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AN ELEMENTARY PROOF OF SFORZA-SANTALÓ RELATION FOR SPHERICAL AND HYPERBOLIC POLYHEDRA

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Abstract. We defined and studied a naturally extended hyperbolic space (see [1] and [2]). In this study, we describe Sforza's formula [7] and Santaló's formula [6], which were rediscovered and later discussed by many mathematicians (Milnor [4], Suárez-Peiró [8], J. Murakami and Ushijima [5], and Mednykh [3]) in the spherical space in an elementary way. Thereafter, using the extended hyperbolic space, we apply the same method to prove their results in the hyperbolic space.

1. Introduction

The spherical space and hyperbolic space are both spaces of constant sectional curvature ± 1. It is well known that these two spaces have many similar properties and similar formulae. Despite these similarities, there is a major difference between them: The spherical space is a finite space whereas the hyperbolic space is infinite.

Even though the hyperbolic space is infinite, we can further extend the hyperbolic space beyond infinity using the following approach. If we look at the hyperbolic space as a unit disk in the Kleinian model, then using an analytic continuation method (exactly ϵ-approximation technique), we can construct an extended hyperbolic space that unifies the hyperbolic space and Lorentzian component as the subspaces of the unit sphere (see [2]). Thus, we can analyze geometric objects lying across the ideal boundary by a finitely additive measure theory and limit integral type volume integration. We describe this in brief.

We deform the Kleinian metric (on $\mathbb{R}^n$) into an ϵ-Kleinian metric,

$$ds_K^2 = \left(\frac{\Sigma x_j dx_j}{1-|x|^2}\right)^2 + \frac{\Sigma dx_j^2}{1-|x|^2} \quad \to \quad ds_{K,\epsilon}^2 = \left(\frac{\Sigma x_j dx_j}{d_\epsilon^2-|x|^2}\right)^2 + \frac{\Sigma dx_j^2}{d_\epsilon^2-|x|^2},$$

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where $d_\epsilon = 1 - \epsilon i$, and $\epsilon > 0$. Next, we can derive the corresponding Kleinian volume form and the Kleinian ϵ-volume form using the above metrics.

$$dV_K = \frac{dx_1 \wedge \cdots \wedge dx_n}{(1-|x|^2)^{\frac{n+1}{2}}} \quad \rightarrow \quad dV_{K,\epsilon} = \frac{d_\epsilon dx_1 \wedge \cdots \wedge dx_n}{(d_\epsilon^2 - |x|^2)^{\frac{n+1}{2}}}.$$

In the new space, we define the volume of a region U as the limit of the integral of the above ϵ-volume form.

$$\mathrm{vol}(U) = \int_U dV_K \qquad \rightarrow \qquad \mathrm{vol}_H(U) = \lim_{\epsilon \to 0} \int_U dV_{K,\epsilon}.$$

If region U is located inside the hyperbolic space, then the new volume satisfies the canonical property $\mathrm{vol}(U) = \mathrm{vol}_H(U)$. Also, if the region U is lying across the ideal boundary, we can calculate the volume $\mathrm{vol}_H(U)$ under natural conditions, although $\mathrm{vol}(U)$ does not exist.

Notice that if we choose $d_\epsilon = 1 + \epsilon i$ instead of $d_\epsilon = 1 - \epsilon i$, then we have a slightly different geometry that exactly corresponds to a contour integration with counterclockwise orientation around the point $z = 1$.

There exists another description of the extended hyperbolic space. We assign the ϵ-Kleinian metric to $\{\pm 1\} \times \mathbb{R}^n \subset \mathbb{R}^{n+1}$ to obtain the pull-back metric on the Euclidean sphere $\{x = (x_0, x_1, \ldots, x_n) \mid x_0^2 + x_1^2 + \cdots + x_n^2 = 1\}$ by radial projection with respect to the origin; this space is called the hyperbolic sphere $\mathbb{S}^n_H$, and the volume in $\mathbb{S}^n_H$ is similarly calculated using the limit integration method. The extended hyperbolic space $\mathbb{S}^n_H$ is topologically the same as the spherical space $\mathbb{S}^n$ and has many properties that resemble those of the spherical space (see [2] and [1]).

As a powerful application of the extended hyperbolic space, we show that the several results of Sforza [7], Santaló [6], Milnor [4], Suárez-Peiró [8], and Murakami-Ushijima [5] regarding spherical or hyperbolic polyhedra are simply restated in a unified and comprehensive form. We prove their results using an elementary method that (importantly) uses Euler characteristic.

2. Sforza-Santaló relation in the spherical space and in the extended hyperbolic space

On the standard sphere $\mathbb{S}^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$, we consider a convex spherical triangle $\triangle(1, 2, 3)$ (such that the three angles are larger than 0 and smaller than π) with three vertices 1, 2, 3, and make a dual triangle $\triangle(1', 2', 3')$, where the three vertices $1', 2', 3'$ are located $\frac{\pi}{2}$ (outward direction with respect to $\triangle(1, 2, 3)$) from the three edges $\overline{23}, \overline{13}, \overline{12}$ of $\triangle(1, 2, 3)$, respectively.

Next, we can divide the sphere $\mathbb{S}^2$ into 8 convex triangles and classify the triangles in 4 classes, $\mathrm{I}, \mathrm{II}, \mathrm{III}$, and IV, with respect to successive facial connectivity to $\triangle(1, 2, 3)$ (see Fig. 1(a)). If we denote the angles as A, B, C and the edges as a, b, c on the original triangle $\triangle(1,2,3)$, then we easily get the dual triangle angles as $\pi-a, \pi-b, \pi-c$ and the dual triangle edges as $\pi-A, \pi-B, \pi-C$ (*the spherical dual principle*).

By calculating the area of each triangle in the same class and taking the sum of the areas, we easily obtain the following two formulae:

$$\text{vol (I)} + \text{vol (III)} = \text{vol (II)} + \text{vol (IV)} = 2\pi \tag{1}$$

or equivalently,

$$\text{vol (I)} - \text{vol (II)} + \text{vol (III)} - \text{vol (IV)} = 0. \tag{2}$$

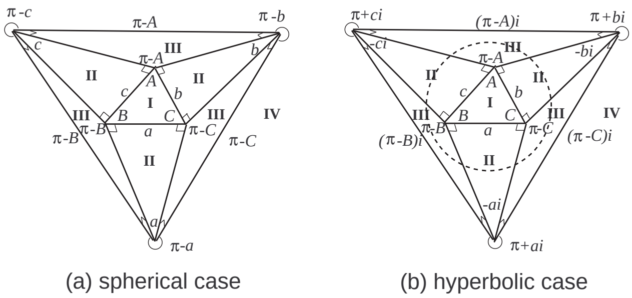

FIGURE 1

Similarly, in the 2-dimensional extended hyperbolic space $\mathbb{S}^2_H$, we can consider a convex hyperbolic triangle and its dual triangle (we need $\frac{\pi i}{2}$ instead of $\frac{\pi}{2}$ in the construction). We then divide the space $\mathbb{S}^2_H$ into 8 triangles and classify the triangles into 4 classes, as done previously (see Fig. 1(b)). If we denote the angles as A, B, C and the edges as a, b, c for the original triangle, then we also easily find the dual triangle angles as $\pi + ai, \pi + bi, \pi + ci$ and the dual triangle edges as $(\pi - A)i, (\pi - B)i, (\pi - C)i$ (*the hyperbolic dual principle*).

In the case of the hyperbolic triangle, we also derive the same formula (2) and formula (1), replacing 2π with -2π. The spherical (or hyperbolic) convex polygons still satisfy both formula (2) and the spherical (or the hyperbolic version of) formula (1).

For convenience, from now on, we will use the Roman number instead of vol(Roman number) as well as other geometric objects.

We studied 2-dimensional case. From now, we study 3-dimensional case. First, we consider a spherical tetrahedron or a spherical convex polyhedron P on the standard 3-sphere $\mathbb{S}^3$. We can make a dual spherical tetrahedron or a dual polyhedron P^* and divide $\mathbb{S}^3$ into several sections of a polyhedron, as in the 2-dimensional case. In this case, we obtain 5 classes of polyhedra and find the following Sforza-Santaló relation, similar to formula (2),

$$\mathrm{I} - \mathrm{II} + \mathrm{III} - \mathrm{IV} + \mathrm{V} = 0, \tag{3}$$

where I = vol (P), II = $\sum_{f\text{:face of }P}$ vol (P_f), III = $\sum_{e\text{:edge of }P}$ vol (P_e), IV = $\sum_{v\text{:vertex of }P}$ vol (P_v), and V = vol (P^*). Here P_f denotes the convex hull of face f of P and dual vertex f^* (corresponding to f) of P^*, P_e denotes the convex hull of edge e of P and dual edge e^* of P^*, and P_v denotes the convex hull of vertex v of P and dual face v^* of P^*. Let us prove the spherical Sforza-Santaló relation by using elementary geometry.

Theorem 2.1. *Any spherical convex polyhedron satisfies the following condition*:

$$P + \mathrm{III} + P^* = \mathrm{II} + \mathrm{IV} = \pi^2 \quad \textit{or simply} \quad \mathrm{I} - \mathrm{II} + \mathrm{III} - \mathrm{IV} + \mathrm{V} = 0.$$

Proof. To prove formula (3), we must show

$$P + \mathrm{III} + P^* = \pi^2 = \frac{\text{vol }(\mathbb{S}^3)}{2} \quad \text{or} \quad \mathrm{II} + \mathrm{IV} = \pi^2.$$

All polyhedra in class III are tetrahedra with four $\frac{\pi}{2}$ long edges and two opposite edges a, a^* in length. Note that the tetrahedron with two opposite edge lengths a and a^* has two opposite dihedral angles, a^* and a (see Fig. 2(a)). Any dihedron with angle θ has a volume vol $(\mathbb{S}^3) \times \frac{\theta}{2\pi}$. Therefore, each tetrahedron in class III, which has two opposite dihedral angles a and a^*, has a volume vol $(\mathbb{S}^3) \times \frac{a}{2\pi} \times \frac{a^*}{2\pi} = \frac{aa^*}{2}$ by the symmetry of space $\mathbb{S}^3$.

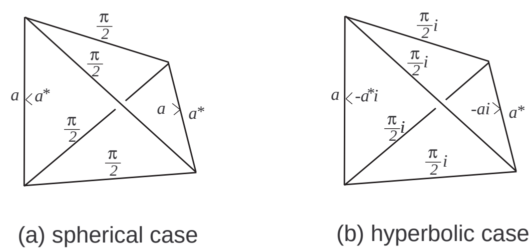

FIGURE 2

Therefore, we have III = $\sum_{a\text{:edge of }P} \frac{aa^*}{2}$, and the formula $P + \mathrm{III} + P^* = \pi^2$ becomes (the already well known) formula (see [3], [4], [6], and [7])

$$P + \sum_{a\text{:edge of }P} \frac{aa^*}{2} + P^* = \pi^2. \tag{4}$$

As the classes are non-overlapping, formula $\mathrm{II} + \mathrm{IV} = \pi^2$ implies formula (4). All polyhedra in class II (resp. IV) are polyhedra constructed by geodesic joining between points of face f (resp. a dual face v^*) in P (resp. P^*) and the dual vertex f^* (resp. the vertex v). If we denote the total area of the faces of P (resp. P^*) as S (resp. S^*), the total volume of the polyhedra of type II (resp.

IV) becomes $\frac{S}{\text{vol }(\mathbb{S}^2)} \times \frac{\text{vol }(\mathbb{S}^3)}{2} = \frac{\pi}{4} S$ (resp. $\frac{\pi}{4} S^*$). Therefore, $\text{II} + \text{IV} = \pi^2$ and $S + S^* = 4\pi$ are equivalent conditions. Thus, all we must do is prove the following formula.

$$S + S^* = 4\pi. \tag{5}$$

If the spherical polyhedron P with a number of faces f has a k-gonal face and the k-gon has angles $\theta_1, \ldots, \theta_k$, then the area of the face becomes $\sum_{l=1}^{k} \theta_l - (k-2)\pi$ from the spherical triangle area formula $A + B + C - \pi$. Taking the sum of the area of the faces of P, we can obtain the total area of the faces of P and P^*,

$$S = \sum_{j=1}^{\mathsf{f}} \left(\sum_{l=1}^{k_j} \theta_{j,l} - (k_j - 2)\pi \right) \quad \text{and} \quad S^* = \sum_{j=1}^{\mathsf{f}^*} \left(\sum_{l=1}^{k_j^*} \theta_{j,l}^* - (k_j^* - 2)\pi \right).$$

The total number of facial angles $\theta_{j,l}$ (resp. $\theta_{j,l}^*$) of P (resp. P^*) is double the total number of edges of P (resp. P^*), i.e., $2\mathsf{e}$ (resp. $2\mathsf{e}^*$). Additionally, we know $\mathsf{v} = \mathsf{f}^*, \mathsf{e} = \mathsf{e}^*, \mathsf{f} = \mathsf{v}^*$ based on the duality between P and P^*. Therefore, the total number of $\theta_{j,l}$ angles in P is the same as the total number of $\theta_{j,l}^*$ angles in P^*.

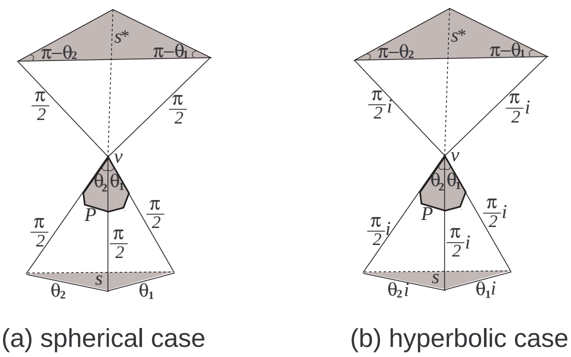

FIGURE 3

Now we consider the polyhedron of type IV at a vertex v of P and make a polyhedron by extending every geodesic arc from v to a point in P with length $\frac{\pi}{2}$ (see Fig. 3(a)). These two polyhedra have two faces that are two spherical polygons s and s^* (= a face of P^* corresponding to the vertex v) which are located in a 2-sphere embedded in $\mathbb{S}^3$ with center v. If we denote the facial angles around v by $\theta_1, \theta_2, \ldots$, then the spherical polygon s has edge lengths $\theta_1, \theta_2, \ldots$. The angles of s^* become $\pi - \theta_1, \pi - \theta_2, \ldots$ by the spherical dual principle. Note that there is a one-to-one correspondence between the facial angles (θ_j) of P and the facial angles $(\pi - \theta_j)$ of P^*.

Denote f_k the number of k-gonal faces of P. Thus, we find (trivially) $\mathsf{f} = \mathsf{f}_3 + \mathsf{f}_4 + \mathsf{f}_5 + \cdots$ and (easily) $2\mathsf{e} = 3\mathsf{f}_3 + 4\mathsf{f}_4 + 5\mathsf{f}_5 + \cdots$. Then we have

$$\sum_{j=1}^{\mathsf{f}}(k_j - 2) = \mathsf{f}_3 + 2\mathsf{f}_4 + 3\mathsf{f}_5 + \cdots = 2\mathsf{e} - 2\mathsf{f} \quad \text{and}$$
$$\sum_{j=1}^{\mathsf{f}^*}(k_j^* - 2) = \mathsf{f}_3^* + 2\mathsf{f}_4^* + 3\mathsf{f}_5^* + \cdots = 2\mathsf{e}^* - 2\mathsf{f}^*.$$

Thus we derive

$$\begin{aligned} S + S^* &= \sum \theta_{j,l} - \pi \sum_{j=1}^{\mathsf{f}}(k_j - 2) + \sum(\pi - \theta_{j,l}) - \pi \sum_{j=1}^{\mathsf{f}^*}(k_j^* - 2) \\ &= \sum_{1}^{2\mathsf{e}} \pi - 2\pi(\mathsf{e} - \mathsf{f} + \mathsf{e}^* - \mathsf{f}^*) \\ &= 2\pi(\mathsf{f} - \mathsf{e} + \mathsf{v}) = 4\pi. \end{aligned}$$

Here, we used the fact that the surface of polyhedron P (homeomorphic to $\mathbb{S}^2$) has an Euler characteristic of 2. As a result, we have proven the theorem. □

Similarly, in the extended hyperbolic space $\mathbb{S}^3_H$, we can consider a convex hyperbolic polyhedron and its dual polyhedron. We can divide the space $\mathbb{S}^3_H$ into several polyhedral pieces and 5 classes, just as in the spherical case. Now let us prove the hyperbolic Sforza-Santaló relation by using elementary geometry.

Theorem 2.2. *All hyperbolic convex polyhedra satisfy the following condition*

$$P + \mathrm{III} + P^* = \mathrm{II} + \mathrm{IV} = \pi^2 i^3 \quad \textit{or simply} \quad \mathrm{I} - \mathrm{II} + \mathrm{III} - \mathrm{IV} + \mathrm{V} = 0.$$

Proof. To prove the Sforza-Santaló relation (3), we must prove $\mathrm{II} + \mathrm{IV} = \frac{\mathrm{vol}\,(\mathbb{S}^3_H)}{2} = \pi^2 i^3$.

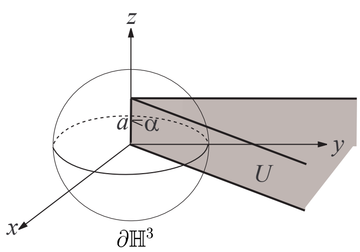

FIGURE 4

All polyhedra in class III are tetrahedra with four $\frac{\pi i}{2}$ long edges and two opposite edges of length a, a^* and dihedral angles $-a^* i, -ai$ (see Fig. 2(b) and

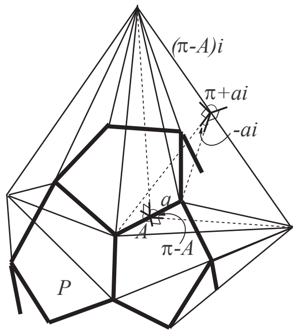

FIGURE 5

the angle definition in [1]). If we consider the given polyhedron in terms of the Kleinian model, we can express it more simply (see Fig. 4) and the volume is represented by

$$\lim_{\epsilon\to 0}\int_U \frac{dx\wedge dy\wedge dz}{(d_\epsilon^2-(x^2+y^2+z^2))^2}=\lim_{\epsilon\to 0}\int_0^{z_0}\int_0^{\alpha}\int_0^{\infty}\frac{r}{(d_\epsilon^2-(r^2+z^2))^2}drd\theta dz,$$

where z_0 is the z-coordinate value corresponding to the edge length a. The existence of the value is supported by Proposition 3.2 in [2]. Note that we can calculate the volume of a dihedron (by considering $\alpha = 2\pi$ in Fig. 4) with dihedral angle β using Corollary 4.9 in [2], that is, $-\beta\pi i$. Therefore, we can derive the volume of the polyhedron in Fig. 2(b) as $-(-ai)\pi i \times \frac{-a^* i}{2\pi} = \frac{aa^*}{2}i$ using the rotational symmetry of the volume around the z-axis. Thus, we find the following formula (the hyperbolic version of formula (4)) in $\mathbb{S}^3_H$:

$$P+\sum_{a:\text{edge of }P}\frac{aa^*}{2}i+P^*=\pi^2 i^3. \tag{6}$$

If we consider the model space as the Kleinian model instead of $\mathbb{S}^3_H$, we have to replace $\pi^2 i^3$ with 0 in (6), since the Kleinian model lose the half volume of $\mathbb{S}^3_H$. We can also show the hyperbolic dual principle for a convex polyhedron; we find $a^* = (\pi - A)i$ (see Fig. 5). Therefore, the identity (6) can be rewritten as

$$P-\sum_{a:\text{edge of }P}\frac{a(\pi-A)}{2}+P'=0, \tag{7}$$

where $P' = P^* \cap K^3$ and A is the corresponding dihedral angle of P at edge a (see [5], [6], and [8]).

To prove the formula (6), we must show that $\mathrm{II}+\mathrm{IV} = \pi^2 i^3$. Any polyhedron of type II or IV has a volume of $\frac{\text{vol }(\mathbb{S}^3_H)}{2\text{vol }(\mathbb{S}^2_H)}s = \frac{\pi i}{4}s$ by Theorem 4.1, given in [2]. By taking the sum of the volumes of type II and type IV polyhedra, we find $\frac{\pi i}{4}(S+S^*) = \pi^2 i^3$. Therefore $\mathrm{II}+\mathrm{IV} = \pi^2 i^3$ and $S+S^* = 4\pi i^2$ are equivalent conditions. Thus all we must do is prove the following formula,

$$S + S^* = -4\pi. \tag{8}$$

We know that the area of the spherical triangle is $A+B+C-\pi$ and that the area of the triangle on the extended hyperbolic space is $\pi - A - B - C$ (see Proposition 4.4 given in [2]), i.e., there is only sign difference between them. Thus a similar process can applied to the hyperbolic case to derive formula (8).

Notice that if we denote the facial angles around v of P as $\theta_1, \theta_2, \ldots$, then the corresponding polygon s in $\mathbb{S}^3_H$ has edge lengths of $\theta_1 i, \theta_2 i, \ldots$. Additionally, the angles of s^* become $\pi - \theta_1, \pi - \theta_2, \ldots$ by the hyperbolic dual principle (see Fig. 3(b)). Therefore we have the same relation between the facial angles (θ_i) of P and the facial angles $(\pi - \theta_i)$ of P^* in the spherical space $\mathbb{S}^3$ and in the extended hyperbolic space $\mathbb{S}^3_H$.

As a result, we can conclude the theorem. □

Remark 2.3. If we choose $d_\epsilon = 1 + \epsilon i$ instead of $d_\epsilon = 1 - \epsilon i$, many geometric quantities are changed; however, the formulae (6), (7), (8) and Theorem 2.2 are satisfied only change of $\pi^2 i^3$ with $\pi^2(-i)^3$.

In this paper, we considered only a hyperbolic convex polygon or a polyhedron. If we draw a general polyhedron lying across the ideal boundary, then we need precise definitions of the dual polygon and dual polyhedron. Naturally, we expect that our two theorems will be satisfied for more general cases from the exact definitions. Additionally, we suspect that Theorems 2.1 and 2.2 could be easily generalized to the higher dimensional case. We leave these problems as the object of further research by readers. We suspect that the n-dimensional version of the formula (3) is the same as Proposition 4.1 in [8].

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