} z^n}{\{(\alpha+p)n+\beta+s+1\}\Gamma_k((\alpha+p)n+\beta+s)n!}. \end{aligned}$$

For convenience, we denote the last summation in (2.3) by S:

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} z^n}{((\alpha+p)n+\beta+s+1)\Gamma_k((\alpha+p)n+\beta+s)n!} \\ &= E^{\gamma}_{k,\alpha+p,\beta+s+1}(z) - E^{\gamma}_{k,\alpha+p,\beta+s+2}(z). \end{aligned} \tag{2.4}$$

Applying a simple identity

$$\frac{1}{u} = \frac{1}{u(u+1)} + \frac{1}{u+1}, \text{ where } u = (\alpha+p)n+\beta+s+1, \text{ we obtain}$$

(2.5)

$$\begin{aligned} S = &\sum_{n=0}^{\infty} \frac{\{(\alpha+p)n+\beta+s\}(\gamma)_{n,k} z^n}{\Gamma_k((\alpha+p)n+\beta+s+3)n!} + \sum_{n=0}^{\infty} \frac{\{(\alpha+p)n+\beta+s\}\{(\alpha+p)n+\beta+s+1\}(\gamma)_{n,k} z^n}{\Gamma_k((\alpha+p)n+\beta+s+3)n!} \\ &= (\alpha+p) \sum_{n=1}^{\infty} \frac{(\gamma)_{n,k} z^n}{\Gamma_k((\alpha+p)n+\beta+s+3)(n-1)!} + (\beta+s) \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} z^n}{\Gamma_k((\alpha+p)n+\beta+s+3)n!} \\ &\quad + (\alpha+p)^2 \sum_{n=1}^{\infty} \frac{n(\gamma)_{n,k} z^n}{\Gamma_k((\alpha+p)n+\beta+s+3)(n-1)!} + b \sum_{n=1}^{\infty} \frac{(\gamma)_{n,k} z^n}{\Gamma_k((\alpha+p)n+\beta+s+3)(n-1)!} \\ &\quad + c \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} z^n}{\Gamma_k((\alpha+p)n+\beta+s+3)n!}, \end{aligned}$$

where $b = (\alpha+p) + 2\beta + 2s + 1$ and $c = (\beta+s) + \beta + s + 1$.

We now express each summation in the right hand side of (2.5) as follows

$$\frac{d^2}{dz^2}\{z^2E^{\gamma}_{k,\alpha+p,\beta+s+3}(z)\} = \sum_{n=0}^{\infty}\frac{(n+2)(n+1)(\gamma)_{n,k}z^n}{\Gamma_k((\alpha+p)n+\beta+s+3)n!} \tag{2.6}$$

we find from (2.6) that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{n(\gamma)_{n,k}z^n}{\Gamma_k((\alpha+p)n+\beta+s+3)(n-1)!} &= z^2\ddot{E}^{\gamma}_{k,\alpha+p,\beta+s+3}(z)+4z\dot{E}^{\gamma}_{k,\alpha+p,\beta+s+3}(z)\\ &\quad -3\sum_{n=1}^{\infty}\frac{(\gamma)_{n,k}z^n}{\Gamma_k((\alpha+p)n+\beta+s+3)(n-1)!}\end{aligned} \tag{2.7}$$

considering

$$\frac{d}{dz}\{zE^{\gamma}_{k,\alpha+p,\beta+s+3}(z)\} = \sum_{n=0}^{\infty}\frac{(n+1)(\gamma)_{n,k}z^n}{\Gamma_k((\alpha+p)n+\beta+s+3)n!}.$$

Similarly, we have

$$\sum_{n=1}^{\infty}\frac{(\gamma)_{n,k}z^n}{\Gamma_k((\alpha+p)n+\beta+s+3)(n-1)!} = z\dot{E}^{\gamma}_{k,\alpha+p,\beta+s+3}(z). \tag{2.8}$$

Applying (2.7) and (2.8) yields

$$\sum_{n=1}^{\infty}\frac{(\gamma)_{n,k}z^n}{\Gamma_k((\alpha+p)n+\beta+s+3)(n-1)!} = z\dot{E}^{\gamma}_{k,\alpha+p,\beta+s+3}(z)+z^2\ddot{E}^{\gamma}_{k,\alpha+p,\beta+s+3}(z). \tag{2.9}$$

Applying (2.8) and (2.9) to (2.5), we get

$$\begin{aligned}s &= (\alpha+p)^2z^2\ddot{E}^{\gamma}_{k,\alpha+p,\beta+s+3}(z)+\{(\alpha+p)^2+(\alpha+p)+b\}z\dot{E}^{\gamma}_{k,\alpha+p,\beta+s+3}(z)\\ &\quad +(\beta+s+c)E^{\gamma}_{k,\alpha+p,\beta+s+3}(z).\end{aligned}$$

Now setting the last identity into (2.4) computes the proof of Theorem 2.1. □

3. Integral representations

Theorem 3.1. *The following result holds good,*

$$\begin{aligned}&\int_0^1 t^{\beta+s}E^{\gamma}_{k,\alpha+p,\beta+s}(t^{\alpha+p})\,dt\\ &= E^{\gamma}_{k,\alpha+p,\beta+s+1}(1)-E^{\gamma}_{k,\alpha+p,\beta+s+2}(1)\ (R(\alpha+p)>0, R(\beta+s)>0), R(\gamma)>0).\end{aligned} \tag{3.1}$$

By setting $\alpha+p=v$ and $\beta+s=m\in\mathbb{N}$ in (3.1), we get the following corollary.

Corollary 3.1.

$$\int_0^1 t^{\beta+s}E^{\gamma}_{k,v,m}(t^v)\,dt = E^{\gamma}_{k,v,m+1}(1)-E^{\gamma}_{k,v,m+2}(1),\ v,m\in\mathbb{N}. \tag{3.2}$$

Proof of Theorem 3.1. Putting $z = 1$ in (2.4) gives

(3.3)
$$\sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} z^n}{\{(\alpha+p)n+\beta+s+1\}\Gamma_k((\alpha+p)n+\beta+s)n!} = E^{\gamma}_{k,\alpha+p,\beta+s+1}(1) - E^{\gamma}_{k,\alpha+p,\beta+s+2}(1).$$

It is easy to find that

$$\int_0^1 t^{\beta+s} E^{\gamma}_{k,\alpha+p,\beta+s}(t^{\alpha+p})dt = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} z^{(\alpha+p)n+\beta+s+1}}{\{(\alpha+p)n+\beta+s+1\}\Gamma_k((\alpha+p)n+\beta+s)n!}. \tag{3.4}$$

Comparing (3.3) with the identity obtaining by setting $z = 1$ in (3.4) is seen to yield (3.1) in Theorem 3.1. □

4. Special cases

(1) Setting $k = 1$ in (2.1) reduces to known recurrence relation of $E^{\gamma}_{\alpha,\beta}(z)$ (see [12]).
(2) Setting $p = 0$, $\gamma = k = 1$ and $\beta + s = m \in \mathbb{N}$ in (2.1) reduces to a known recurrence relation of $E_{\alpha+\beta}(z)$ (see [7]).
(3) Setting $v = m = k = \gamma = 1$ and $v = m = k = \gamma = 2$ in (3.2), respectively, yields

$$\int_0^1 t\, e^t\, dt = E^1_{1,2}(1) - E^1_{1,3}(1) \text{ and } \int_0^1 tE^2_{1,1}(t)\, dt = E^2_{1,2}(1) - E^2_{1,3}(1).$$

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