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ON QUASI-$A(n,k)$ CLASS OPERATORS

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Abstract. To study the operator inequalities, the notions of class A operators and quasi-class A operators are developed up to recently. In this paper, quasi-$A(n,k)$ class operator for $n \geq 2$ and $k \geq 0$ is introduced as a new notion, which generalizes the quasi-class A operator. We obtain some structural properties of these operators. Also we characterize quasi-$A(n,k)$ classes for n and k via backward extension of weighted shift operators. Finally, we give a simple example of quasi-$A(n,k)$ operators with two variables.

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *p-hyponormal* if $(T^*T)^p - (TT^*)^p \geq 0$, $p \in (0,\infty)$. If $p = 1$, then T is *hyponormal*. In particular, T is said to be ∞-*hyponormal* if T is p-hyponormal for every $p > 0$ ([3]). It follows from the Löwner-Heinz inequality that every p-hyponormal operator is a q-hyponormal operator for $q \leq p$. An operator T belongs to *class A* if $|T^2| \geq |T|^2$, where $|T| = (T^*T)^{1/2}$ ([4]). The class A operator is developed as a nice application of the Furuta inequality and there are many generalized classes of operators of class A operator (cf. [2], [4], [9], [12]-[14]).

An operator T is *quasi-class A operator* if $T^*|T^2|T \geq T^*|T|^2T$ ([11]). In [5], Gao-Fang generalized this operator as k-quasiclass A for a positive integer k, i.e., $T^{*k}|T^2|T^k \geq T^{*k}|T|^2T^k$, and showed some useful inequalities of this class of operators. An operator T is a *normaloid* if $\|T\| = r(T)$, where $r(T)$ is the spectral radius of T. In [11], some structural properties of quasi-class A operator are developed in several notions, and they provide some examples which are quasi-class A operator but not normaloid. Recall that an operator T is *spectraloid* if $w(T) = r(T)$, where $w(T)$ is the numerical radius of T.

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The inclusion relationships among these classes are well known as follows:

- quasinormal $\Rightarrow$ ∞-hyponormal $\Rightarrow$ p-hyponormal $(0 < p < \infty)$ $\Rightarrow$ class A $\Rightarrow$ normaloid $\Rightarrow$ spectraloid;
- class A $\Rightarrow$ quasiclass A $\Rightarrow$ k-quasiclass A $\Rightarrow$ spectraloid.

In general the reverse of above implications are not true. The study of partial normalities such as p-hyponormality and other weak hyponormalities has been considered for more than 20 years (see [3]). But until now, we obtained a few models which characterize the properties among above classes of operators. So it is worthwhile to investigate a model to show distinctions between classes of operators.

In this paper, we consider a new notion, namely quasi-$A(n,k)$ class operator, which generalizes class A and k-quasiclass A operators.

Definition 1.1. For integers $n \geq 2$ and $k \geq 0$, a bounded operator T is called a *quasi-$A(n,k)$ class* if

$$T^{*k}|T^n|T^k \geq T^{*k}|T|^n T^k. \tag{1.1}$$

An operator T is called the *quasi-$A(\infty,k)$ class* if the inequality in (1.1) holds for all $n \geq 2$.

Obviously the quasi-$A(2,0)$ (or $A(2,1)$, respectively) class operator is referred as the class A (or quasi-class A, respectively) operator ([11]). For $k \geq 1$, the quasi-$A(2,k)$ class operator is k-quasiclass A operator ([5]). Suppose that T is ∞-hyponormal, by the Löwner-Heinz inequality, obviously we have that T has the quasi-$A(n,0)$ class for all $n \geq 2$. An easy property of operator inequality shows that if T is quasi-$A(n,k)$ class, then T is quasi-$A(n,k+1)$ class for all $k \geq 0$. Recall that if T is a p-hyponormal for $p > 0$, then $T^{*n}T^n \geq (T^*T)^n$ for all positive integer $n \leq p$ ([4]). By using Löwner-Heinz inequality, we easily show that if T is ∞-hyponormal, then T is quasi-$A(n,k)$ class operator for all $n \geq 2$ and $k \geq 0$. In fact the study of quasi-$A(n,k)$ class operators contributes to the operator gaps related to various classes of operators locating among quasinormal operators and spectraloid operators (see Remark 3.3).

This paper consists of three parts as follows. In Section 2, we show some properties of quasi-$A(n,k)$ class operators via the Hansen inequality and Hölder-McCarthy inequality. In Section 3, we consider a model to distinguish with quasi-$A(n,k)$ class operators relative to $n \geq 2$ and $k \geq 0$. As the main tool in this note, we use a backward extension of weighted shift operators. Finally, we show mutually disjoint ranges of quasi-$A(n,k)$ class operators for $n \geq 2$ and $k \geq 0$ in 2-dimensional space.

Throughout this paper, we write $\mathbb{R}_+$ for the set of positive real numbers.

2. Some properties of quasi-$A(n,k)$ class operators

We begin our work with the following lemmas.

Lemma 2.1 (Hansen inequality ([7])). *If bounded operators A and B satisfy $A \geq 0$ and $\|B\| \leq 1$, then $(B^*AB)^\delta \geq B^*A^\delta B$ for all $0 < \delta \leq 1$.*

The following lemma is a slight modification of [11, Theorem 2.2] and [5, Lemma 2.1].

Lemma 2.2. *Let T be a quasi-$A(n,k)$ class for $n \geq 2$ and $k \geq 1$ and let*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \textit{relative to } \overline{ranT^k} \oplus kerT^{*k}.$$

Assume that $ranT^k$ is not dense. Then T_1 has the inequality $|T_1^n| \geq |T_1|^n$ $(n \geq 2)$ on $\overline{ranT^k}$ and $T_3^k = 0$. Moreover, $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proof. This idea comes from proof of [11, Theorem 2.2]. Let P be the orthogonal projection of $\mathcal{H}$ onto $\overline{\mathrm{ran}T^k}$. Then $T_1 = TP = PTP$. Since T is a quasi-$A(n,k)$ class, $P(|T^n| - |T|^n)P \geq 0$. Using facts that $TP = PTP$, $PT^* = PT^*P$ and Hansen inequality, we have

$$\begin{aligned} |T_1^n| &= (T_1^{*n}T_1^n)^{1/2} = (\underbrace{PT^* \cdots PT^*PT^*PT^*}_{n}\underbrace{TPTPTP\cdots TP}_{n})^{1/2} \\ &= (PT^* \cdots PT^*PT^*T^*TTPTP \cdots TP)^{1/2} \\ &= (PT^* \cdots PT^*T^*T^*TTTP \cdots TP)^{1/2} = \cdots \\ &= (P|T^n|^2P)^{1/2} \geq P|T^n|P. \end{aligned}$$

Since $|T|P = P|T|P$ on $\overline{\mathrm{ran}T^k}$, we get $(P|T|^2P)^{1/2} = P|T|P$ on $\overline{\mathrm{ran}T^k}$. So

$$|T_1|^n = (PT^*TP)^{n/2} = (P|T|P)^n = P|T|^nP.$$

Hence $|T_1^n| \geq P|T^n|P \geq P|T|^nP = |T_1|^n$ for $n \geq 2$.

For any $x = (x_1, x_2) \in \overline{\mathrm{ran}T^k} \oplus \ker T^{*k}$, we have

$$\langle T_3^k x_2, x_2 \rangle = \langle T^k(I-P)x, (I-P)x \rangle = \langle (I-P)x, T^{*k}(I-P)x \rangle = 0,$$

which implies that $T_3^k = 0$.

Since $\sigma(T) \cup \mathfrak{G} = \sigma(T_1) \cup \sigma(T_3)$, where $\mathfrak{G}$ is the union of holes in $\sigma(T)$ which happens to be a subset of $\sigma(T_1) \cap \sigma(T_3)$ (see [6]), $\sigma(T_3) = \{0\}$, and $\sigma(T_1) \cap \sigma(T_3)$ has no interior points, we have $\sigma(T) = \sigma(T_1) \cup \{0\}$. □

Lemma 2.3 (Hölder-McCarthy inequality ([3]))**.** *Let $A \geq 0$. Then the following assertions hold.*

(i) $\langle A^r x, x \rangle \geq \langle Ax, x \rangle^r \|x\|^{2(1-r)}$ *for $r \geq 1$ and all $x \in \mathcal{H}$.*

(ii) $\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r \|x\|^{2(1-r)}$ *for $0 \leq r \leq 1$ and all $x \in \mathcal{H}$.*

The following results are slight improvements of [5, Theorem 2.2].

Proposition 2.4. *Let T be a quasi-$A(n,k)$ class for $n \geq 2$ and $k \geq 0$. Then*

(i) $\|T^{n+m}x\|\|T^m x\| \geq \|T^{m+1}x\|^n \|T^m x\|^{2-n}$ *for all $x \in \mathcal{H}$ and all $m \geq k$.*

(ii) *If $T^m = 0$ for some $m \geq k$, then $T^{k+1} = 0$.*

Proof. (i) From the operator inequality, it is clear that quasi-$A(n,k)$ class operators are quasi-$A(n,k+1)$ class operators, and so we will prove for the case $m = k$. Without loss of generality, we assume that $T^k x \neq 0$.

Using Hölder-McCarthy inequality, we have that for all $x \in \mathcal{H}$,

$$\langle T^{*k}|T^n|T^k x, x\rangle \leq \langle |T^n|^2 T^k x, T^k x\rangle^{1/2} \|T^k x\|^{2(1-1/2)} = \|T^{n+k}x\| \|T^k x\|,$$

$$\langle T^{*k}|T|^n T^k x, x\rangle \geq \langle |T|^2 T^k x, T^k x\rangle^{n/2} \|T^k x\|^{2(1-n/2)} = \|T^{k+1}x\|^n \|T^k x\|^{2-n}.$$

Hence we obtain that

$$\|T^{n+k}x\| \ \|T^k x\| \geq \|T^{k+1}x\|^n \ \|T^k x\|^{2-n}, \ x \in \mathcal{H}.$$

(ii) If $T^k = 0$, then it is obvious that $T^{k+1} = 0$. Suppose that $T^k \neq 0$ and $T^{k+j} = 0$ for some $j \geq 2$. Take $n = j$ and $m = k$ in Proposition 2.4(i). It follows from (i) that $T^{k+1} = 0$. □

Theorem 2.5. *Let T be a quasi-$A(n,k)$ class for $n \geq 2$ and $k \geq 1$. If $(T - \lambda)x = 0$ for some $\lambda \neq 0$, then $(T - \lambda)^* x = 0$.*

Proof. We may assume that $x \neq 0$. Let $\mathcal{M} = \operatorname{span}\{x\}$. Then $\mathcal{M}$ is an invariant subspace of T and

$$T = \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on } \mathcal{M} \oplus \mathcal{M}^\perp.$$

Let P be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. For the proof, we claim that $T_2 = 0$. Since T is a quasi-$A(n,k)$ class and $x = T^k(x/\lambda^k) \in \overline{\operatorname{ran} T^k}$, we have $P(|T^n| - |T|^n)P \geq 0$. It follows from $TP = PTP$ and $PT^* = PT^*P$ that

$$\begin{aligned} P|T^n|^2 P &= P \underbrace{T^* \cdots T^*}_{n} \underbrace{T \cdots T}_{n} P = PT^*PT^* \cdots T^*T \cdots TPTP \\ &= \quad \cdots \quad = \underbrace{PT^* \cdots PT^*}_{n} \underbrace{TP \cdots TP}_{n} = \begin{pmatrix} |\lambda|^{2n} & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Then using Hansen inequality, we have that

$$\begin{pmatrix} |\lambda|^n & 0 \\ 0 & 0 \end{pmatrix} = (P|T^n|^2 P)^{1/2} \geq P|T^n|P \geq P|T|^n P = \begin{pmatrix} |\lambda|^n & 0 \\ 0 & 0 \end{pmatrix}.$$

So we may write

$$|T^n| = \begin{pmatrix} |\lambda|^n & A \\ A^* & B \end{pmatrix}.$$

Hence

$$\begin{aligned} \begin{pmatrix} |\lambda|^{2n} & 0 \\ 0 & 0 \end{pmatrix} &= P|T^n||T^n|P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} |\lambda|^n & A \\ A^* & B \end{pmatrix} \begin{pmatrix} |\lambda|^n & A \\ A^* & B \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} |\lambda|^{2n} + AA^* & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

which implies that $A = 0$ and

$$|T^n|^2 = \begin{pmatrix} |\lambda|^{2n} & 0 \\ 0 & B^2 \end{pmatrix}.$$

On the other hand,

$$|T^n|^2 = \begin{pmatrix} \overline{\lambda} & 0 \\ T_2^* & T_3^* \end{pmatrix} \cdots \begin{pmatrix} \overline{\lambda} & 0 \\ T_2^* & T_3^* \end{pmatrix} \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix} \cdots \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix}$$
$$= \begin{pmatrix} |\lambda|^{2n} & \overline{\lambda}^n C \\ \lambda^n C^* & |C|^2 + |T_3^n|^2 \end{pmatrix},$$

where $C = \sum_{i=0}^{n-1} T_2 T_3^i \lambda^{n-1-i}$. Hence $C = 0$ and $B = |T_3^n|$. Also using Lemma 2.2, we have

$$P|T|^{2n}P = P(T^*T)^n P = \begin{pmatrix} |\lambda|^{2n} & 0 \\ 0 & 0 \end{pmatrix}.$$

From $|T|^2 = T^*T$ and some computations, we may write for $n \geq 1$

$$|T|^{2n} = \begin{pmatrix} A_{2n} & B_{2n} \\ B_{2n}^* & C_{2n} \end{pmatrix},$$

where

$$\begin{aligned} A_{2m} &= A_{2(m-1)}A_2 + B_{2(m-1)}B_2^*, \ A_2 \equiv |\lambda|^2, \\ B_{2m} &= A_{2(m-1)}B_2 + B_{2(m-1)}C_2, \ B_2 \equiv \overline{\lambda}T_2, \\ C_{2m} &= B_{2(m-1)}^*B_2 + C_{2(m-1)}C_2, \ C_2 \equiv |T_2|^2 + |T_3|^2 \end{aligned} \tag{2.1}$$

for all $m \geq 2$. For each $n \geq 2$,

$$\begin{pmatrix} |\lambda|^{2n} & 0 \\ 0 & 0 \end{pmatrix} = P|T|^{2n}P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_{2n} & B_{2n} \\ B_{2n}^* & C_{2n} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A_{2n} & 0 \\ 0 & 0 \end{pmatrix}.$$

Using the recurrence formula (2.1), we have

$$\begin{aligned} A_{2n} &= A_{2(n-1)}|\lambda|^2 + B_{2(n-1)}B_2^* \\ &= A_{2(n-2)}|\lambda|^4 + \left(|\lambda|^2 B_{2(n-2)} + B_{2(n-1)}\right) B_2^* \ = \ \cdots \\ &= |\lambda|^2|\lambda|^{2(n-1)} + \left(|\lambda|^{2(n-2)}B_2 + |\lambda|^{2(n-3)}B_4 + \cdots + B_{2(n-1)}\right) B_2^* \\ &= |\lambda|^{2n} + \lambda \left(|\lambda|^{2(n-2)}B_2 + |\lambda|^{2(n-3)}B_4 + \cdots + B_{2(n-1)}\right) T_2^*, \end{aligned}$$

which implies that $T_2 = 0$ for all $n \geq 2$ because of $|\lambda|^{2n} = A_{2n}$ $(\lambda \neq 0)$. □

3. Distinctions of quasi-$A(n,k)$ class operators

In this section we characterize the quasi-$A(n,k)$ class weighted shift for all $n \geq 2$ and $k \geq 0$ for being distinction of quasi-$A(n,k)$ class operators. For this purpose, we consider a backward extension of weighted shift (cf. [8]).

For a sequence $\alpha = \{\alpha_i\}_{i=0}^{\infty}$ of positive real numbers, a weighted shift W_α is defined by $W_\alpha e_n = \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_i\}_{i=0}^{\infty}$ is an orthonormal basis for $\ell^2(\mathbb{Z}_+)$. Obviously, W_α is normal if and only if $\alpha_n = 0$ for all $n \geq 0$, and W_α is quasinormal if and only if $\alpha_n(\alpha_{n+1}^2 - \alpha_n^2) = 0$ for all $n \geq 0$. Moreover, W_α is p-hyponormal for all [some] $0 < p < \infty$ if and only if α is monotone increasing, i.e., $\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots$. By a simple calculation, we see that W_α is of class

A if and only if the sequence α is monotone increasing. But there exists a sequence $\{\alpha_n\}_{n=0}^{\infty} \subset \mathbb{R}_+$ such that $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \cdots$ and W_α is quasi-class A but not normaloid ([11]). Moreover, W_α is k-quasiclass A operator if and only if $\alpha_k \leq \alpha_{k+1} \leq \alpha_{k+2} \leq \cdots$ for a positive integer k ([5]).

Now we obtain some conditions of quasi-$A(n,k)$ class property for any $n \geq 2$ and $k \geq 0$ for a weighted shift operator.

Lemma 3.1. *Let W_α be a weighted shift with weight sequence $\alpha = \{\alpha_i\}_{i=0}^{\infty}$. Then W_α is quasi-$A(n,k)$ class if and only if*

$$\prod_{i=1}^{n-1} \alpha_{k+j+i} \geq \alpha_{k+j}^{n-1} \quad \textit{for all } j \geq 0. \tag{3.1}$$

Proof. By direct computations, we have that

$$\begin{aligned} & W_\alpha^{*k}|W_\alpha^n|W_\alpha^k \geq W_\alpha^{*k}|W_\alpha|^n W_\alpha^k \\ & \qquad \Leftrightarrow \operatorname{Diag}\left\{ \prod_{i=j}^{k-1+j} \alpha_i \left(\prod_{i=k+j}^{k+n-1+j} \alpha_i - \alpha_{k+j}^n \right) \right\}_{j=0}^{\infty} \geq 0, \end{aligned}$$

which is equivalent to (3.1). □

Corollary 3.2. *Let W_α be a weighted shift with weight $\alpha = \{\alpha_i\}_{i=0}^{\infty}$. If α is an increasing sequence, then W_α is quasi-$A(n,k)$ class for all $n \geq 2$ and $k \geq 0$.*

Proof. Using Lemma 3.1, it is obvious. □

Remark 3.3. Consider a weighted shift W_α with $\alpha_0 = 2$ and $\alpha_n = 1$ $(n \geq 1)$. From simple computations, W_α turns to be quasi-$A(2,k)$ class for all $k \geq 1$. i.e., W_α is ∞-quasiclass A, equivalently, quasi-$A(2,\infty)$ class. However, since

$$r(W_\alpha) = \lim_{n\to\infty} \|W_\alpha^n\|^{\frac{1}{n}} = 1,$$

it turns out that W_α is not normaloid.

To find gaps among classes of quasi-$A(n,k)$ class operators with respect to n and k, we consider a backward extension weighted shift operator (cf. [8]). For a weighted shift W_α with weight sequence $\alpha = \{\alpha_i\}_{i=0}^{\infty}$ and $\ell \in \mathbb{N}$, let

$$\alpha(x_1, \ldots, x_\ell) : x_1, x_2, \ldots, x_\ell, \alpha_0, \alpha_1, \ldots$$

be an augmented sequence with positive real number x_j, $1 \leq j \leq \ell$. Such a weighted shift $W_{\alpha(x_1,\ldots,x_\ell)}$ is called an *ℓ-step backward extension weighted shift operator* of W_α (cf. [10]). Write the set

$$\mathbb{R}_+^\ell := \{(x_1, \ldots, x_\ell) : x_i > 0, \quad 1 \leq i \leq \ell\}.$$

For finding the region for distinction of quasi-$A(n,k)$ class operators, we consider the following sets

$$\mathbb{A}(n,k;\ell) := \{(x_1, \ldots, x_\ell) \in \mathbb{R}_+^\ell : W_{\alpha(x_1,\ldots,x_\ell)} \text{ is quasi-}A(n,k) \text{ class}\} \tag{3.2}$$

for $n \geq 2$ and $k \geq 0$.

The following example proves easily that the increasing condition of weight sequence is essential in order to show gaps between classes of quasi-$A(n,k)$ class operators each other for all $n \geq 2$ and $k \geq 0$.

Example 3.4. Let $\alpha : 2, 2, 1, 1, 1, \dots$ and let $\alpha(x) : x, 2, 2, 1, 1, 1, \dots$. We consider the corresponding weighted shift operators W_α and $W_{\alpha(x)}$ with weight sequences α and $\alpha(x)$, respectively. A straightforward calculation shows that W_α can be never quasi-$A(n,k)$ class operator for all $n \geq 2$ and $k \geq 0$. Moreover, we can obtain that $\mathbb{A}(n,k;1) = \emptyset$ for all $n \geq 2$ and $k \geq 0$.

Proposition 3.5. *For a positive integer ℓ, let $W_{\alpha(x_1,\dots,x_\ell)}$ be an ℓ-step backward extension of weighted shift operator W_α with an increasing sequence α. Then $\mathbb{A}(n,k;\ell) = \mathbb{R}_+^\ell$ for all $n \geq 2$ and $k \geq \ell$.*

Proof. For brevity, we consider a sequence $\beta = \{\beta_m\}_{m=0}^\infty$, where

$$\beta_m = \begin{cases} x_{m+1} & (0 \leq m \leq \ell - 1), \\ \alpha_{m-\ell} & (m \geq \ell). \end{cases}$$

For all $n \geq 2$ and $k \geq 0$, it follows from Lemma 3.1 that

$$W_{\alpha(x_1,\dots,x_\ell)} \text{ is quasi-}A(n,k) \text{ class } \iff \Pi_{i=0}^{n-1} \beta_{k+j+i} \geq \beta_{k+j}^n \quad (j \geq 0).$$

For $k \geq \ell$, using the increasing property of α in (3.1), we have that

$$\Pi_{i=0}^{n-1} \beta_{k+j+i} = \alpha_{k+j-\ell} \alpha_{k+j-\ell+1} \cdots \alpha_{k+j-\ell+n-1} \geq \alpha_{k+j-\ell}^n = \beta_{k+j}^n$$

for all $j \geq 0$ and all $(x_1, \dots, x_\ell) \in \mathbb{R}_+^\ell$. Hence $\mathbb{A}(n,k;\ell) = \mathbb{R}_+^\ell$ for all $k \geq \ell$. $\square$

Theorem 3.6. *For a positive integer ℓ, let $W_{\alpha(x_1,\dots,x_\ell)}$ be an ℓ-step backward extension of weighted shift W_α with an increasing sequence α. Suppose that $m, n \geq 2$ and $0 \leq p,\ q < \ell$. Then*

(i) *it holds that*

$$\mathbb{A}(m,p;\ell)$$
$$= \left\{ (x_1, ..., x_\ell) : 0 < x_i \leq \left(\prod_{j_1=i+1}^{\ell} x_{j_1} \prod_{j_2=0}^{m-l+i-2} \alpha_{j_2} \right)^{\frac{1}{m-1}},\ i = p+1, \dots, \ell \right\},$$

(ii) *it holds that $\mathbb{A}(m,p;\ell) \neq \mathbb{A}(n,q;\ell) \iff (m,p) \neq (n,q)$.*

Proof. (i) Using the condition (3.1) about $W_{\alpha(x_1,\dots,x_\ell)}$ and the condition of increasing sequence α, we have that

$$\begin{aligned} & W_{\alpha(x_1,\dots,x_\ell)} \text{ is quasi-}A(m,p;\ell) \text{ class} \\ \iff & x_{p+j+2} \cdots x_\ell \cdot \alpha_0 \cdots \alpha_{p+j+m-1-\ell} \geq x_{p+j+1}^{m-1} \ (0 \leq j \leq \ell - p - 1) \\ \iff & x_i^{m-1} \leq \prod_{j_1=i+1}^{\ell} x_{j_1} \prod_{j_2=0}^{m-\ell+i-2} \alpha_{j_2} \ (p+1 \leq i \leq \ell). \end{aligned}$$

(ii) We first show that $\mathbb{A}(m,p;l) \neq \mathbb{A}(m,q;l)$ for $m=n$ and $0 \leq p \neq q \leq \ell$. From (3.1) and the increasing condition of α, the following holds:

$$\begin{aligned}&(x_1,\ldots,x_\ell) \in \mathbb{A}(m,p;\ell)\\ \Longleftrightarrow\ & x_{i+1}\cdots x_\ell \cdot \alpha_0 \cdots \alpha_{i+m-\ell-2} \geq x_i^{m-1} \ (p+1 \leq i \leq \ell).\end{aligned}$$

This holds for all $x_j > 0$ $(j=1,\ldots,p)$, which implies that $\mathbb{A}(m,p;\ell) \neq \mathbb{A}(m,q;\ell)$ for $0 \leq p \neq q < \ell$.

Next, we claim that if $m \neq n$ and $0 \leq k < \ell$, then $\mathbb{A}(m,k;\ell) \neq \mathbb{A}(n,k;\ell)$. In fact, from Proposition 3.5, we see that $\mathbb{A}(m,k;\ell) = \mathbb{A}(n,k;\ell)$ for $k \geq \ell$ and all $m,n \geq 2$. It is sufficient to show that for $0 \leq k < \ell$. To show the distinctions of the classes $\mathbb{A}(m,k;l)$, we consider the last two cases of $i=\ell-1$ and $i=\ell$. In fact, we can verify that

$$\frac{x_{\ell-1}^{\eta-1}}{\alpha_0\cdots\alpha_{\eta-3}} \leq x_\ell \leq (\alpha_0\cdots\alpha_{\eta-2})^{\frac{1}{\eta-1}}$$

for $\eta=m$ and $\eta=n$. Suppose $2 \leq m \lneq n$. From the increasing condition of α, we obtain that $1/(\alpha_0\cdots\alpha_{n-3}) < 1/(\alpha_0\cdots\alpha_{m-3})$. If we denote $a := (\alpha_0\cdots\alpha_{m-2})^{\frac{1}{m-1}}$ and $b := (\alpha_0\cdots\alpha_{n-2})^{\frac{1}{n-1}}$, then

$$\begin{aligned}(m-1)(n-1)(\ln b - \ln a) &= (m-n)\sum_{i=0}^{m-2} \ln\alpha_i + (m-1)\sum_{i=m-1}^{n-2} \ln\alpha_i\\ &\geq (m-n)(m-1)\ln\alpha_{m-2} + (m-1)(n-m)\ln\alpha_{m-1}\\ &= (n-m)(m-1)\,(\ln\alpha_{m-1} - \ln\alpha_{m-2}) > 0.\end{aligned}$$

Hence $b > a$ for $2 \leq m < n$, which induces that $\mathbb{A}(m,k;\ell) \neq \mathbb{A}(n,k;\ell)$ for $2 \leq m < n$ and $0 \leq k < \ell$. Also we observe that $\mathbb{A}(m,k;\ell) \subset \mathbb{A}(n,k;\ell)$ for $m \leq n$ and $\mathbb{A}(m,p;\ell) \subset \mathbb{A}(m,q;\ell)$ for $p \leq q < \ell$. This proves the arbitrary cases by the above two cases. □

We now close this note with an example of the quasi-$A(n,k)$ class operators with two positive variables which provides a distinction for such classes.

Example 3.7. Consider an augmented weight sequence $\alpha(\sqrt{x},\sqrt{y})$: $\sqrt{x}$, $\sqrt{y}$, α_0, α_1, ... with $\alpha_n = \sqrt{\frac{n+3}{n+4}}$ $(n \geq 0)$ and positive real variables x and y. Let $W_{\alpha(\sqrt{x},\sqrt{y})}$ be a 2-step backward extension of a weighted shift W_α. From some easy computations, we can obtain that

$$\{(x,y) \in \mathbb{R}_+^2 : W_{\alpha(\sqrt{x},\sqrt{y})} \text{ is quasinormal}\} = \varnothing.$$

But our model $\mathbb{A}(n,k;\ell)$ as in (3.2) has a region below the classes of quasinormal operators. For $n \geq 2$ and $k \geq 0$, we denote

$$\mathbb{A}(n,k) := \mathbb{A}(n,k;2) = \{(x,y) \in \mathbb{R}_+^2 : W_{\alpha(\sqrt{x},\sqrt{y})} \text{ is quasi-}A(n,k) \text{ class}\}.$$

For $k \geq 2$, the increasing property of $\{\alpha_i\}_{i=0}^{\infty}$ proves that the condition (3.1) holds for all $x, y \geq 0$, i.e.,

$$\mathbb{A}(n,k) = \mathbb{R}_+^2 \text{ for all } k \geq 2.$$

Now we only consider the cases such that $k = 0$ and 1 for each $n \geq 2$. Also we note that if $k + j \geq 2$ for all $k, j \geq 0$, then the condition (3.1) holds for all $x, y > 0$. In order to show the distinctions among the sets $\mathbb{A}(n,k)$ for $n \geq 2$ and $k = 0, 1$, we find equivalent conditions to (3.1) for each n and k. By some calculations in condition (3.1), we can have the followings:

$$\mathbb{A}(n,0) = \left\{(x,y) \in \mathbb{R}_+^2 : x^{n-1} \leq \frac{3y}{n+1},\ y^{n-1} \leq \frac{3}{n+2}\right\};$$
$$\mathbb{A}(n,1) = \left\{(x,y) \in \mathbb{R}_+^2 :\ y^{n-1} \leq \frac{3}{n+2}\right\},$$

which imply that the sets $\mathbb{A}(n,k)$ are distinct with respect to n and k. Further, we also have that $\mathbb{A}(m,k) \subsetneq \mathbb{A}(n,k)$ for $2 \leq m \lneq n$ and $k = 0, 1$ (Indeed, since the function $f(x) = \left(\frac{3}{x+2}\right)^{1/(x-1)}$ is strictly increasing for $x \geq 2$, we have that $\left(\frac{3}{m+2}\right)^{1/(m-1)} < \left(\frac{3}{n+2}\right)^{1/(n-1)}$ for $2 \leq m < n$). Hence

$$W_{\alpha(\sqrt{x},\sqrt{y})} \text{ is class } A \text{ operator} \Longleftrightarrow (x,y) \in \mathbb{A}(2,0) \Longleftrightarrow 0 < x \leq y \leq 3/4,$$
$$W_{\alpha(\sqrt{x},\sqrt{y})} \text{ is quasi-class } A \text{ operator} \Longleftrightarrow (x,y) \in \mathbb{A}(2,1) = \mathbb{R}_+ \times (0, 3/4].$$

And we have

$$W_{\alpha(\sqrt{x},\sqrt{y})} \text{ is quasi-}A(\infty,0) \text{ class} \Longleftrightarrow (x,y) \in \cup_{n=2}^{\infty}\mathbb{A}(n,0) = (0,1) \times (0,1),$$
$$W_{\alpha(\sqrt{x},\sqrt{y})} \text{ is quasi-}A(\infty,1) \text{ class} \Longleftrightarrow (x,y) \in \cup_{n=2}^{\infty}\mathbb{A}(n,1) = \mathbb{R}_+ \times (0,1).$$

References

[1] M. Cho and T. Yamazaki, *An operator transform from class A to the class of hyponormal operators and its application*, Integral Equations Operator Theory **53** (2005), no. 4, 497–508.

[2] B. Duggal, I. Jeon, and C. Kubrusly, *Contractions satisfying the absolute value property* $|A|^2 \leq |A^2|$, Integral Equations Operator Theory **49** (2004), no. 2, 141–148.

[3] T. Furuta, *Invitation to Linear Operators*, Taylor and Francis Inc. London and New York, 2001.

[4] T. Furuta, M. Ito, and T. Yamazaki, *A subclass of paranormal operators including class of log-hyponormal and several related classes*, Sci. Math. **1** (1998), no. 3, 389–403.

[5] F. Gao and X. Fang, *On k-quasiclass A operators*, J. Inequal. Appl. **2009** (2009), Article ID 921634, 10 pages.

[6] J. K. Han, H. Y. Lee, and W. Y. Lee, *Invertible completions of* 2×2 *upper triangular operator matrices*, Proc. Amer. Math. Soc. **128** (2000), no. 1, 119–123.

[7] F. Hansen, *An operator inequality*, Math. Ann. **246** (1980), no. 3, 249–250.

[8] T. Hoover, I. B. Jung, and A. Lambert, *Moment sequences and backward extensions of subnormal weighted shifts*, J. Aust. Math. Soc. **73** (2002), no. 1, 27–36.

[9] M. Ito, *On classes of operators generalizing class A and paranormality*, Sci. Math. Jpn. **57** (2003), no. 2, 287–297.

[10] Z. Jablonski, I. B. Jung, and J. Stochel, *Backward extensions of hyperexpansive operators*, Studia Math. **173** (2006), no. 3, 233–257.
[11] I. H. Jeon and I. H. Kim, *On operators satisfying* $T^*|T^2|T \geq T^*|T|^2T^*$, Linear Algebra Appl. **418** (2006), no. 2-3, 854–862.
[12] S. Jung, E. Ko, and M. Lee, *On class A operators*, Studia Math. **198** (2010), no. 3, 249–260.
[13] A. Uchiyama and K. Tanahashi, *On the Riesz idempotent of class A operators*, Math. Inequal. Appl. **5** (2002), no. 2, 291–298.
[14] A. Uchiyama, K. Tanahashi, and J. Lee, *Spectrum of class* $A(s,t)$ *operators*, Acta Sci. Math. (Szeged) **70** (2004), no. 1-2, 279–287.

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