Commun. Korean Math. Soc. **28** (2013), No. 4, pp. 751–766
http://dx.doi.org/10.4134/CKMS.2013.28.4.751

LONG-TIME BEHAVIOR FOR SEMILINEAR DEGENERATE PARABOLIC EQUATIONS ON $\mathbb{R}^N$

Cung The Anh and Le Thi Thuy

Abstract. We study the existence and long-time behavior of solutions to the following semilinear degenerate parabolic equation on $\mathbb{R}^N$:

$$\frac{\partial u}{\partial t} - \operatorname{div}(\sigma(x)\nabla u) + \lambda u + f(u) = g(x),$$

under a new condition concerning a variable non-negative diffusivity $\sigma(\cdot)$. Some essential difficulty caused by the lack of compactness of Sobolev embeddings is overcome here by exploiting the tail-estimates method.

1. Introduction

The understanding of the asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics. One way to attack the problem for a dissipative dynamical system is to study its global attractor. This is an invariant compact set that attracts all the trajectories of the system. In the last decades, many authors have paid attention to this problem and obtained many results for a large class of nondegenerate partial differential equations (see e.g. [8, 17] and references therein). In recent years, there has been an increasing interest for the study of the existence of global attractors for a class of degenerate parabolic equations with weights of Caldiroli-Musina type, in both semilinear case (cf. [1, 4, 10, 11, 12]) and quasilinear case (cf. [2, 3, 5, 6]). However, all of the above results are in the compact case, that is the case where the weights were assumed to satisfy certain conditions which ensure the compactness of some Sobolev embeddings, and this plays an essential role in the study of these works. To the best of our knowledge, little seems to be known about the existence and asymptotic behavior of solutions to these equations in the non-compact case, the more complicated case. This is a motivation of the present paper.

Received March 13, 2012.
2010 *Mathematics Subject Classification.* 35B41, 35K65, 35D30.
Key words and phrases. semilinear degenerate parabolic equation, weak solution, global attractor, non-compact case, tail estimates method.

In this paper we study the following semilinear degenerate parabolic equation with a variable, nonnegative coefficient $\sigma(\cdot)$ in $\mathbb{R}^N$, $N \geq 2$,

$$\begin{aligned} \frac{\partial u}{\partial t} - \operatorname{div}(\sigma(x)\nabla u) + \lambda u + f(u) = g(x),\ x \in \mathbb{R}^N, t > 0, \\ u(x,0) = u_0(x),\ x \in \mathbb{R}^N, \end{aligned} \tag{1.1}$$

where $\lambda > 0$, u_0 and g belong to $L^2(\mathbb{R}^N)$ given, $f : \mathbb{R} \to \mathbb{R}$ is a C^1-function satisfying some conditions specified later.

Problem (1.1) can be derived as a simple model for neutron diffusion (feedback control of nuclear reactor) (see [9]). In this case u and σ stand for the neutron flux and neutron diffusion, respectively. The degeneracy of problem (1.1) is considered in the sense that the measurable, nonnegative diffusion coefficient $\sigma(\cdot)$, is allowed to have at most a finite number of (essential) zeroes at some points.

Problem (1.1) in a general (bounded or unbounded) domain $\Omega \subset \mathbb{R}^N$ was studied extensively in [1, 4, 10, 11, 12], in which the diffusivity $\sigma(\cdot)$ was assumed to satisfy one of the following conditions which ensure important compactness properties:

- $(\mathcal{H}_\alpha)$ $\sigma \in L^1_{\text{loc}}(\Omega)$ and for some $\alpha \in (0,2), \liminf_{x\to z} |x-z|^{-\alpha}\sigma(x) > 0$ for every $z \in \overline{\Omega}$, when the domain Ω is bounded;
- $(\mathcal{H}^\infty_{\alpha,\beta})$ σ satisfies condition $(\mathcal{H}_\alpha)$ and $\liminf_{|x|\to\infty} |x|^{-\beta}\sigma(x) > 0$ for some $\beta > 2$, when the domain Ω is unbounded.

Both assumptions have a strong physical significance which is related to the existence of regions occupied by perfect insulators or perfect conductors (see [7, 9, 10, 11, 12]). The natural phase space for problem (1.1) in these cases involves $\mathcal{D}^1_0(\Omega,\sigma)$, which is defined as the closure of $C^\infty_0(\Omega)$ in the norm

$$\|u\|_{\mathcal{D}^1_0(\Omega,\sigma)} := \Big(\int_\Omega \sigma(x)|\nabla u|^2 dx\Big)^{1/2}.$$

Then under either assumption $(\mathcal{H}_\alpha)$ or $(\mathcal{H}^\infty_{\alpha,\beta})$, the embedding $\mathcal{D}^1_0(\Omega,\sigma) \hookrightarrow L^2(\Omega)$ is compact and this property plays an essential role for the investigation in [1, 4, 10, 11, 12]. Observe, however, that when Ω is unbounded, the function $\sigma(\cdot)$ must grow faster than quadratically at infinity for this property to hold (see [7] for more details).

In this paper we would like to find a new condition concerning the diffusivity $\sigma(\cdot)$ which ensures the asymptotic compactness of the semigroup generated by problem (1.1) and as a result, the existence of a global attractor, without restricting the limiting behavior of $\sigma(\cdot)$ at infinity. As it turns out, such a condition can be found with careful tails estimates as in [18] (see the proof of Lemma 3.2 below). More precisely, we assume that the function $\sigma : \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption:

($\mathcal{H}^\infty$) σ is a nonnegative measurable function such that $\sigma \in L^1_{\rm loc}(\mathbb{R}^N)$, and for some $\alpha \in (0,2)$, $\liminf_{x\to z} |x-z|^{-\alpha}\sigma(x) > 0$ for every $z \in \mathbb{R}^N$, and σ satisfies one of the following two conditions:

i) there exists $K_0 > 0$ such that $\sup\limits_{k\geq K_0} \sup\limits_{k\leq|x|\leq\sqrt{2}k} \sigma(x) < \infty$;

ii) there exists $K_0 > 0$ such that $\sup\limits_{k\geq K_0} \int\limits_{k\leq|x|\leq\sqrt{2}k} |\sigma(x)|^{\frac{N}{2-\alpha}} dx < \infty$.

Let us give a comment on the condition ($\mathcal{H}^\infty$). Observe that the absence of a specific limiting behavior at infinity for $\sigma(\cdot)$ (cf. condition ($\mathcal{H}^\infty_{\alpha,\beta}$)) is now compensated by a higher local integrability. A simple example in which ($\mathcal{H}^\infty$) is fulfilled but ($\mathcal{H}^\infty_{\alpha,\beta}$) is not, is provided by the function $\sigma(x) \equiv 1$ (the nondegenerate case) or $\sigma(x) = e^{-|x|}\big(|x|^\alpha + |x|^\gamma\big)$ with $\alpha, \gamma \in (0,2)$.

In this paper we assume that the nonlinearity f and the external force g satisfy the following conditions:

(F) $f : \mathbb{R} \to \mathbb{R}$ is a C^1-function satisfying

$$f(u)u \geq -\mu u^2,\ \mu < \lambda, \tag{1.2}$$

$$f'(u) \geq -C, \tag{1.3}$$

$$|f(u)| \leq C|u|. \tag{1.4}$$

It is noticed that assumption (**F**) covers two significant cases arising in physics. The first one corresponds to the sinusoidal nonlinearity $f(u) = \sin u$, while the second to the so-called saturated nonlinearity $f(u) = \frac{u}{1+u^2}$.

(G) $g \in L^2(\mathbb{R}^N)$.

Let us introduce some function spaces related to problem (1.1). For a domain $\Omega \subset \mathbb{R}^N$, we define the space $\mathcal{H}^1_0(\Omega,\sigma)$ as the closure of $C^\infty_0(\Omega)$ with respect to the norm

$$\|u\|_{\mathcal{H}^1_0(\Omega,\sigma)} := \Big(\int_\Omega \sigma(x)|\nabla u|^2 dx + \int_\Omega |u|^2 dx\Big)^{1/2}.$$

Hence, the natural energy spaces for problem (1.1) involves the space $\mathcal{H}^1_0(\mathbb{R}^N,\sigma)$ and its dual space $\mathcal{H}^{-1}(\mathbb{R}^N,\sigma)$. It is noticed that in the compact case, that is the case where the assumption ($\mathcal{H}_\alpha$) or ($\mathcal{H}^\infty_{\alpha,\beta}$) holds, then $\mathcal{H}^1_0(\Omega,\sigma) \equiv \mathcal{D}^1_0(\Omega,\sigma)$ and the embedding $\mathcal{H}^1_0(\Omega,\sigma) \hookrightarrow L^2(\Omega)$ is compact. However, this property is no longer true in the non-compact case, that is the case $\sigma(\cdot)$ satisfies ($\mathcal{H}^\infty$). This introduces some new essential difficulty when studying problem (1.1), both in the existence of solutions and the existence of global attractors, when compared with the compact case in [1, 4, 10, 11, 12].

This paper is an attempt to study the existence and long-time behavior of solutions to problem (1.1) in the non-compact case. Let us explain the method used in the paper. First, using the Galerkin method, we prove the global existence of a unique weak solution and then construct the continuous semigroup $S(t) : L^2(\mathbb{R}^N) \to \mathcal{H}^1_0(\mathbb{R}^N,\sigma)$ associated to problem (1.1). Next, we use *a priori* estimates to show the existence of a bounded absorbing set in $\mathcal{H}_0^1(\mathbb{R}^N, \sigma)$ for the semigroup. In the compact case, that is the case $\sigma(\cdot)$ satisfies $(\mathcal{H}_\alpha)$ or $(\mathcal{H}_{\alpha,\beta}^\infty)$, since the embedding $\mathcal{H}_0^1(\Omega, \sigma) \equiv \mathcal{D}_0^1(\Omega, \sigma) \hookrightarrow L^2(\Omega)$ is compact, this immediately implies the asymptotic compactness of the semigroup in $L^2(\Omega)$, and therefore the existence of a global attractor in $L^2(\Omega)$. Here, because the embedding is no longer compact, the proof of the asymptotic compactness in $L^2(\mathbb{R}^N)$ is much more involved. To do this, we exploit the tail estimates method introduced by B. Wang in [18], and as a result, we obtain the existence of a global attractor in $L^2(\mathbb{R}^N)$. Finally, under an additional condition of the nonlinearity f (see hypothesis $(\mathbf{F}')$ in Section 4), we show that the global attractor exists in $\mathcal{H}_0^1(\mathbb{R}^N, \sigma)$. The main new feature of the paper is that we are able to prove the existence of global attractors for a class of semilinear degenerate parabolic equations in the non-compact case. To the best of our knowledge, this work seems to be the first one addressing the question about the long-time dynamics of (1.1) in the case lack of compactness of the Sobolev embeddings.

The paper is organized as follows. Section 2 is devoted to the proof of existence and uniqueness of solutions. In Section 3, we prove the existence of a compact global attractor in $L^2(\mathbb{R}^N)$. In the last section, under an additional condition of f, we show that the global attractor is in fact in $\mathcal{H}_0^1(\mathbb{R}^N, \sigma)$.

2. Existence of weak solutions

Definition 2.1. A function $u(t, x)$, $t \in [0, T]$, $x \in \mathbb{R}^N$, is said to be a weak solution of (1.1) on $[0, T]$ if $u \in L^2(0, T; \mathcal{H}_0^1(\mathbb{R}^N, \sigma))$, $\frac{du}{dt} \in L^2(0, T; \mathcal{H}^{-1}(\mathbb{R}^N, \sigma))$, $u(0) = u_0$, and u satisfies equation (1.1) in the distribution sense, that is,

$$\int_0^T (u_t, v)_{L^2(\mathbb{R}^N)} dt + \int_0^T \int_{\mathbb{R}^N} \sigma(x) \nabla u \nabla v dx dt + \lambda \int_0^T (u, v)_{L^2(\mathbb{R}^N)} dx dt + \int_0^T (f(u), v)_{L^2(\mathbb{R}^N)} dx dt = \int_0^T (g, v)_{L^2(\mathbb{R}^N)} dx dt$$

for all test functions $v \in L^2(0, T; \mathcal{H}_0^1(\mathbb{R}^N, \sigma))$.

We now prove the existence theorem.

Theorem 2.1. *Let hypotheses $(\mathcal{H}^\infty) - (\mathbf{F}) - (\mathbf{G})$ hold. Then, for any $u_0 \in L^2(\mathbb{R}^N)$ and $T > 0$ given, problem (1.1) has a unique weak solution u on $[0, T]$. Moreover, for each $t \in [0, T]$, the map $u_0 \mapsto u(t)$ is continuous on $L^2(\mathbb{R}^N)$.*

Proof. i) *Existence.* For $m \geq 1$, we put

$$\Omega_m = \mathbb{R}^N \cap \{x \in \mathbb{R}^N : |x|_{\mathbb{R}^N} < m\},$$

where $|\cdot|_{\mathbb{R}^N}$ denotes the Euclidean norm in $\mathbb{R}^N$. For each integer $n \geq 1$, we denote by

$$u_n(t) = \sum_{j=1}^n \gamma_{nj}(t) \omega_j$$

a solution of

$$\frac{d}{dt}(u_n(t),\omega_j)-(\mathrm{div}(\sigma(x)\nabla u),\omega_j)+\lambda(u_n(t),\omega_j)+(f(u_n(t)),\omega_j)=(g,\omega_j),\ \ t>0,$$

$$(u_n(0),\omega_j) = (u_0,\omega_j), \quad j=1,2,\dots,n,$$

where $\{\omega_j\}_{j=1}^{\infty} \subset \mathcal{H}_0^1(\mathbb{R}^N,\sigma)$ is a Hilbert basis of $L^2(\mathbb{R}^N)$ such that $\mathrm{span}\{\omega_j\}_{j=1}^{\infty}$ is dense in $\mathcal{H}_0^1(\mathbb{R}^N,\sigma)$.

Multiplying (1.1) by u_n and integrating over $\mathbb{R}^N$, we have

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{L^2(\mathbb{R}^N)}+\int_{\mathbb{R}^N}\sigma(x)|\nabla u_n|^2dx+\lambda\|u_n\|^2_{L^2(\mathbb{R}^N)}+\int_{\mathbb{R}^N}f(u_n)u_n dx=\int_{\mathbb{R}^N}gu_n dx.$$

Using (**F**) and the Cauchy inequality, we obtain

$$\frac{d}{dt}\|u_n\|^2_{L^2(\mathbb{R}^N)}+2\int_{\mathbb{R}^N}\sigma(x)|\nabla u_n|^2dx+(\lambda-\mu)\|u_n\|^2_{L^2(\mathbb{R}^N)} \le \frac{1}{\lambda-\mu}\|g\|^2_{L^2(\mathbb{R}^N)}.$$

Integrating the last inequality over $(0,t), 0<t\le T$, we obtain

$$\|u_n(t)\|^2_{L^2(\mathbb{R}^N)}+2\int_0^t\int_{\mathbb{R}^N}\sigma(x)|\nabla u_n(s)|^2dxds+(\lambda-\mu)\int_0^t\|u_n(s)\|^2_{L^2(\mathbb{R}^N)}ds$$
$$\le \|u_0\|^2_{L^2(\mathbb{R}^N)}+\frac{1}{\lambda-\mu}\|g\|^2_{L^2(\mathbb{R}^N)}T,$$

where we have used the fact that $\|u_n(0)\|_{L^2(\mathbb{R}^N)} \le \|u_0\|_{L^2(\mathbb{R}^N)}$. It follows from the above estimate that

$$\{u_n\} \text{ is bounded in } L^2(0,T;\mathcal{H}_0^1(\mathbb{R}^N,\sigma))\cap C([0,T];L^2(\mathbb{R}^N)). \tag{2.1}$$

Hence, by assumption (**F**), it is easy to check that

$$\{f(u_n)\} \text{ is bounded in } L^2(0,T;L^2(\mathbb{R}^N)).$$

Then, there exists a subsequence $\{u_\mu\}$ such that

$$u_\mu \rightharpoonup u \text{ in } L^2(0,T;\mathcal{H}_0^1(\mathbb{R}^N,\sigma)), \tag{2.2}$$
$$f(u_\mu) \rightharpoonup \chi \text{ in } L^2(0,T;L^2(\mathbb{R}^N)).$$

Hence, (2.2) implies that

$$-\mathrm{div}(\sigma(x)\nabla u_\mu)+\lambda u_\mu \rightharpoonup -\mathrm{div}(\sigma(x)\nabla u)+\lambda u \text{ in } L^2(0,T;\mathcal{H}^{-1}(\mathbb{R}^N,\sigma)).$$

On the other hand, to prove that $\chi(t)=f(u(t))$, we argue similarly to [15]. As in [15], we first deduce that

$$\lim_{a\to 0}\sup_{\mu}\int_0^{T-a}\|u_\mu(t+a)-u_\mu(t)\|^2_{L^2(\mathbb{R}^N)}dt=0. \tag{2.3}$$

Let $\phi \in C^1([0,+\infty))$ be a function such that

$$0\le\phi(s)\le 1,$$
$$\phi(s)=1 \quad \forall s\in[0,1],$$
$$\phi(s)=0 \quad \forall s\ge 2.$$

For each μ and $m \geq 1$, we define

$$v_{\mu,m}(x,t) = \phi\Big(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\Big)u_\mu(t) \quad \forall x \in \Omega_{2m}, \forall \mu, \forall m \geq 1.$$

We obtain from (2.1) that, for all $m \geq 1$, the sequence $\{v_{\mu,m}\}_{\mu\geq 1}$ is bounded in $L^\infty(0,T;L^2(\Omega_{2m})) \cap L^2(0,T;\mathcal{H}^1_0(\Omega_{2m},\sigma))$. In particular, it follows that

$$\lim_{a\to 0}\sup_\mu \Big(\int_0^a \|v_{\mu,m}(x,t)\|^2_{L^2(\Omega_{2m})}dt + \int_{T-a}^T \|v_{\mu,m}(x,t)\|^2_{L^2(\Omega_{2m})}dt\Big) = 0.$$

On the other hand, from (2.3) we deduce that for all $m \geq 1$,

$$\lim_{a\to 0}\sup_\mu \Big(\int_0^{T-a} \|v_{\mu,m}(x,t+a) - v_{\mu,m}(x,t)\|^2_{L^2(\Omega_{2m})}dt\Big) = 0.$$

Moreover, as Ω_{2m} is a bounded set, then $\mathcal{H}^1_0(\Omega_{2m},\sigma)$ is included in $L^2(\Omega_{2m})$ with compact injection. Then, by Theorem 13.3 and Remark 13.1 in [16] with $X = L^2(\Omega_{2m})$, $Y = \mathcal{H}^1_0(\Omega_{2m},\sigma)$, $r = 2$ and $\mathcal{G} = \{v_{\mu,m}\}_{\mu\geq 1}$, we obtain that

$$\{v_{\mu,m}\}_{\mu\geq 1} \text{ is relatively compact in } L^2(0,T;L^2(\Omega_{2m})),$$

and thus, taking into account that $v_{\mu,m}(x,t) = u_\mu(x,t)$ for all $x \in \Omega_m$, we deduce that, in particular, for all $m \geq 1$,

$$\{u_{\mu|\Omega_m}\} \text{ is pre-compact in } L^2(0,T;L^2(\Omega_m)). \tag{2.4}$$

By a diagonal procedure, one can conclude from (2.4) and (2.2) that there exists a subsequence $\{u^\mu_\mu\}_{\mu\geq 1} \subset \{u_\mu\}_{\mu\geq 1}$ such that

$$u^\mu_\mu \to u \text{ in } \Omega_m \times (0,T) \text{ as } \mu \to \infty, \forall m \geq 1.$$

Then, as $f(\cdot)$ is continuous,

$$f(u^\mu_\mu) \to f(u) \text{ a.e. in } \Omega_m \times (0,T),$$

and as $\{f(u^\mu_\mu)\}$ is bounded in $L^2(\Omega_m \times (0,T))$, by Lemma 1.3 in [13, Chapter 1], we obtain

$$f(u^\mu_\mu) \rightharpoonup f(u) \text{ in } L^2(0,T;L^2(\Omega_m)).$$

By the uniqueness of the weak limit, we have

$$\chi = f(u) \text{ a.e. in } \Omega_m \times (0,T),\ \forall m \geq 1,$$

and thus, taking into account that $\cup_{m=1}^\infty \Omega_m = \mathbb{R}^N$, we obtain

$$\chi = f(u) \text{ a.e. in } \mathbb{R}^N \times (0,T).$$

It is now straightforward to show that u is a weak solution of problem (1.1) with the initial datum u_0.

ii) *Uniqueness and continuous dependence.* Let u_1, u_2 be two weak solutions to problem (1.1) with the initial data $u_{01}, u_{02} \in L^2(\mathbb{R}^N)$, respectively. Putting $u = u_1 - u_2$, we have

$$\frac{du}{dt} - \operatorname{div}(\sigma\nabla u) + \lambda u + f(u_1) - f(u_2) = 0.$$

Multiplying this equation by u in $L^2(\mathbb{R}^N)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R}^N)}+\int_{\mathbb{R}^N}\sigma|\nabla u|^2dx+\lambda\int_{\mathbb{R}^N}u^2dx+\int_{\mathbb{R}^N}(f(u_1)-f(u_2))(u_1-u_2)dx=0.$$

From hypothesis (**F**), we have

$$\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R}^N)}+2\int_{\mathbb{R}^N}\sigma|\nabla u|^2dx+2\lambda\int_{\mathbb{R}^N}u^2dx\leq 2C\int_{\mathbb{R}^N}u^2dx.$$

In particular, we have

$$\frac{d}{dt}\|u(t)\|^2_{L^2(\mathbb{R}^N)}\leq 2C\|u(t)\|^2_{L^2(\mathbb{R}^N)}.$$

Applying the Gronwall inequality, we get

$$\|u(t)\|^2_{L^2(\mathbb{R}^N)}\leq\|u(0)\|^2_{L^2(\mathbb{R}^N)}e^{2Ct}.$$

This implies the uniqueness (if $u_{01}=u_{02}$), and continuous dependence of the solutions on the initial data. □

3. Existence of a global attractor in $L^2(\mathbb{R})$

Thanks to Theorem 2.1, we can define a continuous semigroup $\{S(t)\}_{t\geq 0}$ as follows

$$S(t):L^2(\mathbb{R}^N)\to\mathcal{H}^1_0(\mathbb{R}^N,\sigma),$$

where $S(t)u_0:=u(t)$ is the unique weak solution of (1.1) subject to u_0 as initial datum.

For the shake of brevity, in the proofs of the following lemmas, we will give some formal calculations, the rigorous proof is done by use of Galerkin approximations (these solutions are smooth enough) and Lemma 11.2 in [14].

We first prove the existence of an absorbing set for $S(t)$ in $\mathcal{H}^1_0(\mathbb{R}^N,\sigma)$.

Lemma 3.1. *Suppose* $(\mathcal{H}^\infty)$ - **(F)** - **(G)** *hold. Then the semigroup* $S(t)$ *generated by* (1.1) *has a bounded absorbing set in* $\mathcal{H}^1_0(\mathbb{R}^N,\sigma)$*, that is, there exists a positive constant* ρ*, such that for every bounded subset* B *in* $L^2(\mathbb{R}^N)$*, there is a number* $T=T(B)>0$*, such that for all* $t\geq T$*,* $u_0\in B$*, we have*

$$\|u(t)\|_{\mathcal{H}^1_0(\mathbb{R}^N,\sigma)}\leq\rho.$$

Proof. Taking the inner product of (1.1) with u in $L^2(\mathbb{R}^N)$ and using (**F**) we get

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R}^N)}+\int_{\mathbb{R}^N}\sigma(x)|\nabla u|^2dx+(\lambda-\mu)\|u\|^2_{L^2(\mathbb{R}^N)}=\int_{\mathbb{R}^N}gudx. \tag{3.1}$$

Applying the Cauchy inequality, we have

$$\left|\int_{\mathbb{R}^N}gudx\right|\leq\|g\|_{L^2(\mathbb{R}^N)}\|u\|_{L^2(\mathbb{R}^N)}\leq\frac{\lambda-\mu}{2}\|u\|^2_{L^2(\mathbb{R}^N)}+\frac{1}{2(\lambda-\mu)}\|g\|^2_{L^2(\mathbb{R}^N)}. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R}^N)}+2\int_{\mathbb{R}^N}\sigma(x)|\nabla u|^2dx+(\lambda-\mu)\|u\|^2_{L^2(\mathbb{R}^N)}\le\frac{1}{\lambda-\mu}\|g\|^2_{L^2(\mathbb{R}^N)}. \tag{3.3}$$

Hence,

$$\frac{d}{dt}\|u(t)\|^2_{L^2(\mathbb{R}^N)}+(\lambda-\mu)\|u(t)\|^2_{L^2(\mathbb{R}^N)}\le\frac{1}{\lambda-\mu}\|g\|^2_{L^2(\mathbb{R}^N)}.$$

Using the Gronwall inequality, we obtain

$$\|u(t)\|^2_{L^2(\mathbb{R}^N)}\le e^{-(\lambda-\mu)t}\|u_0\|^2_{L^2(\mathbb{R}^N)}+\frac{1}{(\lambda-\mu)^2}\|g\|^2_{L^2(\mathbb{R}^N)}.$$

Hence we deduce the existence of an absorbing set in $L^2(\mathbb{R}^N)$: There are a positive constant R and a time $t_0=t_0(\|u_0\|_{L^2(\mathbb{R}^N)})$ such that for the solution $u(t)=S(t)u_0$, we have

$$\|u(t)\|_{L^2(\mathbb{R}^N)}\le R \quad \text{for all} \quad t\ge t_0.$$

Integrating (3.3) on $(t,t+1), t\ge t_0$, in particular, we find that

$$\begin{aligned}&\int_t^{t+1}\Big(\int_{\mathbb{R}^N}\sigma(x)|\nabla u(s)|^2dx+(\lambda-\mu)\|u(s)\|^2_{L^2(\mathbb{R}^N)}\Big)ds\\ &\le\|u(t)\|^2_{L^2(\mathbb{R}^N)}+\frac{1}{\lambda-\mu}\|g\|^2_{L^2(\mathbb{R}^N)}\\ &\le R^2+\frac{1}{\lambda-\mu}\|g\|^2_{L^2(\mathbb{R}^N)}.\end{aligned} \tag{3.4}$$

On the other hand, multiplying the first equation in (1.1) by $-\mathrm{div}(\sigma(x)\nabla u)+(\lambda-\mu)u$ and integrating over $\mathbb{R}^N$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\int_{\mathbb{R}^N}\sigma|\nabla u|^2dx+(\lambda-\mu)\int_{\mathbb{R}^N}|u|^2dx\Big)+\int_{\mathbb{R}^N}|\mathrm{div}(\sigma\nabla u)|^2dx\\ &+(2\lambda-\mu)\int_{\mathbb{R}^N}\sigma|\nabla u|^2dx+\lambda(\lambda-\mu)\|u\|^2_{L^2(\mathbb{R}^N)}\\ =&\int_{\mathbb{R}^N}f(u)\mathrm{div}(\sigma\nabla u)dx-(\lambda-\mu)\int_{\mathbb{R}^N}f(u)udx\\ &-\int_{\mathbb{R}^N}g\mathrm{div}(\sigma\nabla u)dx+(\lambda-\mu)\int_{\mathbb{R}^N}gudx.\end{aligned} \tag{3.5}$$

Since

$$\begin{aligned}&\int_{\mathbb{R}^N}f(u)\mathrm{div}(\sigma\nabla u)dx=-\int_{\mathbb{R}^N}f'(u)\sigma|\nabla u|^2dx\le C\int_{\mathbb{R}^N}\sigma|\nabla u|^2dx,\\ &-(\lambda-\mu)\int_{\mathbb{R}^N}f(u)udx\le C(\lambda-\mu)\|u\|^2_{L^2(\mathbb{R}^N)},\\ &(\lambda-\mu)\int_{\mathbb{R}^N}gudx\le\lambda(\lambda-\mu)\|u\|^2_{L^2(\mathbb{R}^N)}+C\|g\|^2_{L^2(\mathbb{R}^N)},\end{aligned}$$

and

$$\Big|\int_{\mathbb{R}^N} g\mathrm{div}(\sigma\nabla u)dx\Big| \le \|g\|_{L^2(\mathbb{R}^N)}\Big(\int_{\mathbb{R}^N}|\mathrm{div}(\sigma\nabla u)|^2dx\Big)^{1/2}$$
$$\le \frac{1}{4}\|g\|^2_{L^2(\mathbb{R}^N)} + \|\mathrm{div}(\sigma\nabla u)\|^2_{L^2(\mathbb{R}^N)},$$

it follows from (3.5) that

$$\begin{aligned}&\frac{d}{dt}\Big(\int_{\mathbb{R}^N}\sigma(x)|\nabla u|^2dx + (\lambda-\mu)\int_{\mathbb{R}^N}|u|^2dx\Big)\\ &\le C\Big(\int_{\mathbb{R}^N}\sigma(x)|\nabla u|^2dx + (\lambda-\mu)\int_{\mathbb{R}^N}|u|^2dx\Big) + \tilde{C}\|g\|^2_{L^2(\mathbb{R}^N)}.\end{aligned} \tag{3.6}$$

From (3.4) and (3.6), applying the uniform Gronwall inequality, we deduce that

$$\|u(t)\|_{\mathcal{H}^1_0(\mathbb{R}^N,\sigma)} \le \rho \quad \text{for all } t \ge t_0+1,$$

where $\rho = \rho(\lambda,\mu,\|g\|_{L^2(\mathbb{R}^N)})$. This implies the desired result. □

We now give tail-estimates of the solutions.

Lemma 3.2. *Suppose* $(\mathcal{H}^\infty) - (\mathbf{F}) - (\mathbf{G})$ *hold. Then for any* $\eta > 0$ *and any bounded subset* $B \subset L^2(\mathbb{R}^N)$*, there exist* $T = T(\eta, B) > 0$ *and* $K = K(\eta, B) > 0$ *such that for all* $t \ge T$ *and* $k \ge K$*,*

$$\int_{|x|\ge k}|u(x,t)|^2dx \le \eta,$$

where u *is the weak solution of* (1.1) *subject to the initial condition* $u(0) = u_0 \in B$*.*

Proof. We use a cut-off technique to establish the estimates on the tails of solutions. Let θ be a smooth function satisfying $0 \le \theta(s) \le 1$ for $s \in \mathbb{R}^+$, and

$$\theta(s) = 0 \quad \text{for } 0 \le s \le 1; \quad \theta(s) = 1 \quad \text{for } s \ge 2.$$

Then there exists a constant C such that $|\theta'(s)| \le C$ for all $s \in \mathbb{R}^+$. Taking the inner product of (1.1) with $\theta(\frac{|x|^2}{k^2})u$ in $L^2(\mathbb{R}^N)$ and using hypothesis $(\mathbf{F})$, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}\theta(\frac{|x|^2}{k^2})|u|^2dx - \int_{\mathbb{R}^N}\theta(\frac{|x|^2}{k^2})u\mathrm{div}(\sigma(x)\nabla u)dx\\ &+(\lambda-\mu)\int_{\mathbb{R}^N}\theta(\frac{|x|^2}{k^2})|u|^2dx\\ \le &\int_{\mathbb{R}^N}\theta(\frac{|x|^2}{k^2})g(x)u(x,t)dx.\end{aligned} \tag{3.7}$$

We estimate the right-hand side of (3.7) as follows

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \theta(\frac{|x|^2}{k^2}) g(x) u(x,t) dx \\
&= \int_{|x|\geq k} \theta(\frac{|x|^2}{k^2}) g(x) u(x,t) dx \\
&\leq \frac{\lambda-\mu}{2} \int_{|x|\geq k} \theta^2(\frac{|x|^2}{k^2}) |u|^2 dx + \frac{1}{2(\lambda-\mu)} \int_{|x|\geq k} |g(x)|^2 dx \\
&\leq \frac{\lambda-\mu}{2} \int_{\mathbb{R}^N} \theta^2(\frac{|x|^2}{k^2}) |u|^2 dx + \frac{1}{2(\lambda-\mu)} \int_{|x|\geq k} |g(x)|^2 dx.
\end{aligned}
\tag{3.8}
$$

For the second term on the left-hand side of (3.7), by integrating by parts, we find that

$$
\begin{aligned}
&-\int_{\mathbb{R}^N} \theta(\frac{|x|^2}{k^2}) u \mathrm{div}(\sigma(x)\nabla u) dx \\
&= \int_{\mathbb{R}^N} \theta(\frac{|x|^2}{k^2}) \sigma(x) |\nabla u|^2 dx + \int_{\mathbb{R}^N} \theta'(\frac{|x|^2}{k^2})(\frac{2x}{k^2} \cdot \sigma(x)\nabla u) u dx \\
&\geq \int_{k\leq |x|\leq \sqrt{2}k} \theta'(\frac{|x|^2}{k^2})(\frac{2x}{k^2} \cdot \sigma(x)\nabla u) u dx.
\end{aligned}
\tag{3.9}
$$

It follows from (3.7)-(3.9) that

$$
\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^N} \theta(\frac{|x|^2}{k^2}) |u|^2 dx + (\lambda-\mu) \int_{\mathbb{R}^N} \theta(\frac{|x|^2}{k^2}) |u|^2 dx \\
&\leq \frac{1}{\lambda-\mu} \int_{|x|\geq k} |g(x)|^2 dx + 2\int_{k\leq |x|\leq \sqrt{2}k} |\theta'(\frac{|x|^2}{k^2})| \frac{2|x|}{k^2} \sigma(x) |\nabla u||u| dx.
\end{aligned}
\tag{3.10}
$$

We have

$$
\begin{aligned}
&\int_{k\leq |x|\leq \sqrt{2}k} |\theta'(\frac{|x|^2}{k^2})| \frac{2|x|}{k^2} \sigma(x) |\nabla u||u| dx \\
&\leq \frac{C}{k} \int_{k\leq |x|\leq \sqrt{2}k} \sigma(x) |\nabla u||u| dx \\
&\leq \frac{C}{k} \Big(\int_{k\leq |x|\leq \sqrt{2}k} \sigma(x) |u|^2 dx \Big)^{1/2} \Big(\int_{k\leq |x|\leq \sqrt{2}k} \sigma(x) |\nabla u|^2 dx \Big)^{1/2},
\end{aligned}
\tag{3.11}
$$

where C is independent of k. We now estimate the term

$$\int_{k\leq |x|\leq \sqrt{2}k} \sigma(x) |u|^2 dx.$$

Case 1: σ satisfies condition i) in $(\mathcal{H}^\infty)$. We have for all $k \geq K_0$,

$$\int_{k\leq |x|\leq \sqrt{2}k} \sigma(x) |u|^2 dx \leq C \int_{k\leq |x|\leq \sqrt{2}k} |u|^2 dx.$$

Case 2: σ satisfies condition ii) in $(\mathcal{H}^\infty)$. By Hölder's inequality, we obtain

$$\begin{aligned}\int_{k\le|x|\le\sqrt{2}k}\sigma(x)|u|^2dx &\le \Big(\int_{k\le|x|\le\sqrt{2}k}\sigma(x)^{\frac{N}{2-\alpha}}dx\Big)^{\frac{2-\alpha}{N}}\Big(\int_{k\le|x|\le\sqrt{2}k}|u|^{2^*_\alpha}dx\Big)^{\frac{1}{2^*_\alpha}}\\ &\le C\Big(\int_{k\le|x|\le\sqrt{2}k}|u|^{2^*_\alpha}dx\Big)^{\frac{1}{2^*_\alpha}}.\end{aligned}$$

In both these cases, since $\mathcal{D}^1_0(\Omega,\sigma)\subset L^p(\Omega)$ for all $p\in[1,2^*_\alpha]$ when Ω is a bounded domain (see [7]), we have for all $t\ge T$, with T is given in Lemma 3.1,

$$\int_{k\le|x|\le\sqrt{2}k}|\theta'(\frac{|x|^2}{k^2})|\frac{2|x|}{k^2}\sigma(x)|\nabla u(t)||u(t)|dx\le\frac{C(\rho)}{k},$$

where we have used the fact that $\|u(t)\|_{\mathcal{H}^1_0(\mathbb{R}^N,\sigma)}\le\rho$ for all $t\ge T$. Thus, from (3.10) we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^N}\theta(\frac{|x|^2}{k^2})|u|^2dx+(\lambda-\mu)\int_{\mathbb{R}^N}\theta(\frac{|x|^2}{k^2})|u|^2dx\\ &\le\frac{1}{(\lambda-\mu)}\int_{|x|\ge k}|g(x)|^2dx+\frac{C(\rho)}{k}.\end{aligned}\tag{3.12}$$

Multiplying (3.12) by $e^{(\lambda-\mu)t}$ and integrating over (T,t), after some simple computations, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}\theta(\frac{|x|^2}{k^2})|u(t)|^2dx\\ &\le e^{-(\lambda-\mu)t}\|u(T)\|^2_{L^2(\mathbb{R}^N)}+\frac{1}{(\lambda-\mu)^2}\int_{|x|\ge k}|g(x)|^2dx+\frac{C(\rho)}{k}.\end{aligned}\tag{3.13}$$

For given $\eta>0$, there exists $T_1=T_1(\eta)>0$ such that for all $t\ge T_1$,

$$e^{-(\lambda-\mu)t}\|u(T)\|^2_{L^2(\mathbb{R}^N)}\le\frac{\eta}{3}.\tag{3.14}$$

Since $g\in L^2(\mathbb{R}^N)$, there is $K_2=K_2(\eta)>K_1$ such that for all $k\ge K_2$,

$$\frac{1}{(\lambda-\mu)^2}\int_{|x|\ge k}|g(x)|^2dx\le\frac{\eta}{3}.\tag{3.15}$$

For the last term on the right-hand side of (3.13), there is $K_3=K_3(\eta)>K_2$ such that for all $k\ge K_3$,

$$\frac{C(\rho)}{k}\le\frac{\eta}{3}.\tag{3.16}$$

Let $T_0=\max\{T,T_1\}$. Then by (3.13) - (3.16) we find that for all $k\ge K_3$ and $t\ge T_0$,

$$\int_{\mathbb{R}^N}\theta(\frac{|x|^2}{k^2})|u(t)|^2dx\le\eta,$$

and hence for all $k \geq K_3$ and $t \geq T_0$,

$$\int_{|x|\geq\sqrt{2}k} |u(t)|^2 dx \leq \int_{\mathbb{R}^N} \theta(\frac{|x|^2}{k^2})|u(t)|^2 dx \leq \eta.$$

This completes the proof. □

We are now ready to show the asymptotic compactness of $S(t)$ in $L^2(\mathbb{R}^N)$.

Lemma 3.3. *Suppose* $(\mathcal{H}^\infty)$ - **(F)** - **(G)** *hold. Then* $S(t)$ *is asymptotically compact in* $L^2(\mathbb{R}^N)$*, that is, for any bounded sequence* $\{u_{0n}\}_{n=1}^\infty \subset L^2(\mathbb{R}^N)$ *and* $t_n \geq 0, t_n \to \infty$, $\{S(t_n)u_{0n}\}_{n=1}^\infty$ *has a convergent subsequence with respect to the topology of* $L^2(\mathbb{R}^N)$.

Proof. We use the uniform estimates on the tails of solutions to establish the precompactness of $\{u_n(t_n) := S(t_n)u_{0n}\}$, that is, we prove that for every $\eta > 0$, the sequence $u_n(t_n)$ has a finite covering of balls of radii less than η. Given $K > 0$, denote

$$\Omega_K = \{x : |x| \leq K\} \quad \text{and} \quad \Omega_K^c = \{x : |x| > K\}.$$

Then by Lemma 3.2, for the given $\eta > 0$, there exist $K = K(\eta) > 0$ and $T = T(\eta) > 2$ such that for $t \geq T$,

$$\|u(t)\|_{L^2(\Omega_K^c)} \leq \frac{\eta}{4}.$$

Since $t_n \to \infty$, there is $N_1 = N_1(\eta) > 0$ such that $t_n \geq T$ for all $n \geq N_1$, and hence we obtain that, for all $n \geq N_1$,

$$\|u_n(t_n)\|_{L^2(\Omega_K^c)} \leq \frac{\eta}{4}. \tag{3.17}$$

Let $\zeta(\cdot) \in C^\infty(\mathbb{R}^N)$ be a function such that $0 \leq \zeta(s) \leq 1$ for any $s \geq 0$, and

$$\zeta(s) = 1 \text{ for } 0 \leq s \leq 1, \zeta(s) = 0 \text{ for } s \geq 2.$$

Furthermore, define $\zeta_k(x) = \zeta(\frac{|x|^2}{k^2})$. Then $\{\zeta_K u_n(t_n)\}$ belongs to $\mathcal{H}_0^1(\Omega_{\sqrt{2}K}, \sigma)$. By Lemma 3.1, there exist $C > 0$ and $N_2 > 0$ such that for all $n \geq N_2$,

$$\|\zeta_K u_n(t_n)\|_{\mathcal{H}_0^1(\Omega_{\sqrt{2}K}, \sigma)} \leq C.$$

By the compactness of embedding $\mathcal{H}_0^1(\Omega_{\sqrt{2}K}, \sigma) \equiv \mathcal{D}_0^1(\Omega_{\sqrt{2}K}, \sigma) \hookrightarrow L^2(\Omega_{\sqrt{2}K})$ (see [7]), the sequence $\{\zeta_K u_n(t_n)\}$ is precompact in $L^2(\Omega_{\sqrt{2}K})$. This in particular implies that $\{u_n(t_n)\}$ is precompact in $L^2(\Omega_K)$. Therefore, for the given $\eta > 0$, $\{u_n(t_n)\}$ has a finite covering in $L^2(\Omega_K)$ of balls of radii less than η, which along with (3.17) shows that $\{u_n(t_n)\}$ has a finite covering in $L^2(\mathbb{R}^N)$ of balls of radii less than η, and thus $\{u_n(t_n)\}$ is precompact in $L^2(\mathbb{R}^N)$. □

We now prove the existence of a global attractor for $S(t)$ in $L^2(\mathbb{R}^N)$.

Theorem 3.1. *Suppose* $(\mathcal{H}^\infty)$ - **(F)** - **(G)** *hold. Then the semigroup* $S(t)$ *generated by problem* (1.1) *has a compact connected global attractor* $\mathcal{A}$ *in* $L^2(\mathbb{R}^N)$.

Proof. Denote by

$$B = \Big\{u : \|u\|_{L^2(\mathbb{R}^N)} \le R\Big\},$$

where R is the positive constant in the proof of Lemma 3.1. Noting that B is a bounded absorbing set for $S(t)$ in $L^2(\mathbb{R}^N)$. In addition, $S(t)$ is asymptotically compact in $L^2(\mathbb{R}^N)$ since Lemma 3.3. Thus, the existence of a global attractor in $L^2(\mathbb{R}^N)$ for $S(t)$ follows immediately. □

4. Existence of a global attractor in $\mathcal{H}_0^1(\mathbb{R}^N, \sigma)$

In this section, we assume that the nonlinearity f satisfies the following condition:

(F') $f : \mathbb{R} \to \mathbb{R}$ is a C^1-function satisfying (**F**) and

$$F(u) \ge -\frac{\mu}{2}u^2,\ \mu < \lambda,$$

where $F(u) = \int_0^u f(s)ds$ is a primitive of f.
Hence it is easy to check that $|F(u)| \le Cu^2$ for all $u \in \mathbb{R}$.

We now derive uniform estimates of the derivatives of solutions in time.

Lemma 4.1. *Suppose* $(\mathcal{H}^\infty)$ - **(F')** - **(G)** *hold. Then for every bounded subset* B *in* $L^2(\mathbb{R}^N)$, *there exists a constant* $T = T(B) > 0$ *such that,*

$$\|u_t(s)\|^2_{L^2(\mathbb{R}^N)} \le \rho_1 \text{ for all } u_0 \in B, \text{ and } s \ge T,$$

where $u_t(s) = \frac{d}{dt}(S(t)u_0)|_{t=s}$ *and* ρ_1 *is a positive constant independent of* B.

Proof. By differentiating (1.1) in time and denoting $v = u_t$, we get

$$\frac{\partial v}{\partial t} - \operatorname{div}(\sigma(x)\nabla v) + \lambda v + f'(u)v = 0.$$

Taking the inner product of this equality with v in $L^2(\mathbb{R}^N)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \sigma(x)|\nabla v|^2 dx + \lambda\|v\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} f'(u)|v|^2 dx = 0. \tag{4.1}$$

By hypothesis (**F**), it follows from (4.1) that

$$\frac{d}{dt}\|v\|^2_{L^2(\mathbb{R}^N)} \le 2C\|v\|^2_{L^2(\mathbb{R}^N)}. \tag{4.2}$$

On the other hand, multiplying (1.1) by $u_t(s)$ and integrating over $\mathbb{R}^N$, we obtain

$$\|u_t(s)\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{2}\frac{d}{ds}\Big(\int_{\mathbb{R}^N} \sigma(x)|\nabla u(s)|^2 dx + \lambda\|u(s)\|^2_{L^2(\mathbb{R}^N)} + 2\int_{\mathbb{R}^N} F(u(s))dx\Big)$$
$$= \int_{\mathbb{R}^N} g u_t(s)dx \le \frac{1}{2}\|g\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{2}\|u_t(s)\|^2_{L^2(\mathbb{R}^N)}.$$

Hence

(4.3)

$$\|u_t(s)\|^2_{L^2(\mathbb{R}^N)} + \frac{d}{ds}\Big(\int_{\mathbb{R}^N}\sigma(x)|\nabla u(s)|^2dx + \lambda\|u(s)\|^2_{L^2(\mathbb{R}^N)} + 2\int_{\mathbb{R}^N}F(u(s))dx\Big)$$
$$\leq \|g\|^2_{L^2(\mathbb{R}^N)}.$$

Integrating (4.3) from t to $t+1$ and using **(F')**, we obtain

$$\int_t^{t+1}\|u_t(s)\|^2_{L^2(\mathbb{R}^N)}ds + \int_{\mathbb{R}^N}\sigma(x)|\nabla u(t+1)|^2dx + (\lambda-\mu)\|u(t+1)\|^2_{L^2(\mathbb{R}^N)}$$
$$\leq \int_{\mathbb{R}^N}\sigma(x)|\nabla u(t)|^2dx + \lambda\|u(t)\|^2_{L^2(\mathbb{R}^N)} + 2\int_{\mathbb{R}^N}F(u(t))dx + \|g\|^2_{L^2(\mathbb{R}^N)}$$
$$\leq C\Big(\int_{\mathbb{R}^N}\big(\sigma(x)|\nabla u(t)|^2 + |u(t)|^2\big)dx\Big) + \|g\|^2_{L^2(\mathbb{R}^N)}.$$

Hence

$$\int_t^{t+1}\|u_t(s)\|^2_{L^2(\mathbb{R}^N)}ds \leq C(\rho, \|g\|^2_{L^2(\mathbb{R}^N)}) \tag{4.4}$$

as t large enough, where ρ is the constant in Lemma 3.1. Combining (4.2) with (4.4), and using the uniform Gronwall inequality, we have

$$\|u_t(s)\|^2_{L^2(\mathbb{R}^N)} \leq C(\rho, \|g\|^2_{L^2(\mathbb{R}^N)}).$$

The proof is complete. □

Lemma 4.2. *Suppose* $(\mathcal{H}^\infty)$ - **(F')** - **(G)** *hold. Then the semigroup* $S(t)$ *is asymptotically compact in* $\mathcal{H}^1_0(\mathbb{R}^N,\sigma)$.

Proof. Let B be a bounded subset in $L^2(\mathbb{R}^N)$, we will show that for any $\{u_{0n}\}\subset B$ and $t_n\to\infty$, $\{u_n(t_n) := S(t_n)u_{0n}\}$ is precompact in $\mathcal{H}^1_0(\mathbb{R}^N,\sigma)$. By Lemma 3.3, we can assume that $\{u_n(t_n)\}$ is a Cauchy sequence in $L^2(\mathbb{R}^N)$. For any $n,m\geq 1$, it follows from (1.1) that

$$\begin{aligned} &-\operatorname{div}(\sigma(x)\nabla(u_n(t_n)-u_m(t_m))) + \lambda(u_n(t_n)-u_m(t_m))\\ &+ f(u_n(t_n)) - f(u_m(t_m)) = -\frac{d}{dt}u_n(t_n) + \frac{d}{dt}u_m(t_m). \end{aligned} \tag{4.5}$$

Multiplying (4.5) by $u_n(t_n)-u_m(t_m)$ and using **(F)** we get

$$\begin{aligned} &\int_{\mathbb{R}^N}\sigma(x)|\nabla u_n(t_n)-u_m(t_m)|^2dx + \lambda\|u_n(t_n)-u_m(t_m)\|^2_{L^2(\mathbb{R}^N)}\\ &\leq \|u_{nt}(t_n)-u_{mt}(t_m)\|_{L^2(\mathbb{R}^N)}\|u_n(t_n)-u_m(t_m)\|_{L^2(\mathbb{R}^N)}\\ &\quad + C\|u_n(t_n)-u_m(t_m)\|^2_{L^2(\mathbb{R}^N)}. \end{aligned} \tag{4.6}$$

By Lemma 4.1, for any bounded subset B in $L^2(\mathbb{R}^N)$, there exists $T = T(B)$ such that for all $t_n\geq T$,

$$\|u_{nt}(t_n)\|_{L^2(\mathbb{R}^N)} \leq C.$$

Combining this with (4.6), it implies that, for all $n, m \geq N$,

$$\int_{\mathbb{R}^N} \sigma(x)|\nabla u_n(t_n) - u_m(t_m)|^2 dx + \lambda \|u_n(t_n) - u_m(t_m)\|^2_{L^2(\mathbb{R}^N)}$$
$$\leq C\|u_n(t_n) - u_m(t_m)\|_{L^2(\mathbb{R}^N)} + C\|u_n(t_n) - u_m(t_m)\|^2_{L^2(\mathbb{R}^N)}.$$

Hence, by Lemma 3.3, we deduce that $\{u_n(t_n)\}$ is a Cauchy sequence in $\mathcal{H}^1_0(\mathbb{R}^N, \sigma)$. □

Theorem 4.1. *Suppose* $(\mathcal{H}^\infty)$ - **(F')** - **(G)** *hold. Then the semigroup* $S(t)$ *generated by problem* (1.1) *has a compact connected global attractor* $\mathcal{A}_{\mathcal{H}^1_0}$ *in* $\mathcal{H}^1_0(\mathbb{R}^N, \sigma)$.

Proof. By Lemma 3.1, there exists a bounded absorbing set for $S(t)$ in $\mathcal{H}^1_0(\mathbb{R}^N, \sigma)$. In addition, $S(t)$ is asymptotically compact in $\mathcal{H}^1_0(\mathbb{R}^N, \sigma)$ since Lemma 4.2. Thus, there exists a global attractor for $S(t)$ in $\mathcal{H}^1_0(\mathbb{R}^N, \sigma)$. □

Remark 4.1. The global attractors $\mathcal{A}_{L^2}$ and $\mathcal{A}_{\mathcal{H}^1_0}$ obtained in Theorems 3.1 and 4.1 are of course the same object and will be denoted by $\mathcal{A}$. In particular, $\mathcal{A}$ is a compact connected set in $\mathcal{H}^1_0(\mathbb{R}^N, \sigma)$.

Acknowledgments. This research is funded by Vietnam National Foundation for Science and Technology Development(NAFOSTED) under grant number 101.01-2012.04.

References

[1] C. T. Anh, N. D. Binh, and L. T. Thuy, *On the global attractors for a class of semilinear degenerate parabolic equations*, Ann. Polon. Math. **98** (2010), no. 1, 71–89.

[2] ———, *Attractors for quasilinear parabolic equations involving weighted p-Laplacian operators*, Vietnam J. Math. **38** (2010), no. 3, 261–280.

[3] C. T. Anh, N. M. Chuong, and T. D. Ke, *Global attractor for the m-semiflow generated by a quasilinear degenerate parabolic equation*, J. Math. Anal. Appl. **363** (2010), no. 2, 444–453.

[4] C. T. Anh and P. Q. Hung, *Global existence and long-time behavior of solutions to a class of degenerate parabolic equations*, Ann. Polon. Math. **93** (2008), no. 3, 217–230.

[5] C. T. Anh and T. D. Ke, *Long-time behavior for quasilinear parabolic equations involving weighted p-Laplacian operators*, Nonlinear Anal. **71** (2009), no. 10, 4415–4422.

[6] ———, *On quasilinear parabolic equations involving weighted p-Laplacian operators*, Nonlinear Differential Equations Appl. **17** (2010), no. 2, 195–212.

[7] P. Caldiroli and R. Musina, *On a variational degenerate elliptic problem*, Nonlinear Differential Equations Appl. **7** (2000), no. 2, 187–199.

[8] V. V. Chepyzhov and M. I. Vishik, *Attractors for Equations of Mathematical Physics*, Amer. Math. Soc. Colloq. Publ., Vol. 49, Amer. Math. Soc., Providence, RI, 2002.

[9] R. Dautray and J. L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology, Vol. I: Physical origins and classical methods*, Springer-Verlag, Berlin, 1985.

[10] N. I. Karachalios and N. B. Zographopoulos, *Convergence towards attractors for a degenerate Ginzburg-Landau equation*, Z. Angew. Math. Phys. **56** (2005), no. 1, 11–30.

[11] ———, *Global attractors and convergence to equilibrium for degenerate Ginzburg-Landau and parabolic equations*, Nonlinear Anal. **63** (2005), 1749–1768.

[12] ______, *On the dynamics of a degenerate parabolic equation: Global bifurcation of stationary states and convergence*, Calc. Var. Partial Differential Equations **25** (2006), no. 3, 361–393.

[13] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969.

[14] J. C. Robinson, *Infinite-Dimensional Dynamical Systems*, Cambridge University Press, Cambridge, 2001.

[15] R. Rosa, *The global attractor for the* 2*D Navier-Stokes flow on some unbounded domains*, Nonlinear Anal. **32** (1998), no. 1, 71–85.

[16] R. Temam, *Navier-Stokes Equations and Nonlinear Functional Analysis*, 2nd edition, Philadelphia, 1995.

[17] ______, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, 2nd edition, Springer-Verlag, 1997.

[18] B. Wang, *Attractors for reaction-diffusion equations in unbounded domains*, Physica D **179** (1999), no. 1, 41–52.

Cung The Anh
Department of Mathematics
Hanoi National University of Education
136 Xuan Thuy, Cau Giay, Hanoi, Vietnam
E-mail address: anhctmath@hnue.edu.vn

Le Thi Thuy
Department of Mathematics
Electric Power University
235, Hoang Quoc Viet, Tu Liem, Hanoi, Vietnam
E-mail address: thuylephuong@gmail.com