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HYERS-ULAM STABILITY OF MAPPINGS FROM A RING A INTO AN A-BIMODULE

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To the memory of Bwa Hmad Khoudda n Ait Guetto

Abstract. We deal with the Hyers-Ulam stability problem of linear mappings from a vector space into a Banach one with respect to the following functional equation:

$$f\left(\frac{-x+y}{3}\right)+f\left(\frac{x-3z}{3}\right)+f\left(\frac{3x-y+3z}{3}\right)=f(x).$$

We then combine this equation with other ones and establish the Hyers-Ulam stability of several kinds of linear mappings, among which the algebra ($*$-) homomorphisms, the derivations, the multipliers and others. We thus repair and improve some previous assertions in the literature.

1. Introduction and preliminaries

The Hyers-Ulam stability problem of a functional equation, or of a kind of mappings with respect to such an equation, is by now folklore. Whether an approximate solution of a functional equation can be approximated by an exact solution of the same equation is the soul of the Hyers-Ulam stability problem. This problem was reformulated in the frame of groups by S. M. Ulam [20] as follows: if G_1 is a group, (G_2, d) is a metric group and $\epsilon > 0$ is a scalar, does it exist a number $\delta > 0$ such that, whenever a function $f : G_1 \to G_2$ satisfies the inequality

$$d(f(xy), f(x)f(y)) < \delta,\ \forall x, y \in G_1,$$

there exists a group homomorphism $T : G_1 \to G_2$ such that

$$d(f(x), T(x)) < \epsilon,\ \forall x \in G_1.$$

Whenever the answer to this problem is in the affirmative, one says that the group homomorphisms from G_1 into G_2 are stable with respect to the equation $f(xy) = f(x)f(y)$, $\forall x, y \in G_1$, and the uniform (or the Ulam) approximation.

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A partial solution of Ulam's problem is given by D. H. Hyers [9]. The latter showed that, under a weak continuity condition on f, the additive mappings from a real Banach space E into another one F are stable with respect to the Cauchy equation

$$f(x+y) = f(x) + f(y)$$

and the Ulam approximation. Later, the Hyers-Ulam stability problem has known a wide expansion in different directions. On one hand, the approximation condition has been improved by different authors. For instance, T. M. Rassias [18] considered a non uniform approximation given by the so-called bounded Cauchy differences. This new approximation was generalized by K. W. Jun and H. M. Kim [11]. It seems that, up to now, the best approximation condition was given by P. Găvruta [8], it is worth, however, to mention also that given by L. Cadariu and V. Radu in [4, 5]. On the other hand, different proofs of the Hyers-Ulam stability problem mainly of additive mappings between Banach spaces were produced [8, 9, 15, 18]. A third direction consists of the functional equation with respect to which a kind of functions are shown to be stable [9, 10, 18]. The first one considered was the Cauchy equation $f(x+y) = f(x)+f(y)$, then [4, 12] the Jensen equation $2f(\frac{x+y}{2}) = f(x)+f(y)$ and so on. Further, such equations were combined with others in order to show the Hyers-Ulam stability of (linear or) additive mappings with further properties such as being a ring homomorphism, a derivation or so on (see for details [1, 3, 7, 13, 14, 17]). In [15], it is given a general equation depending on two parameters so that many of the results prior to [15] correspond to special values of these parameters.

In this paper, after the foregoing introductory section, we will first give in Section 2 several comments on the paper [6]. We will show that some assertions therein are inexact. We then provide rectifications and improvements to such assertions (see Proposition 2.2).

In Section 3, we deal with additive mappings from a linear space A into a Banach space B and establish (see Theorem 3.1) their Hyers-Ulam-Găvruta-stability with respect to the following equation introduced in [6]:

$$\text{(E)} \qquad f\left(\frac{-x+y}{3}\right) + f\left(\frac{x-3z}{3}\right) + f\left(\frac{3x-y+3z}{3}\right) = f(x).$$

In particular (see Remark 3.2(3)), if a mapping $f : A \to B$ satisfies (E) up to a function φ bounded by some $M > 0$, then there exists a unique additive mapping $h : A \to B$ such that:

$$\|h(x) - f(x)\| \leq \frac{3}{2}M, \quad \forall x \in A.$$

As a matter of fact (see Theorem 3.1), when (E) is combined with another equation such as $f \circ \sigma = \gamma \circ f$ for some positively homogeneous mappings $\sigma : A \to A$ and $\gamma : B \to B$ with γ continuous (involutions for instance), then h turns out to satisfy $h \circ \sigma = \gamma \circ h$. In case of involutions, h is a $*$-mapping.

We then establish the Hyers-Ulam-Găvruta-stability of linear mappings with respect to an equation (13) similar to (E).

In Section 4, we are concerned with additive (linear) mappings from a ring (algebra) A into a Banach space (or Algebra) B satisfying additional properties such as being a homomorphism, a derivation, a multiplier and so on. We provide general results (see Theorem 4.1 and Theorem 4.3) covering a wide class of such mappings. As special cases, we show the Hyers-Ulam stability of ring ($*$-) homomorphisms, ring derivations and several other types of mappings with respect to (E), combined with appropriate equations, and the bounded Cauchy differences as approximation. We also get similar results but with the Cadariu-Radu approximation. Since our results depend on arbitrary mappings σ and the γ_j's, a wide range of other applications can be obtained according to such maps one considers.

In all what follows, the vector spaces and algebras in consideration will have $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ as basic field. Unless the contrary is stated, A will stand for an arbitrary vector space and B for a Banach space with respect to some norm $|| \, ||$. We will denote by f a function from A into B and by $\mathbb{K}^*$ the set $\mathbb{K} \setminus \{0\}$. We will assume that $f(0) = 0$ although this is sometimes automatically satisfied under the conditions in consideration. We will also denote by $\mathbb{N}$ (resp. $\mathbb{N}^*$) the set of all non negative (resp. positive) integers. If A happens to be a unital C^*-algebra, we will denote by $\mathcal{U}(A)$, the set of all unitary elements of A. This is $\mathcal{U}(A) = \{a \in A : a^*a = aa^* = e\}$, where obviously e stands for the unit of A. Finally, we will denote by $\mathbb{T}_1$ the set $\{z \in \mathbb{K} : |z| = 1\}$. In the real case this is only $\{-1, 1\}$.

2. Remarks on [6]

In [6], Eshaghi Gordji et al. give the following result asserting the supper stability of Jordan $*$-homomorphisms between C^*-algebras:

Theorem 2.1 ([6, Theorem 2.2]). *Let A and B be unital C^*-algebras, $p < 1$ and θ nonnegative real numbers, and $f : A \to B$ a mapping satisfying $f(0) = 0$ and*

$$f(3^n ux) = f(3^n u)f(x), \quad \forall u \in U(A), \qquad x \in A, \tag{0.1}$$

$$\left\| f\left(\frac{b-a}{3}\right) + f\left(\frac{a-3\mu c}{3}\right) + \mu f\left(\frac{3a+3c-b}{3}\right) \right\| \leq \|f(a)\|, \tag{0.2}$$
$$\forall \mu \in \mathbb{T}_1, \ \forall (a,b,c) \in A^3,$$

$$\|f(3^n \ u^*) - f(3^n u)^*\| \leq 2\theta 3^{np}, \quad \forall u \in \mathcal{U}(A), n \in \mathbb{N}. \tag{0.3}$$

Then f is a Jordan $*$-homomorphism.

Actually the only mapping f satisfying (0.2) is the null function. For, the additivity of f is guarantied by Lemma 2.1 of the same paper whenever $\mu = 1$.

But $\mu = -1$ and $a = c = 0$ in (0.2) gives $f(\frac{b}{3}) = 0$ for every $b \in A$ which ensures $f = 0$ everywhere.

It seems that, under these conditions, nothing guaranties the homogeneity of f. The best thing one can expect of (0.2), without any μ, is that f is additive. Moreover, in presence of (0.2), the conditions (0.1) and (0.3) imply respectively $f(ux) = f(u)f(x)$, and $f(u^*) = f(u)^*$ for every $u \in \mathcal{U}(A)$ and $x \in A$. The latter equality is ensured by the fact that $p < 1$. If $p \geq 1$ however, the sequence $(3^{-n}\theta 3^{np})_n$ does not converge and this equality may fail to hold. For this reason, Theorem 2.3, of [6] need not hold.

The following proposition shows the super stability of linear mappings with respect to an equation similar to (E). It improves and repairs Theorem 2.2 and Theorem 2.3 therein.

Proposition 2.2. *If f satisfies, for every $(a, b, c) \in A^3$ and every $\mu \in \mathbb{T}_1$:*

$$\left\| f\left(\mu \frac{b-a}{3}\right) + f\left(\mu \frac{a-3c}{3}\right) + \mu f\left(\frac{3a+3c-b}{3}\right) \right\| \leq \|f(a)\|, \tag{1}$$

then f is linear. Moreover, if A is a C^*-algebra and B is an algebra with an involution *, then f is a ring *-homomorphism provided it enjoys $f(uv^*) = f(u)f(v)^*$ for all $u, v \in \mathcal{U}(A)$.

In the real case, we additionally assume that, for every $x \in A$, the mapping $f_x : \mathbb{R} \to B$, $t \mapsto f(tx)$ is bounded on some $]-\epsilon_x, \epsilon_x[$, $\epsilon_x > 0$.

Proof. We first fix $\mu = 1$. If we take in (1), $a = b = c = 0$, we will obtain that $f(0) = 0$. Now taking in (1) $a = b = 0$, we get $f(-c) = -f(c)$ for every $c \in A$. Further, $a = 0$ in (1) and the oddness of f yield $f(\frac{b}{3}) - f(c) + f(c - \frac{b}{3}) = 0$. Whence the additivity of f.

For the homogeneity of f, let μ be arbitrary in $\mathbb{T}_1$. If we take in (1) $a = b = 0$ and c arbitrary in A, we get $f(\mu c) = \mu f(c)$. Since f is additive, $f(rx) = rf(x)$ for all $x \in A$ and $r \in \mathbb{Q}$. Therefore the homogeneity of f reduces to $f(rx) = rf(x)$ for every real r with $0 < r < 1$. But in the complex case, for such a number r, $2r$ is the sum of two numbers from $\mathbb{T}_1$. Hence, by additivity, f is homogeneous.

In the real case, if $(r_n)_n$ is a sequence of rational numbers converging to r, then we have, for every integer p: $||f(rx) - r_n f(x)|| = \frac{1}{p}||f(p(r - r_n)x)||$. But for n large enough, $p(r - r_n) \in]-\epsilon_x, \epsilon_x[$. Hence $||f(p(r - r_n)x)|| \leq M$ for some $M > 0$. Letting n tend to infinity yields $||f(rx) - rf(x)|| \leq \frac{1}{p}M$. Now if p tends to infinity, we get $f(rx) = rf(x)$, whence the homogeneity of f and the linearity follows.

In order to show that f is a $*$-homomorphism whenever A is a C^*-algebra and B is an algebra with an involution * such that $f(uv^*) = f(u)f(v)^*$ for all $u, v \in \mathcal{U}(A)$, notice that, for every $u \in \mathcal{U}(A)$, $f(u) = f(u)f(e)^*$ and similarly $f(u^*) = f(e)f(u)^*$. Then, applying * to the second equality, we get $f(u)f(e)^* = f(u^*)^*$, whereby $f(u^*) = f(u)^*$ for every $u \in \mathcal{U}(A)$. Since, in any unital C^*-algebra, every element is a finite combination of unitary elements ([2], p.

70), f is a $*$-linear mapping. On the other hand, for every $u,\ v \in \mathcal{U}(A)$, $f(uv) = f(u)f(v^*)^* = f(u)f(v)$. Again, by the decomposability of elements of a C^*-algebra as finite combinations of unitary elements, $f(xy) = f(x)f(y)$ for every $x, y \in A$. Whereby f is a $*$-homomorphism and the proof is achieved. $\square$

In the same paper [6], Eshaghi Gordji et al. give the following assertion:

Theorem 2.3 ([6, Theorem 2.4]). *Let A and B be unital C^*-algebras and $f : A \to B$ a mapping for which there exists a function $\varphi : A^3 \to \mathbb{R}^+$ such that:*

$$\sum_{i=0}^{+\infty} 3^n \varphi\left(\frac{a}{3^n}\frac{b}{3^n}\frac{c}{3^n}\right) < \infty, \quad \forall (a,b,c) \in A^3, \tag{0.4}$$

$$\lim_{n\to\infty} 3^{2n}\varphi\left(\frac{a}{3^n}, \frac{b}{3^n}, \frac{c}{3^n}\right) = 0, \quad \forall (a,b,c) \in A^3, \tag{0.5}$$

$$\|f(3^n u^*) - f(3^n u)^*\|_B \le \varphi(3^n u, 3^n u, 3^n u), \quad \forall u \in \mathcal{U}(A), \tag{0.6}$$

(0.7)
$$\left\| f\left(\frac{\mu b - a}{3}\right) + f\left(\frac{a-3c}{3}\right) + \mu f\left(\frac{3a+3c-b}{3}\right) - f(a) + f(c^2) - f(c)^2 \right\|$$
$$\le \varphi(a,b,c), \quad \forall \mu \in \mathbb{T}_1,\ \forall (a,b,c) \in A^3.$$

Then there exists a unique Jordan $*$-homomorphism $h : A \to B$ such that

$$\|h(a) - f(a)\|_B \le \sum_{i=0}^{+\infty} 3^n \varphi\left(\frac{a}{3^n}, \frac{2a}{3^n}, 0\right), \ \forall a \in A.$$

The first thing to be noticed here is that the condition (0.5) implies the other (0.4). Actually, as we will see later on, the latter is sufficient to show the desired results. Moreover, the only $*$-homomorphism k satisfying the equation

$$k\left(\frac{\mu b - a}{3}\right) + k\left(\frac{a-3c}{3}\right) + \mu k\left(\frac{3a+3c-b}{3}\right) - k(a) + k(c^2) - k(c)^2 = 0$$

is the null one. For, if we put $a = b = 0$, we obtain $k(c) = \mu k(c)$ for every $\mu \in \mathbb{T}_1$ and every $c \in A$. But this is true only if $k = 0$. Hence the assertion above is also vacuous.

In the following section, we will provide results repairing and improving this assertion.

3. Hyers-Ulam stability of additive mappings

We start this section with the following lemma showing the Hyers-Ulam stability of additive mappings with respect to the equation:

$$f\left(\frac{b-a}{3}\right) + f\left(\frac{a-3c}{3}\right) + f\left(\frac{3a+3c-b}{3}\right) = f(a).$$

Actually it is a generalization of the Hyers-Ulam-Găvruta stability since, in the approximation condition, a large range of mappings φ are allowed, while the classical Hyers-Ulam-Găvruta result is given for φ's corresponding in our result to the case $\epsilon = 1$ (see the lemma bellow).

In all the following, we will say that a mapping σ from a linear space E into another one F is homogeneous if, for every $r \in \mathbb{K}$ and $x \in E$, $\sigma(rx) = r\sigma(x)$. It is said to be positively homogeneous of order $p > 0$ if, for every $r \geq 0$ and $x \in E$, $\sigma(rx) = r^p\sigma(x)$. Whenever $p = 1$, we will only say that σ is positively homogeneous.

Lemma 3.1. *Suppose that $f(0) = 0$ and that there exist $\epsilon = \pm 1$ and a function $\varphi : A^3 \to \mathbb{R}^+$ such that, for all $(a, b, c) \in A^3$,*

$$\sum_{n=0}^{\infty} 3^{\epsilon n}\varphi\left(\frac{a}{3^{\epsilon n}}, \frac{b}{3^{\epsilon n}}, \frac{c}{3^{\epsilon n}}\right) < +\infty \tag{2}$$

and

$$\left\|f\left(\frac{b-a}{3}\right) + f\left(\frac{a-3c}{3}\right) + f\left(\frac{3a+3c-b}{3}\right) - f(a)\right\| \leq \varphi(a, b, c). \tag{3}$$

Then there exists a unique additive mapping $h : A \to B$ such that

$$\|h(x) - f(x)\| \leq \sum_{n=0}^{\infty} 3^{\epsilon n}\varphi\left(\frac{x}{3^{\epsilon n}}, \frac{2x}{3^{\epsilon n}}, 0\right), \quad \forall x \in A. \tag{4}$$

Moreover, if $\sigma : A \to A$ and $\delta : B \to B$ are positively homogeneous mappings with δ continuous and

$$\|f(\sigma(a)) - \delta(f(a))\| \leq \varphi(a, 0, 0), \quad \forall a \in A, \tag{5}$$

then $h(\sigma(a)) = \delta(h(a))$, $\forall a \in A$.

Proof. For an arbitrary $a \in A$, if we apply (3) to a, $b = 2a$ and $c = 0$, we will get:

$$\left\|3f\left(\frac{a}{3}\right) - f(a)\right\| \leq \varphi(a, 2a, 0), \tag{6}$$

or equivalently

$$\left\|f(a) - \frac{1}{3}f(3a)\right\| \leq \frac{1}{3}\varphi(3a, 2(3a), 0). \tag{7}$$

Hence, for each integer $k \geq 1$,

$$\left\|3^{k+1}f\left(\frac{a}{3^{k+1}}\right) - 3^k f\left(\frac{a}{3^k}\right)\right\| \leq 3^k\varphi\left(\frac{a}{3^k}, \frac{2a}{3^k}, 0\right) \tag{8}$$

and

$$\left\|\frac{1}{3^{k+1}}f(3^{k+1}a) - \frac{1}{3^k}f(3^k a)\right\| \leq \frac{1}{3^{k+1}}\varphi\left(3^{k+1}a, 2.3^{k+1}a, 0\right). \tag{9}$$

By the triangular inequality, for every $(n,m) \in \mathbb{N}^2$,

$$\left\| 3^{\epsilon n} f\left(\frac{a}{3^{\epsilon n}}\right) - 3^{\epsilon m} f\left(\frac{a}{3^{\epsilon m}}\right) \right\| \leq \sum_{k=m}^{n} 3^{\epsilon k} \varphi\left(\frac{a}{3^{\epsilon k}}, \frac{2a}{3^{\epsilon k}}, 0\right). \tag{10}$$

Since the series

$$\sum_{n=0}^{\infty} 3^{\epsilon n} \varphi\left(\frac{a}{3^{\epsilon n}}, \frac{2a}{3^{\epsilon n}}, 0\right)$$

converges in $\mathbb{R}$, for every $a \in A$, the sequence $\left(3^{\epsilon n} f\left(\frac{a}{3^{\epsilon n}}\right)\right)_n$ is Cauchy in B. Hence it converges to some element $h(a) \in B$ satisfying (4), this is:

$$\|h(a) - f(a)\| \leq \sum_{k=0}^{\infty} 3^{\epsilon k} \varphi\left(\frac{a}{3^{\epsilon k}}, \frac{2a}{3^{\epsilon k}}, 0\right), \quad \forall a \in A.$$

Let us show that the so defined mapping h from A into B is additive. To this aim, take in (3) $a = b = 0$, and $c = \frac{x}{3^{\epsilon n}}$ with $x \in A$, then multiply both sides by $3^{\epsilon n}$ and get:

$$\left\| 3^{\epsilon n} f\left(\frac{x}{3^{\epsilon n}}\right) + 3^{\epsilon n} f\left(-\frac{x}{3^{\epsilon n}}\right) \right\| \leq 3^{\epsilon n} \varphi\left(0, 0, \frac{x}{3^{\epsilon n}}\right). \tag{11}$$

Making n tend to infinity, we get $h(-x) = -h(x)$ for every $x \in A$. Now, if we take in (3) $a = 0$, $b = \frac{x}{3^{\epsilon n}}$ and $c = \frac{y}{3^{\epsilon n}}$, with $x, y \in A$, then multiply again both sides by $3^{\epsilon n}$, we get:

$$\left\| 3^{\epsilon n} f\left(\frac{x}{3^{\epsilon n}}\right) + 3^{\epsilon n} f\left(\frac{y}{3^{\epsilon n}}\right) + 3^{\epsilon n} f\left(-\frac{x}{3^{\epsilon n}} - \frac{y}{3^{\epsilon n}}\right) \right\| \leq 3^{\epsilon n} \varphi\left(0, \frac{x}{3^{\epsilon n}}, \frac{y}{3^{\epsilon n}}\right). \tag{12}$$

As n tends on infinity, we obtain the additivity of h, using the fact that $h(-x) = -h(x)$ for $x \in A$.

Now, let us show the unicity of h. Assume that there is some other mapping $k : A \to B$ satisfying (4). Then, for every $a \in A$ and every $n \in \mathbb{N}$, one has:

$$\begin{aligned}
\|h(a) - k(a)\| &\leq 3^{\epsilon n} \|h\left(\frac{a}{3^{\epsilon n}}\right) - f\left(\frac{a}{3^{\epsilon n}}\right)\| + 3^{\epsilon n} \|k\left(\frac{a}{3^{\epsilon n}}\right) - f\left(\frac{a}{3^{\epsilon n}}\right)\| \\
&\leq 2\left(3^{\epsilon n} \sum_{i=0}^{\infty} 3^{\epsilon i} \varphi\left(\frac{a}{3^{\epsilon(i+n)}}, \frac{2a}{3^{\epsilon(i+n)}}, 0\right)\right) \\
&\leq 2\sum_{i=0}^{\infty} 3^{\epsilon(i+n)} \varphi\left(\frac{a}{3^{\epsilon(i+n)}}, \frac{2a}{3^{\epsilon(i+n)}}, 0\right) \\
&\leq 2\sum_{i=n}^{\infty} 3^{\epsilon n} \varphi\left(\frac{a}{3^{\epsilon n}}, \frac{2a}{3^{\epsilon n}}, 0\right).
\end{aligned}$$

Since the last member tends to 0 as n tends to infinity, $h(a) = k(a)$. As a is arbitrary, we conclude that $h = k$.

Now, take in (5) $a = \frac{x}{3^{\epsilon n}}$ and get:

$$\left\| 3^{\epsilon n} f\left(\sigma\left(\frac{x}{3^{\epsilon n}}\right)\right) - 3^{\epsilon n} \delta\left(f\left(\frac{x}{3^{\epsilon n}}\right)\right) \right\| \leq 3^{\epsilon n} \varphi\left(\frac{x}{3^{\epsilon n}}, 0, 0\right), \quad \forall x \in A.$$

Hence, using the homogeneity condition on σ and δ,

$$\left\|3^{\epsilon n} f\left(\frac{\sigma(x)}{3^{\epsilon n}}\right) - \delta\left(3^{\epsilon n} f\left(\frac{x}{3^{\epsilon n}}\right)\right)\right\| \leq 3^{\epsilon n}\varphi\left(\frac{x}{3^{\epsilon n}}, 0, 0\right), \quad \forall x \in A.$$

Since δ is continuous as n tends to infinity, we get the required conclusion. □

Remark 3.2. 1. In the lemma above, in case $\epsilon = 1$, the condition $f(0) = 0$ is automatically satisfied since, from (2) derives immediately $\varphi(0,0,0) = 0$ and by taking $a = b = c = 0$ in (3), we conclude that also $f(0) = 0$.

2. Under the conditions of the lemma above, if A and B are endowed with linear involutions both denoted by * and if that of B is continuous such that,

$$\|f(a^*)) - (f(a))^*\| \leq \varphi(a, 0, 0), \quad \forall a \in A,$$

then the mapping h given by the lemma is involutive, in the sense that $h(x^*) = (h(x))^*$, $\forall x \in A$. In particular, if A and B are C^*-algebras and h is an algebra homomorphism, then it is also a star algebra homomorphism. We will give later on conditions under which h is an algebra homomorphism.

3. Whenever $\epsilon = -1$, every bounded function φ satisfies (2). Hence the lemma above holds for such a mapping.

The following theorem gives the (generalized) Hyers-Ulam-Găvruta stability of linear mapping.

Theorem 3.3. *Suppose that* $f(0) = 0$ *and that there exist* $\epsilon = \pm 1$ *and a function* $\varphi : A^3 \to \mathbb{R}^+$ *satisfying* (2) *and*

$$\left\|f\left(\mu\frac{b-a}{3}\right) + f\left(\mu\frac{a-3c}{3}\right) + \mu f\left(\frac{3a+3c-b}{3}\right) - \mu f(a)\right\| \leq \varphi(a,b,c), \tag{13}$$
$$\forall (a,b,c) \in A^3, \quad \mu \in \mathbb{T}_1.$$

Then there exists a unique linear mapping $h : A \to B$ *such that* (4) *holds.*
In the real case, we additionally assume that, for every $x \in A$, *the mappings*

$$\begin{array}{llcll} f_x : & \mathbb{R} \to B & \textit{and} & \widetilde{\varphi}_x : & \mathbb{R} \to B \\ & t \mapsto f(tx) & & & t \mapsto \sum_{i=0}^{+\infty} 3^{\epsilon n}\varphi(\frac{tx}{3^{\epsilon n}}, \frac{2tx}{3^{\epsilon n}}, 0) \end{array}$$

are bounded on some interval $]-\epsilon_x, \epsilon_x[$, $\epsilon_x > 0$.

Proof. By Lemma 3.1, whenever $\mu = 1$, there exists a unique additive mapping h satisfying (4). Moreover h is given by the formula $h(x) = \lim_{n\to\infty} 3^{\epsilon n} f(\frac{x}{3^{\epsilon n}})$. Then taking in (13), $b = a = 0$ and $c = \frac{x}{3^{\epsilon n}}$, with $x \in A$, we get:

$$\left\|3^{\epsilon n} f\left(-\mu\frac{x}{3^{\epsilon n}}\right) + \mu 3^{\epsilon n} f\left(\frac{x}{3^{\epsilon n}}\right)\right\| \leq 3^{\epsilon n}\varphi\left(0, 0, \frac{x}{3^{\epsilon n}}\right). \tag{14}$$

Letting n tend to infinity, we obtain, for every $x \in A$ and $\mu \in \mathbb{T}_1$, $h(\mu x) = \mu h(x)$. Hence, by additivity, the homogeneity of h reduces to $h(rx) = rh(x)$ for every real r with $0 < r < 1$. But in the complex case $2r$ is the sum of two elements from $\mathbb{T}_1$ and the homogeneity follows.

In the real case, assume that some sequence $(r_n)_{n\in\mathbb{N}}$ of rational numbers converges to r and consider an arbitrary $p>0$. Then

$$\exists n_p \in \mathbb{N},\quad \forall n \geq n_p: \quad p(r-r_n) \in]-\epsilon_x, \epsilon_x[.$$

Then, by the hypotheses on $\widetilde{\varphi}_x$ and f_x, there is some $M>0$ so that for $n\geq n_p$

$$\sum_{i=0}^{\infty} 3^{\epsilon i}\varphi\left(\frac{p(r-r_n)x}{3^{\epsilon i}}, \frac{2p(r-r_n)x}{3^{\epsilon i}}, 0\right) \leq M \quad \text{ and } \quad \|f\left(p(r-r_n)x\right)\| \leq M.$$

Hence,

$$\begin{aligned}
&\|h(rx) - r_n h(x)\| \\
&= \frac{1}{p}\|h(p(r-r_n)x)\| \\
&\leq \frac{1}{p}\left(\|h(p(r-r_n)x) - f(p(r-r_n)x)\| + \|f(p(r-r_n)x)\|\right) \\
&\leq \frac{1}{p}\left(\sum_{i=0}^{\infty} 3^{\epsilon i}\varphi\left(\frac{p(r-r_n)x}{3^{\epsilon i}}, \frac{2p(r-r_n)x}{3^{\epsilon i}}, 0\right) + \|f(p(r-r_n)x)\|\right) \\
&\leq \frac{2}{p}M.
\end{aligned}$$

Therefore, when n tends to infinity, we get $\|h(rx) - rh(x)\| \leq \frac{1}{p}M$. Letting p tend to infinity, we come to $h(rx) = rh(x)$ and then to the homogeneity of h. □

4. Hyers-Ulam stability of mappings between rings

In this section, we will combine (3) or (13) with other equations in order to establish the Hyers-Ulam stability of several types of mappings. To this purpose, let $L(E,F)$ denote the space of all linear mappings from a vector space E into another F.

Theorem 4.1. *Given $n\in\mathbb{N}^*$. For every $1\leq j\leq n$, consider the spaces C_j and D_j and the mappings $g_j : A^2 \to C_j$, $k_j : A^2 \to D_j$ and $\gamma_j : C_j \times D_j \to B$ such that $\{C_j, D_j\} \subset \{A, B\}$, g_j and k_j are (independently) linear or the composition of f with some element of $L(A^2, A)$ and γ_j is positively homogeneous of order 2 and continuous in the variable(s) in B. Let also $\sigma : A^2 \to A$ be positively homogeneous of order 2.*

If there exist $\epsilon = \pm 1$ and a function $\varphi : A^3 \to \mathbb{R}^+$ satisfying (2), (3) and the following two conditions:

$$\lim_{k\to\infty} 3^{2\epsilon k}\varphi\left(\frac{a_1}{3^{\epsilon k}}, \frac{a_2}{3^{\epsilon k}}, 0\right) = 0, \quad \forall (a_1, a_2) \in A^2, \tag{15}$$

$$\left\| f\left(\sigma(a_1,a_2)\right) - \sum_{j=1}^{n} \gamma_j\left(g_j(a_1,a_2), k_j(a_1,a_2)\right) \right\| \leq \varphi(a_1,a_2,0), \quad \forall (a_1,a_2) \in A^2, \tag{16}$$

then there exists a unique additive mapping $h : A \to B$ *such that* (4) *holds and*

$$h(\sigma(x,y)) = \sum_{j=1}^{n} \gamma_j(\tilde{g}_j(x,y), \tilde{k}_j(x,y)), \quad \forall (x,y) \in A^2. \tag{17}$$

Here, $\tilde{g}_j = g_j$ *if* g_j *is linear and* $\tilde{g}_j = h \circ l$ *if* $g_j = f \circ l$ *for some* $l \in L(A^2, A)$. *Similarly,* $\tilde{k}_j = k_j$ *if* k_j *is linear and* $\tilde{k}_j = h \circ m$ *if* $k_j = f \circ m$ *for some* $m \in L(A^2, A)$.

Proof. By Lemma 3.1, there exists a unique additive mapping $h : A \to B$ such that (4) holds. Now, if we take in (16), $a_1 = \frac{x}{3^{\epsilon k}}$ and $a_2 = \frac{y}{3^{\epsilon k}}$, then, using the homogeneity condition on the maps σ and the γ_j's, we get, for every $(x,y) \in A^2$:

$$\left\| 3^{2\epsilon k} f\left(\frac{\sigma(x,y)}{3^{2\epsilon k}}\right) - \sum_{j=1}^{n} \gamma_j \left(3^{\epsilon k} g_j\left(\frac{x}{3^{\epsilon k}}, \frac{y}{3^{\epsilon k}}\right), 3^{\epsilon k} k_j\left(\frac{x}{3^{\epsilon k}}, \frac{y}{3^{\epsilon k}}\right)\right) \right\|$$
$$\leq 3^{2\epsilon k} \varphi\left(\frac{x}{3^{\epsilon k}}, \frac{y}{3^{\epsilon k}}, 0\right).$$

But if g_j is linear, then, for every k and every $(x,y) \in A^2$, $3^{\epsilon k} g_j(\frac{x}{3^{\epsilon k}}, \frac{y}{3^{\epsilon k}})) = g_j(x,y)$; otherwise, $g_j = f \circ l$ for some linear l and then

$$3^{\epsilon k} g_j\left(\frac{x}{3^{\epsilon k}}, \frac{y}{3^{\epsilon k}}\right) = 3^{\epsilon k} f\left(\frac{l(x,y)}{3^{\epsilon k}}\right).$$

The same hold also for k_j. Moreover, by hypothesis $3^{2\epsilon k}\varphi(\frac{x}{3^{\epsilon k}}, \frac{y}{3^{\epsilon k}}, 0)$ tends to 0 as k tends to infinity. Hence, by the continuity condition on the γ_j's, we obtain as k tends to ∞:

$$h(\sigma(x,y)) = \sum_{j=1}^{n} \gamma_j \left(\tilde{g}_j(x,y), \tilde{k}_j(x,y)\right), \forall (x,y) \in A^2.$$

This achieves the proof. □

Remark 4.2. The "continuity in the variable(s) in B" means that, whenever for instance $C_j = A$ and $D_j = B$, for every $a \in A$, the partial mapping $\gamma_{j,a} : b \mapsto \gamma_j(a,b)$ is continuous from B into itself. Similarly, if $D_j = A$ and $C_j = B$, the partial mapping $\gamma_j^a : b \mapsto \gamma_j(b,a)$ is continuous. Finally, if $C_j = D_j = B$, then γ_j is continuous in both variables.

In the following, we give a slightly different version of this theorem. Its proof is omitted since it is similar to the one above.

Theorem 4.3. *Let* n, j, C_j, D_j *and* γ_j *be as in Theorem 4.1 and let* $g_j : A \to C_j$, $k_j : A \to D_j$ *and* $\sigma : A \to A$ *be mappings such that* σ *is positively homogeneous of order 2 and* g_j *and* k_j *are* (*independently*) *linear or equal to* f.

If there exist $\epsilon = \pm 1$ *and a function* $\varphi : A^3 \to \mathbb{R}^+$ *satisfying* (2), (3) *and*

$$\lim_{k \to \infty} 3^{2\epsilon k} \varphi\left(\frac{a}{3^{\epsilon k}}, 0, 0\right) = 0, \quad \forall a \in A, \tag{18}$$

$$\left\| f\left(\sigma(a)\right) - \sum_{j=1}^{n} \gamma_j \left(g_j(a), k_j(a)\right) \right\| \leq \varphi(a,0,0), \quad \forall a \in A, \tag{19}$$

then there exists a unique additive mapping $h : A \to B$ such that (4) *holds and*

$$h(\sigma(x)) = \sum_{j=1}^{n} \gamma_j \left(\tilde{g}_j(x), \tilde{k}_j(x)\right), \quad \forall x \in A. \tag{20}$$

Here, $\tilde{g}_j = g_j$ (resp. $\tilde{k}_j = k_j$) if g_j (resp. k_j) is linear and $\tilde{g}_j = h$ (resp. $\tilde{k}_j = h$) otherwise.

As applications, let us first give some corollaries of Theorem 4.1 or Theorem 4.3 for general Găvruta type approximation. We will then take special φ's in order to get results of type Cadariu-Radu or Rassias. Before that, if B is an A-bimodule (then for every $a \in A$, the mappings $b \mapsto ab$ and $b \mapsto ba$ are continuous form B into itself) and $h : A \to B$ an additive mapping, we will say that h is an Anansa derivation (resp. left multiplier, right multiplier, multiplier) if, for every element $x \in E$, $h(x^2) = h(x)x + xh(x)$ (resp. $h(x^2) = h(x)x$, $h(x^2) = xh(x)$, $h(x^2) = xh(x) = h(x)x$). If A and B are algebras, h will be said to be an Anansa ring homomorphism if, for every element $x \in A$, $h(x^2) = h(x)^2$.

Corollary 4.4. *Assume that A is an algebra and B is a Banach algebra and that $\varphi : A^3 \to \mathbb{R}_+$ satisfies* (2), (3), (18) *and*

$$\left\| f\left(a^2\right) - f(a)^2 \right\| \leq \varphi(a,0,0), \quad \forall a \in A. \tag{21}$$

Then there exists a unique Anansa ring homomorphism $h : A \to B$ enjoying (4).

Proof. It suffices to take in Theorem 4.3: $n = 1$, $C_1 = D_1 = B$, $g_1 = k_1 = f$, $\sigma(a) = a^2$ and $\gamma_1 : B^2 \to B$; $(b_1, b_2) \mapsto b_1 b_2$. Since B is a Banach algebra, γ_1 is continuous. ☐

Corollary 4.5. *Assume that A is an algebra, B an A-bimodule and that $\varphi : A^3 \to \mathbb{R}_+$ satisfies* (2), (3), (18) *and, for some $\alpha_i \in \mathbb{K}$, $i = 1, 2$:*

$$\left\| f\left(a^2\right) - \alpha_1 a f(a) - \alpha_2 f(a) a \right\| \leq \varphi(a,0,0), \quad \forall a \in A. \tag{22}$$

Then there exists a unique additive map $h : A \to B$ enjoying (4) *and*

$$h\left(a^2\right) = \alpha_1 a h(a) + \alpha_2 h(a) a, \quad \forall a \in A. \tag{23}$$

Proof. It suffices to take in Theorem 4.3: $n = 2$, $\sigma(a) = a^2$, $C_1 = D_2 = A$, $D_1 = C_2 = B$, $g_1 = k_2 = Id_A$, $k_1 = g_2 = f$ and $\gamma_1 : A \times B \to B$; $(a, b) \mapsto \alpha_1 ab$ and $\gamma_2 : B \times A \to B$; $(b, a) \mapsto \alpha_2 ba$. Since B is an A-bimodule, the mappings $\gamma_{1,a}$ and γ_2^a are continuous for every $a \in A$. ☐

Remark 4.6. In the corollary above, if $\alpha_1 = \alpha_2 = 1$, then h is an Anansa derivation. If $\alpha_1 = 1$ and $\alpha_2 = 0$, then h is an Anansa right multiplier. Finally if $\alpha_2 = 1$ and $\alpha_1 = 0$, then h is an Anansa left multiplier.

A kind of generalization of this corollary in the spirit of Theorem 4.1 is the following:

Corollary 4.7. *Assume that A is an algebra, B an A-bimodule and that $\varphi : A^3 \to \mathbb{R}_+$ satisfies* (2), (3), (15) *and, for some $\alpha_i \in \mathbb{K}$, $i = 1, 2$:*

$$\|f(xy) - \alpha_1 x f(y) - \alpha_2 f(x) y\| \leq \varphi(x, y, 0), \quad \forall (x, y) \in A^2. \tag{24}$$

Then there exists a unique additive map $h : A \to B$ enjoying (4) *and*

$$h(xy) = \alpha_1 x h(y) + \alpha_2 h(x) y, \quad \forall (x, y) \in A^2.$$

Proof. It suffices to take in Theorem 4.1: $n = 2$, $\sigma(a, b) = ab$, $C_1 = D_2 = A$, $D_1 = C_2 = B$, $g_1 = p_1$, $k_1 = f \circ p_2$, $g_2 = f \circ p_1$, $k_2 = p_2$ and $\gamma_1 : A \times B \to B$; $(a, b) \mapsto \alpha_1 ab$ and $\gamma_2 : B \times A \to B$; $(b, a) \mapsto \alpha_2 ba$. Here p_i stands for the i^{th} projection, $i = 1, 2$. Once again, $\gamma_{1,a}$ and γ_2^a are continuous for every $a \in A$. □

In [4] Cadariu and Radu considered a control by mappings φ enjoying the following condition:

$$\exists L \in]0, 1[, : \varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{2}\varphi(x, y), \ \forall (x, y) \in A^2.$$

Actually a mapping φ satisfying a similar condition in the present framework, namely:

$$\exists L \in]0, 1[, \exists \epsilon = \pm 1 \ : \varphi\left(\frac{x}{3^\epsilon}, \frac{y}{3^\epsilon}, \frac{z}{3^\epsilon}\right) \leq \frac{L}{3^\epsilon}\varphi(x, y, z), \quad \forall (x, y, z) \in A^3 \tag{25}$$

turns out to satisfy also (2). We therefore get the following corollary of Cadariu-Radu type:

Corollary 4.8. *Assume that A is an algebra, B is an A-bimodule and that $\varphi : A^3 \to \mathbb{R}_+$ satisfies* (3) *and* (25). *Then there exists a unique additive map $h : A \to B$ enjoying* (4). *Moreover, if $3^\epsilon L < 1$ and, for some $\alpha_i \in \mathbb{K}$, $i = 1, 2$:*

$$\|f(xy) - \alpha_1 x f(y) - \alpha_2 f(x) y\| \leq \varphi(x, y, 0), \quad \forall (x, y) \in A^2, \tag{26}$$

then

$$h(xy) = \alpha_1 x h(y) + \alpha_2 h(x) y, \quad \forall (x, y) \in A^2.$$

Proof. Just note that, whenever $3^\epsilon L < 1$, the sequence $3^{2\epsilon n}\varphi(\frac{x}{3^{\epsilon n}}, \frac{y}{3^{\epsilon n}}, \frac{z}{3^{\epsilon n}})$ tends to 0 as n tends to infinity, (x, y, z) being arbitrary in A^3. □

Remark 4.9. In all the results above, if one wants h to be linear, one has to assume that f and φ satisfy (13) instead of (3) and that, in the real case, the mappings f_x and $\widetilde{\varphi}_x$ are bounded on some neighborhood of 0, x being arbitrary in A.

Now, we come to applications where the approximation is relative to a kind of "bounded Cauchy differences" as by T. M. Rassias [18]. Here the condition $f(0) = 0$ is automatically satisfied.

Corollary 4.10. *Let $(A, \| \ \|)$ be a normed space, p_1, p_2, p_3 and θ be positive scalars such that $q := \min(p_1, p_2, p_3) > 1$ or $r := \max(p_1, p_2, p_3) < 1$. Assume that*:

$$\begin{aligned}(27)\qquad &\left\| f\left(\mu\frac{b-a}{3}\right) + f\left(\mu\frac{a-3c}{3}\right) + \mu f\left(\frac{3a+3c-b}{3}\right) - \mu f(a)\right\| \\ &\leq \theta\left(\|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3}\right), \quad \forall \mu \in \mathbb{T}_1,\ \forall (a,b,c) \in A^3.\end{aligned}$$

Then there exists a unique linear mapping $h : A \to B$ such that

$$(28)\qquad \|h(x) - f(x)\| \leq \begin{cases} \frac{\theta(\|x\|^{p_1} + \|2x\|^{p_2})}{1-3^{(1-\min(p_1,p_2))}}, & q > 1, \\ \frac{\theta(\|x\|^{p_1} + \|2x\|^{p_2})}{1-3^{(\max(p_1,p_2)-1)}}, & r < 1. \end{cases}$$

In the real case, we additionally suppose that, for every $x \in A$, f_x is bounded on some neighborhood of 0.

Whenever $r < 1$ or $q > 1$ and $\min(p_1, p_2) > 2$, if A happens to be an algebra and B an A-bimodule so that, for some $\alpha_i \in \mathbb{K}$, $i = 1, 2$:

$$(29)\qquad \|f(xy) - \alpha_1 x f(y) - \alpha_2 f(x) y\| \leq \theta\left(\|x\|^{p_1} + \|y\|^{p_2}\right), \quad \forall (x,y) \in A^2,$$

then h enjoys

$$h(xy) = \alpha_1 x h(y) + \alpha_2 h(x) y, \quad \forall (x,y) \in A^2.$$

Proof. Let us show that the mapping φ defined by:

$$\varphi(a,b,c) := \theta\left(\|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3}\right), (a,b,c) \in A^3$$

satisfies (2). To this aim, let $(a,b,c) \in A^3$ be given. If $q > 1$, taking $\epsilon = 1$, we have:

$$\begin{aligned}\sum_{n=0}^{\infty} 3^n \varphi\left(\frac{a}{3^n}, \frac{b}{3^n}, \frac{c}{3^n}\right) &= \theta \sum_{n=0}^{\infty} 3^n \left(\left\|\frac{a}{3^n}\right\|^{p_1} + \left\|\frac{b}{3^n}\right\|^{p_2} + \left\|\frac{c}{3^n}\right\|^{p_3}\right) \\ &= \theta \sum_{n=0}^{\infty} 3^n \left(\frac{1}{3^{np_1}}\|a\|^{p_1} + \frac{1}{3^{np_2}}\|b\|^{p_2} + \frac{1}{3^{np_3}}\|c\|^{p_3}\right) \\ &\leq \theta(\|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3}) \sum_{n=0}^{\infty} 3^{n(1-q)} \\ &\leq \frac{\theta\left(|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3}\right)}{1-3^{(1-q)}}.\end{aligned}$$

Now, if $r < 1$, set $\epsilon = -1$ and get:

$$\sum_{n=0}^{\infty} 3^{-n} \varphi\left(\frac{a}{3^{-n}}, \frac{b}{3^{-n}}, \frac{c}{3^{-n}}\right)$$

$$= \theta \sum_{n=0}^{\infty} 3^{-n} \left(\left\| \frac{a}{3^{-n}} \right\|^{p_1} + \left\| \frac{b}{3^{-n}} \right\|^{p_2} + \left\| \frac{c}{3^{-n}} \right\|^{p_3} \right)$$
$$= \theta \sum_{n=0}^{\infty} 3^{-n} \left(\frac{1}{3^{np_1}} \|a\|^{p_1} + \frac{1}{3^{np_2}} \|b\|^{p_2} + \frac{1}{3^{np_3}} \|c\|^{p_3} \right)$$
$$\leq \theta (\|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3}) \sum_{n=0}^{\infty} 3^{n(r-1)}$$
$$\leq \frac{\theta \left(|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3} \right)}{1 - 3^{(r-1)}}.$$

It follows that φ enjoys (2). Notice that, in the real case, the map

$$\widetilde{\varphi}_x : t \mapsto \sum_{n=0}^{\infty} 3^{\epsilon n} \theta \left(\left\| \frac{ta}{3^{\epsilon n}} \right\|^{p_1} + \left\| \frac{2ta}{3^{\epsilon n}} \right\|^{p_2} \right)$$

is bounded on the unit interval. Indeed, if $|t| \leq 1$, then, putting $\alpha = q$ if $\epsilon = 1$ and $\alpha = r$ if $\epsilon = -1$, we obtain:

$$\begin{aligned} \sum_{n=0}^{\infty} 3^{\epsilon n} \varphi \left(\frac{tx}{3^{\epsilon n}}, \frac{2tx}{3^{\epsilon n}}, 0 \right) &= \sum_{n=0}^{\infty} 3^{\epsilon n} \theta \left(\left\| \frac{tx}{3^{\epsilon n}} \right\|^{p_1} + \left\| \frac{2tx}{3^{\epsilon n}} \right\|^{p_2} \right) \\ &\leq \theta(\|x\|^{p_1} + \|2x\|^{p_2}) \sum_{n=0}^{\infty} 3^{\epsilon n(1-\alpha)} \\ &\leq \frac{\theta(\|x\|^{p_1} + \|2x\|^{p_2})}{1 - 3^{\epsilon(1-\alpha)}}. \end{aligned}$$

Since the last term does not depend on t and is finite, $\widetilde{\varphi}_x$ is bounded on $]-1, 1[$. Therefore, by Theorem 3.3, there exists a unique linear mapping $h : A \to B$ such that (4) holds. But, in our framework, (4) implies:

$$\|h(x) - f(x)\| \leq \begin{cases} \frac{\theta(\|x\|^{p_1} + \|2x\|^{p_2})}{1 - 3^{(1 - \min(p_1, p_2))}}, & q > 1, \\ \frac{\theta(\|x\|^{p_1} + \|2x\|^{p_2})}{1 - 3^{(\max(p_1, p_2) - 1)}}, & r < 1. \end{cases} \tag{30}$$

Moreover, whenever $r < 1$ or $q > 1$ and $\min(p_1, p_2) > 2$, the equation (15) turns out to be satisfied and Theorem 4.1 permits to conclude. □

A special case of this corollary is obtained by taking all the p_i's equal to a fixed $0 < p \neq 1$.

Corollary 4.11. *Suppose that A is a normed space. If there are $0 < p \neq 1$ and $\theta > 0$ such that, for all $(a, b, c) \in A^3$ and all $\mu \in \mathbb{T}_1$, f satisfies:*

$$\left\| f\left(\mu \frac{b-a}{3}\right) + f\left(\mu \frac{a-3c}{3}\right) + \mu f\left(\frac{3a-b+3c}{3}\right) - \mu f(a) \right\|$$
$$\leq \theta \left(\|a\|^p + \|b\|^p + \|c\|^p \right),$$

then there exists a unique linear mapping $h : A \to B$ *such that, for* $\epsilon = 1$ *if* $p > 1$ *and* $\epsilon = -1$ *if* $p < 1$,

$$\|h(x) - f(x)\| \leq \frac{\theta(1 + 2^p)}{1 - 3^{\epsilon(1-p)}}\|x\|^p, \quad \forall x \in A.$$

If $p > 2$ *or* $p < 1$, A *is an algebra and* B *is an* A*-bimodule so that, for some* $\alpha_i \in \mathbb{K}$, $i = 1, 2$:

$$\|f(xy) - \alpha_1 x f(y) - \alpha_2 f(x) y\| \leq \theta\left(\|x\|^p + \|y\|^p\right), \quad \forall (x, y) \in A^2,$$

then h *enjoys*

$$h(xy) = \alpha_1 x h(y) + \alpha_2 h(x) y, \quad \forall (x, y) \in A^2.$$

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