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COUPLED $\mathcal{N}$-STRUCTURES APPLIED TO IDEALS IN d-ALGEBRAS

Sun Shin Ahn and Jung Mi Ko

Abstract. The notions of coupled $\mathcal{N}$-subalgebra, coupled (positive implicative) $\mathcal{N}$-ideals of d-algebras are introduced, and related properties are investigated. Characterizations of a coupled $\mathcal{N}$-subalgebra and a coupled (positive implicative) $\mathcal{N}$-ideals of d-algebras are given. Relations among a coupled $\mathcal{N}$-subalgebra, a coupled $\mathcal{N}$-ideal and a coupled positive implicative $\mathcal{N}$-ideal of d-algebras are discussed.

1. Introduction

J. Neggers and H. S. Kim ([14]) introduced the idea of a d-algebra as a generalization of BCK-algebras introduced by Y. Imai and K. Iséki ([8]). This class of algebras has been studied in rather great detail and is of current interest to many researchers ([3, 4, 10, 12, 13]). Beside d-algebras, other generalizations of BCK-algebras include the class of BCI-algebras, also introduced by Y. Imai and K. Iséki ([7]), the class of BCH-algebras, introduce by Q. P. Hu and X. Li ([5, 6]), of which the class of BCI-algebras is a proper subclass. The class of BCK-algebras is a proper subclass of the class of d-algebras, and it has been shown that many constructions on the class of d-algebras leave this class invariant, but not so the class of BCK-algebras. For example, the mirror algebra of a d-algebra is a d-algebra, but the mirror algebra of a BCK-algebra is not necessarily a BCK-algebra ([3]). J. Neggers, Y. B. Jun and H. S. Kim ([13]) have studied the ideal theory of d-algebras, introducing the notions of d-subalgebra, d-ideal, $d^{\#}$-ideal and d^*-ideal, and various relations among them. After that some further aspects were studied in [1, 4, 10, 12]. In [1], S. S. Ahn and K. H. Han introduced the notion of $\mathcal{N}$-subalgebras, (positive implicative) $\mathcal{N}$-ideals of d-algebras and investigated their related properties. Jun et al. ([9]) introduced the notion of coupled $\mathcal{N}$-structures and its application in BCK/BCI-algebras was discussed.

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In this paper, we introduce the notion of coupled $\mathcal{N}$-structures, and discuss its application in d-algebras. We introduce the notions of a coupled $\mathcal{N}$-subalgerba, a coupled $\mathcal{N}$-ideal and a coupled positive implicative $\mathcal{N}$-ideal, and investigate their relations. We discuss characterizations of a coupled $\mathcal{N}$-ideal and a coupled positive implicative $\mathcal{N}$-ideal.

2. Preliminaries

A *d-algebra* ([14]) is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms:

(I) $x * x = 0$,
(II) $0 * x = 0$,
(III) $x * y = 0$ and $y * x = 0$ imply $x = y$ for all $x, y \in X$.

For brevity we also call X a *d-algebra*. In X, we can define a binary relation " $\leq$ " by $x \leq y$ if and only if $x * y = 0$.

A BCK-algebra is a d-algebra $(X; *, 0)$ satisfying the additional axioms:

(IV) $((x * y)) * (x * z)) * (z * y) = 0$,
(V) $(x * (x * y)) * y = 0$ for all $x, y, z \in X$.

Example 2.1 ([13])**.** Let $\mathbb{R}$ be the set of all real numbers and define $x * y := -x(x-y), x, y \in \mathbb{R}$, where "$\cdot$" and "$-$" are the ordinary product and subtraction of real numbers. Then $x * x = 0,\ 0 * x = 0$. If $x * y = y * x = 0$, then $x(x-y) = y(y-x)$ and so $(x-y)(x+y) = 0$. Hence $x = y$ or $x = -y$. If $x = -y$, then $0 = x*(-x) = -x(x+x)$ and $0 = -x*x = -(-x)((-x)-x) = x(-2x) = -2x^2$. Thus $x = y = 0$. Hence $x = y$. Therefore $(\mathbb{R}; *, 0)$ is a d-algebra, but not a BCK-algebra, since $5 * 0 = -5^2 = -25 \neq 5$.

Definition 2.2 ([13])**.** Let $(X; *, 0)$ be a d-algebra and $\emptyset \neq I \subseteq X$. I is a d-subalgebra of X if $x * y \in I$ whenever $x \in I$ and $y \in I$. I is called a *BCK-ideal* (briefly, an *ideal*) of X if it satisfies:

(D_0) $0 \in I$,
(D_1) $x * y \in I$ and $y \in I$ imply $x \in I$.

I is called a *d-ideal* of X if it satisfies (D_1) and

(D_2) $x \in I$ and $y \in X$ imply $x * y \in I$, i.e., $I * X \subseteq I$.

I is called a *$d^{\#}$-ideal* of X if it satisfies $(D_1), (D_2)$ and

(D_3) $x * z \in I$ whenever $x * y \in I$ and $y * z \in I$ for any $x, y, z \in X$.

I is called a *d^*-ideal* of X if it satisfies (D_1), (D_2), (D_3) and

(D_4) $x * y \in I$ and $y * x \in I$ imply $(x * z) * (y * z) \in I$ and $(z * x) * (z * y) \in I$ for any $x, y, z \in X$.

Example 2.3 ([14])**.** (a) Let $X := \{0, a, b, c, d\}$ be a d-algebra which is not a BCK-algebra with the following table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	c	0
c	c	c	b	0	c
d	c	c	a	a	0

Then $L := \{0, a\}$ is a d-ideal which is not a $d^{\#}$-ideal of X, since $b * d = 0 \in L, d * c = a \in L$, but $b * c = c \notin L$.

(b) Let $X := \{0, a, b, c\}$ be a d-algebra which is not a BCK-algebra with the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	c	b	0	c
c	c	b	b	0

Then $I := \{0, a\}$ is a $d^{\#}$-ideal, but not a d^*-ideal, since $0 * a = 0 \in I$ and $a * 0 = a \in I$, but $(c * 0) * (c * a) = c * b = b \notin I$.

(c) Let $X := \{0, a, b, c\}$ be a d-algebra which is not a BCK-algebra with the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	a	0

Then $J := \{0, a\}$ is a d^*-ideal of X.

Note that d^*-ideal $\subsetneq$ $d^{\#}$-ideal $\subsetneq$ d-ideal in a d-algebra (see [14]).

Definition 2.4 ([14])**.** Let $(X; *, 0)$ be a d-algebra and $x \in X$. Define $x * X := \{x * a \mid a \in X\}$. X is said to be *edge* if for any $x \in X$, $x * X = \{x, 0\}$.

Lemma 2.5 ([14])**.** *Let $(X; *, 0)$ be an edge d-algebra. Then $x * 0 = x$ for any $x \in X$.*

Definition 2.6 ([13])**.** A d-algebra X is called a d^*-algebra if it satisfies the identity $(x * y) * x = 0$ for all $x, y \in X$.

Clearly, a BCK-algebra is a d^*-algebra, but the converse need not be true (see [13]).

Theorem 2.7 ([13])**.** *In a d^*-algebra, every BCK-ideal is a d-ideal.*

Corollary 2.8 ([13])**.** *In a d^*-algebra, every BCK-ideal is a d-subalgebra.*

Definition 2.9 ([1])**.** Let $(X;*,0)$ be a d-algebra, and let $\emptyset \neq I \subseteq X$. I is called a *positive implicative ideal* if it satisfies, for all $x, y, z \in X$,

(i) $0 \in I$,

(ii) $(x * y) * z \in I$ and $y * z \in I$ imply $x * z \in I$.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$ (briefly, $\mathcal{N}$-*function* on X). By an $\mathcal{N}$-*structure* we mean an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X. We define an order relation "$\ll$" on $[-1, 0] \times [-1, 0]$ as follows:

$$(\forall (r_1, k_1), (r_2, k_2) \in [-1, 0] \times [-1, 0]) \, ((r_1, k_1) \ll (r_2, k_2) \;\Leftrightarrow\; r_1 \leq r_2,\; k_1 \geq k_2).$$

3. Coupled $\mathcal{N}$-structures applied to subalgebras and ideals in d-algebras

Definition 3.1 ([9])**.** A *coupled $\mathcal{N}$-structure* $\mathcal{C}$ in a nonempty set X is an object of the form

$$\mathcal{C} = \{\langle x; f_{\mathcal{C}}, g_{\mathcal{C}} \rangle : x \in X\},$$

where $f_{\mathcal{C}}$ and $g_{\mathcal{C}}$ are $\mathcal{N}$-functions on X such that $-1 \leq f_{\mathcal{C}}(x) + g_{\mathcal{C}}(x) \leq 0$ for all $x \in X$.

A coupled $\mathcal{N}$-structure $\mathcal{C} = \{\langle x; f_{\mathcal{C}}, g_{\mathcal{C}} \rangle : x \in X\}$ in X can be identified to an ordered pair $(f_{\mathcal{C}}, g_{\mathcal{C}})$ in $\mathcal{F}(X, [-1, 0]) \times \mathcal{F}(X, [-1, 0])$. For the sake of simplicity, we shall use the notation $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ instead of $\mathcal{C} = \{\langle x; f_{\mathcal{C}}, g_{\mathcal{C}} \rangle : x \in X\}$.

For a coupled $\mathcal{N}$-structure $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ in X and $t, s \in [-1, 0]$ with $t + s \geq -1$, the set

$$\mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t, s)\} = \{x \in X \mid f_{\mathcal{C}}(x) \leq t,\; g_{\mathcal{C}}(x) \geq s\}$$

is called an $\mathcal{N}(t, s)$-level set of $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$. An $\mathcal{N}(t, t)$-level set of $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is called an $\mathcal{N}$-level set of $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$.

Definition 3.2. A coupled $\mathcal{N}$-structure $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ in a d-algebra X is called a *coupled $\mathcal{N}$-subalgebra* of X if it satisfies:

$$(3.1) \qquad f_{\mathcal{C}}(x * y) \leq \bigvee \{f_{\mathcal{C}}(x), f_{\mathcal{C}}(y)\} \quad \text{and} \quad g_{\mathcal{C}}(x * y) \geq \bigwedge \{g_{\mathcal{C}}(x), g_{\mathcal{C}}(y)\}$$

for all $x, y \in X$.

Theorem 3.3. *A coupled $\mathcal{N}$-structure $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ in a d-algebra X is a coupled $\mathcal{N}$-subalgebra of X if and only if the nonempty $\mathcal{N}(t, s)$-level set $\mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t, s)\}$ is a subalgebra of X for all $t, s \in [-1, 0]$ with $t + s \geq -1$.*

Proof. Assume that $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ is a coupled $\mathcal{N}$-subalgebra of a d-algebra X. Let $t, s \in [-1, 0]$ with $t + s \geq -1$ and $x, y \in \mathcal{N}\{(f_\mathcal{C}, g_\mathcal{C}); (t, s)\}$. Then $f_\mathcal{C}(x) \leq t$, $f_\mathcal{C}(y) \leq t$, $g_\mathcal{C}(x) \geq s$, and $g_\mathcal{C}(y) \geq s$. If follows from (3.1) that

$$f_\mathcal{C}(x * y) \leq \bigvee \{f_\mathcal{C}(x), f_\mathcal{C}(y)\} \leq t \text{ and } g_\mathcal{C}(x * y) \geq \bigwedge \{g_\mathcal{C}(x), g_\mathcal{C}(y)\} \geq s$$

so that $x * y \in \mathcal{N}\{(f_\mathcal{C}, g_\mathcal{C}); (t, s)\}$. Hence the nonempty $\mathcal{N}(t, s)$-level set $\mathcal{N}\{(f_\mathcal{C}, g_\mathcal{C}); (t, s)\}$ is a subalgebra of X for all $t, s \in [-1, 0]$ with $t + s \geq -1$.

Conversely, suppose that the nonempty $\mathcal{N}(t, s)$-level set $\mathcal{N}\{(f_\mathcal{C}, g_\mathcal{C}); (t, s)\}$ is a subalgebra of a d-algebra X for all $t, s \in [-1, 0]$ with $t + s \geq -1$. Let $x, y \in X$ be such that $\mathcal{C}(x) = (t_x, s_x)$ and $\mathcal{C}(y) = (t_y, s_y)$, that is, $f_\mathcal{C}(x) = t_x$, $g_\mathcal{C}(x) = s_x$, $f_\mathcal{C}(y) = t_y$, and $g_\mathcal{C}(y) = s_y$ with $-1 \leq t_x + s_x$ and $-1 \leq t_y + s_y$. Then $x \in \mathcal{N}\{(f_\mathcal{C}, g_\mathcal{C}); (t_x, s_x)\}$ and $y \in \mathcal{N}\{(f_\mathcal{C}, g_\mathcal{C}); (t_y, s_y)\}$. We may assume that $(t_x, s_x) \ll (t_y, s_y)$ without loss of generality. Then

$$\mathcal{N}\{(f_\mathcal{C}, g_\mathcal{C}); (t_x, s_x)\} \subseteq \mathcal{N}\{(f_\mathcal{C}, g_\mathcal{C}); (t_y, s_y)\},$$

and so $x, y \in \mathcal{N}\{(f_\mathcal{C}, g_\mathcal{C}); (t_y, s_y)\}$. Since $\mathcal{N}\{(f_\mathcal{C}, g_\mathcal{C}); (t_y, s_y)\}$ is a subalgebra of X, it follows that $x * y \in \mathcal{N}\{(f_\mathcal{C}, g_\mathcal{C}); (t_y, s_y)\}$ so that

$$f_\mathcal{C}(x * y) \leq t_y = \bigvee \{f_\mathcal{C}(x), f_\mathcal{C}(y)\} \quad \text{and } g_\mathcal{C}(x * y) \geq s_y = \bigwedge \{g_\mathcal{C}(x), g_\mathcal{C}(y)\}.$$

Therefore $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ in X is a coupled $\mathcal{N}$-subalgebra of X. □

Lemma 3.4. *Every coupled $\mathcal{N}$-subalgebra $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ of a d-algebra X satisfies $f_\mathcal{C}(0) \leq f_\mathcal{C}(x)$ and $g_\mathcal{C}(0) \geq g_\mathcal{C}(x)$ for all $x \in X$.*

Proof. For any $x, y \in X$, we have

$$f_\mathcal{C}(0) = f_\mathcal{C}(x * x) \leq \bigvee \{f_\mathcal{C}(x), f_\mathcal{C}(x)\} = f_\mathcal{C}(x)$$

and

$$g_\mathcal{C}(0) = g_\mathcal{C}(x * x) \geq \bigwedge \{g_\mathcal{C}(x), g_\mathcal{C}(x)\} = g_\mathcal{C}(x).$$

This completes the proof. □

Proposition 3.5. *Let X be an edge d-algebra. If every $\mathcal{N}$-subalgebra $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ of X satisfies the inequalities $f_\mathcal{C}(x * y) \leq f_\mathcal{C}(y)$ and $g_\mathcal{C}(x * y) \geq g_\mathcal{C}(y)$ for any $x, y \in X$, then $f_\mathcal{C}$ and $g_\mathcal{C}$ are constant functions.*

Proof. Let $x \in X$. Using Lemma 2.5 and assumption, we have $f_\mathcal{C}(x) = f_\mathcal{C}(x * 0) \leq f_\mathcal{C}(0)$ and $g_\mathcal{C}(x) = g_\mathcal{C}(x * 0) \geq g_\mathcal{C}(0)$. It follows from Lemma 3.4 that $f_\mathcal{C}(x) = f_\mathcal{C}(0)$ and $g_\mathcal{C}(x) = g_\mathcal{C}(0)$. Hence $f_\mathcal{C}$ and $g_\mathcal{C}$ are constant functions. □

Definition 3.6. A coupled $\mathcal{N}$-structure $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ in a d-algebra X is called a *coupled $\mathcal{N}$-ideal* of X if it satisfies:

(c81) $f_\mathcal{C}(0) \leq f_\mathcal{C}(x)$ and $g_\mathcal{C}(0) \geq g_\mathcal{C}(x)$,
(c82) $f_\mathcal{C}(x) \leq \bigvee \{f_\mathcal{C}(x * y), f_\mathcal{C}(y)\}$ and $g_\mathcal{C}(x) \geq \bigwedge \{g_\mathcal{C}(x * y), g_\mathcal{C}(y)\}$

for all $x, y \in X$.

Example 3.7. (1) Let $X = \{0, a, b, c\}$ be a d-algebra ([2]), which is not a BCK/BCI-algebra, with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	a
c	c	c	c	0

Let $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ be a coupled $\mathcal{N}$-structure in X defined by

$$\mathcal{C} = \big\{\langle 0; -0.8, -0.1\rangle, \langle a; -0.4, -0.3\rangle, \langle b; -0.4, -0.3\rangle, \langle c; -0.3, -0.5\rangle\big\}.$$

Then $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is both a coupled $\mathcal{N}$-subalgebra and a coupled $\mathcal{N}$-ideal of X.

(2) Consider a d-algebra $X = \{0, a, b, c\}$ as in Example 2.3(c). Let $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ be a coupled $\mathcal{N}$-structure in X defined by

$$\mathcal{C} = \big\{\langle 0; -0.6, -0.3\rangle, \langle a; -0.6, -0.3\rangle, \langle b; -0.4, -0.5\rangle, \langle c; -0.6, -0.3\rangle\big\}.$$

Then $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is a coupled $\mathcal{N}$-subalgebra of X, but not a coupled $\mathcal{N}$-ideal of X, since

$$f_{\mathcal{C}}(b) = -0.4 \not\leq -0.6 = \bigvee \{f_{\mathcal{C}}(b * c), f_{\mathcal{C}}(c)\}$$

and/or

$$g_{\mathcal{C}}(b) = -0.5 \not\geq -0.3 = \bigwedge \{g_{\mathcal{C}}(b * c), g_{\mathcal{C}}(c)\}.$$

Proposition 3.8. *Every coupled $\mathcal{N}$-ideal of a d-algebra X satisfies the following assertions*:

(i) $(\forall x, y, z \in X)(x * y \leq z \Rightarrow f_{\mathcal{C}}(x) \leq \bigvee \{f_{\mathcal{C}}(y), f_{\mathcal{C}}(z)\},\ g_{\mathcal{C}}(x) \geq \bigwedge \{g_{\mathcal{C}}(y), g_{\mathcal{C}}(z)\})$.

(ii) $(\forall x, y \in X)(x \leq y \Rightarrow f_{\mathcal{C}}(x) \leq f_{\mathcal{C}}(y), g_{\mathcal{C}}(x) \geq g_{\mathcal{C}}(y))$.

Proof. (i) Let $x, y, z \in X$ be such that $x * y \leq z$. Then $(x * y) * z = 0$, and so

$$\begin{aligned} f_{\mathcal{C}}(x) &\leq \bigvee \{f_{\mathcal{C}}(x * y), f_{\mathcal{C}}(y)\} \\ &\leq \bigvee \Big\{\bigvee \{f_{\mathcal{C}}((x * y) * z), f_{\mathcal{C}}(z)\}, f_{\mathcal{C}}(y)\Big\} \\ &= \bigvee \Big\{\bigvee \{f_{\mathcal{C}}(0), f_{\mathcal{C}}(z)\}, f_{\mathcal{C}}(y)\Big\} \\ &= \bigvee \{f_{\mathcal{C}}(y), f_{\mathcal{C}}(z)\} \end{aligned}$$

and

$$\begin{aligned} g_{\mathcal{C}}(x) &\geq \bigwedge \{g_{\mathcal{C}}(x * y), g_{\mathcal{C}}(y)\} \\ &\geq \bigwedge \Big\{\bigwedge \{g_{\mathcal{C}}((x * y) * z), g_{\mathcal{C}}(z)\}, g_{\mathcal{C}}(y)\Big\} \\ &= \bigwedge \Big\{\bigwedge \{g_{\mathcal{C}}(0), g_{\mathcal{C}}(z)\}, g_{\mathcal{C}}(y)\Big\} \\ &= \bigwedge \{g_{\mathcal{C}}(y), g_{\mathcal{C}}(z)\}. \end{aligned}$$

(ii) Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$, and so

$$\begin{aligned} f_{\mathcal{C}}(x) &\leq \bigvee \{f_{\mathcal{C}}(x * y), f_{\mathcal{C}}(y)\} \\ &= \bigvee \{f_{\mathcal{C}}(0), f_{\mathcal{C}}(y)\} \\ &= f_{\mathcal{C}}(y) \end{aligned}$$

and

$$\begin{aligned} g_{\mathcal{C}}(x) &\geq \bigwedge \{g_{\mathcal{C}}(x * y), g_{\mathcal{C}}(y)\} \\ &= \bigwedge \{g_{\mathcal{C}}(0), g_{\mathcal{C}}(y)\} \\ &= g_{\mathcal{C}}(y). \end{aligned}$$

This completes the proof. □

Theorem 3.9. *For a coupled $\mathcal{N}$-structure $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ in a d-algebra X, the following are equivalent*:

(1) *$\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is a coupled $\mathcal{N}$-ideal of X.*
(2) *The nonempty $\mathcal{N}(t, s)$-level set $\mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t, s)\}$ is an ideal of X for all $t, s \in [-1, 0]$ with $t + s \geq -1$.*

Proof. (1) ⇒ (2). Obviously, $0 \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t, s)\}$. Let $x, y \in X$ be such that $x*y \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t, s)\}$ and $y \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t, s)\}$ for all $t, s \in [-1, 0]$ with $t + s \geq -1$. Then $f_{\mathcal{C}}(x*y) \leq t$, $g_{\mathcal{C}}(x*y) \geq s$, $f_{\mathcal{C}}(y) \leq t$, and $g_{\mathcal{C}}(y) \geq s$. Using (c82), we have $f_{\mathcal{C}}(x) \leq \bigvee \{f_{\mathcal{C}}(x * y), f_{\mathcal{C}}(y)\} \leq t$ and $g_{\mathcal{C}}(x) \geq \bigwedge \{g_{\mathcal{C}}(x * y), g_{\mathcal{C}}(y)\} \geq s$ which imply that $x \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t, s)\}$. Hence the nonempty $\mathcal{N}(t, s)$-level set $\mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t, s)\}$ is an ideal of X for all $t, s \in [-1, 0]$ with $t + s \geq -1$.

(2) ⇒ (1). Since $0 \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t, s)\}$, we have the condition (c81). Let $x, y \in X$ be such that $\mathcal{C}(x * y) = (t_x, s_x)$ and $\mathcal{C}(y) = (t_y, s_y)$, that is,

$$f_{\mathcal{C}}(x * y) = t_x,\ g_{\mathcal{C}}(x * y) = s_x,\ f_{\mathcal{C}}(y) = t_y, \text{ and } g_{\mathcal{C}}(y) = s_y.$$

Then $x*y \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t_x, s_x)\}$ and $y \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t_y, s_y)\}$. We may assume that $(t_x, s_x) \ll (t_y, s_y)$ without loss of generality. Then

$$\mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t_x, s_x)\} \subseteq \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t_y, s_y)\},$$

and so $x * y, y \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t_y, s_y)\}$. Since $\mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t_y, s_y)\}$ is an ideal of X, it follows that $x \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t_y, s_y)\}$ so that

$$f_{\mathcal{C}}(x) \leq t_y = \bigvee \{f_{\mathcal{C}}(x * y), f_{\mathcal{C}}(y)\} \quad \text{and } g_{\mathcal{C}}(x) \geq s_y = \bigwedge \{g_{\mathcal{C}}(x * y), g_{\mathcal{C}}(y)\}.$$

Therefore $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ in X is a coupled $\mathcal{N}$-ideal of X. □

Definition 3.10. A coupled $\mathcal{N}$-structure $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ in a d-algebra X is called a *coupled positive implicative $\mathcal{N}$-ideal* of X if it satisfies (c81) and

$$\text{(c83)} \quad \begin{aligned} & f_{\mathcal{C}}(x * z) \leq \bigvee \{f_{\mathcal{C}}((x * y) * z), f_{\mathcal{C}}(y * z)\} \text{ and} \\ & g_{\mathcal{C}}(x * z) \geq \bigwedge \{g_{\mathcal{C}}((x * y) * z), g_{\mathcal{C}}(y * z))\} \end{aligned}$$

for all $x, y \in X$.

Example 3.11. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a d-algebra ([1]), which is not a BCK/BCI-algebra, with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	2
2	2	2	0	0	2	1
3	3	3	3	0	0	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Let $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ be a coupled $\mathcal{N}$-structure in X defined by

$$\mathcal{C} = \big\{\langle 0; -0.8, -0.2\rangle, \langle 1; -0.6, -0.3\rangle, \langle 2; -0.6, -0.3\rangle,\\ \langle 3; -0.6, -0.3\rangle, \langle 4; -0.6, -0.3\rangle, \langle 5; -0.3, -0.5\rangle\big\}.$$

It is easy to check that $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is both a coupled $\mathcal{N}$-ideal of X and a coupled positive implicative $\mathcal{N}$-ideal of X.

Let $\mathcal{D} = (f_{\mathcal{D}}, g_{\mathcal{D}})$ be a coupled $\mathcal{N}$-structure in X defined by

$$\mathcal{D} = \big\{\langle 0; -0.7, -0.2\rangle, \langle 1; -0.5, -0.3\rangle, \langle 2; -0.5, -0.3\rangle,\\ \langle 3; -0.5, -0.3\rangle, \langle 4; -0.5, -0.3\rangle, \langle 5; -0.5, -0.3\rangle\big\}.$$

It is easy to show that $\mathcal{D} = (f_{\mathcal{D}}, g_{\mathcal{D}})$ is both a coupled $\mathcal{N}$-subalgebra and a coupled $\mathcal{N}$-ideal of X, but not a coupled positive implicative $\mathcal{N}$-ideal of X, since

$$f_{\mathcal{D}}(2 * 4) = f_{\mathcal{D}}(2) = -0.5 \not\leq -0.7 = \bigvee \{f_{\mathcal{D}}((2 * 3) * 4), f_{\mathcal{D}}(3 * 4)\}$$

and/or

$$g_{\mathcal{D}}(2 * 4) = g_{\mathcal{D}}(2) = -0.3 \not\geq -0.2 = \bigvee \{g_{\mathcal{D}}((2 * 3) * 4), g_{\mathcal{D}}(3 * 4)\}.$$

Example 3.12. For a coupled $\mathcal{N}$-structure $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ in a d-algebra X, the following are equivalent:

(1) $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is a coupled positive implicative $\mathcal{N}$-ideal of X.
(2) The nonempty $\mathcal{N}(t, s)$-level set $\mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t, s)\}$ is a positive implicative ideal of X for all $t, s \in [-1, 0]$ with $t + s \geq -1$.

Proof. Assume that $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is a coupled positive implicative $\mathcal{N}$-ideal of X. Let $t, s \in [-1, 0]$ be such that $t + s \geq -1$. Obviously, $0 \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t, s)\}$. Let $x, y, z \in X$ be such that $(x * y) * z, y * z \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t, s))\}$. Then $f_{\mathcal{C}}((x * y) * z) \leq t, f_{\mathcal{C}}(y * z) \leq t$ and $g_{\mathcal{C}}((x * y) * z) \geq s, g_{\mathcal{C}}(y * z) \geq s$. It follows from (c83) that $f_{\mathcal{C}}(x * z) \leq \bigvee\{f_{\mathcal{C}}((x * y) * z), f_{\mathcal{C}}(y * z)\} \leq t$ and $g_{\mathcal{C}}(x * z) \geq \bigwedge\{g_{\mathcal{C}}((x * y) * z), g_{\mathcal{C}}(y * z)\} \geq s$, which imply that $x * z \in \mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t, s)\}$. Hence the nonempty $\mathcal{N}(t, s)$-level set of $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is a positive implicative ideal of X for all $t, s \in [-1, 0]$ with $t + s \geq -1$.

Conversely, suppose that the nonempty $\mathcal{N}(t,s)$-level set of $\mathcal{C}=(f_\mathcal{C},g_\mathcal{C})$ is a positive implicative ideal of X for all $t,s\in[-1,0]$ with $t+s\geq -1$. Since $0\in\mathcal{N}\{(f_\mathcal{C},g_\mathcal{C});(t,s)\}$, the condition (c81) is valid. Assume that there exist $a,b,c\in X$ such that $f_\mathcal{C}(a*c)>\bigvee\{f_\mathcal{C}((a*b)*c),f_\mathcal{C}(b*c)\}$ or $g_\mathcal{C}(a*c)<\bigwedge\{g_\mathcal{C}((a*b)*c),g_\mathcal{C}(b*c)\}$. For the case $f_\mathcal{C}(a*c)>\bigvee\{f_\mathcal{C}((a*b)*c),f_\mathcal{C}(b*c)\}$ and $g_\mathcal{C}(a*c)\geq\bigwedge\{g_\mathcal{C}((a*b)*c),g_\mathcal{C}(b*c)\}$, there exist $s_0,t_0\in[-1,0)$ such that $f_\mathcal{C}(a*c)>t_0>\bigvee\{f_\mathcal{C}((a*b)*c),f_\mathcal{C}(b*c)\}$ and $s_0=\bigwedge\{g_\mathcal{C}((a*b)*c),g_\mathcal{C}(b*c)\}$. It follows that $(a*b)*c,b*c\in\mathcal{N}\{(f_\mathcal{C},g_\mathcal{C});(t_0,s_0)\}$, but $a*c\notin\mathcal{N}\{(f_\mathcal{C},g_\mathcal{C});(t_0,s_0)\}$. This is impossible. For the case $f_\mathcal{C}(a*c)\leq\bigvee\{f_\mathcal{C}((a*b)*c),f_\mathcal{C}(b*c)\}$ and $g_\mathcal{C}(a*c)<\bigwedge\{g_\mathcal{C}((a*b)*c),g_\mathcal{C}(b*c)\}$, there exist $s_0,t_0\in[-1,0)$ such that $t_0=\bigvee\{f_\mathcal{C}((a*b)*c),f_\mathcal{C}(b*c)\}$ and $g_\mathcal{C}(a*c)<s_0<\bigwedge\{g_\mathcal{C}((a*b)*c),g_\mathcal{C}(b*c)\}$. Then $(a*b)*c,b*c\in\mathcal{N}\{(f_\mathcal{C},g_\mathcal{C});(t_0,s_0)\}$, but $a*c\notin\mathcal{N}\{(f_\mathcal{C},g_\mathcal{C});(t_0,s_0)\}$. This is a contradiction. If $f_\mathcal{C}(a*c)>\bigvee\{f_\mathcal{C}((a*b)*c),f_\mathcal{C}(b*c)\}$ and $g_\mathcal{C}(a*c)<\bigwedge\{g_\mathcal{C}((a*b)*c),g_\mathcal{C}(b*c)\}$, then $(a*b)*c,b*c\in\mathcal{N}\{(f_\mathcal{C},g_\mathcal{C});(t_0,s_0)\}$, but $a*c\notin\mathcal{N}\{(f_\mathcal{C},g_\mathcal{C});(t_0,s_0)\}$, where $t_0:=\frac{1}{2}(f_\mathcal{C}(a*c)+\bigvee\{f_\mathcal{C}((a*b)*c),f_\mathcal{C}(b*c)\})$ and $s_0:=\frac{1}{2}(g_\mathcal{C}(a*c)+\bigwedge\{g_\mathcal{C}((a*b)*c),g_\mathcal{C}(b*c)\})$. This is a contradiction. Therefore $\mathcal{C}=(f_\mathcal{C},g_\mathcal{C})$ is a coupled positive implicative $\mathcal{N}$-ideal of X. □

Proposition 3.13. *For any d^*-algebra X, every coupled $\mathcal{N}$-ideal is a coupled $\mathcal{N}$-subalgebra of X.*

Proof. Let a coupled $\mathcal{N}$-structure $\mathcal{C}=(f_\mathcal{C},g_\mathcal{C})$ be a coupled $\mathcal{N}$-ideal of a d^*-algebra X and let $x,y\in X$. Then

$$f_\mathcal{C}(x*y)\leq\bigvee\{f_\mathcal{C}((x*y)*x),f_\mathcal{C}(x)\}=\bigvee\{f_\mathcal{C}(0),f_\mathcal{C}(x)\}\leq\bigvee\{f_\mathcal{C}(x),f_\mathcal{C}(y)\}$$

and

$$g_\mathcal{C}(x*y)\geq\bigwedge\{g_\mathcal{C}((x*y)*x),g_\mathcal{C}(x)\}=\bigwedge\{g_\mathcal{C}(0),g_\mathcal{C}(x)\}\geq\bigwedge\{g_\mathcal{C}(x),g_\mathcal{C}(y)\}.$$

Hence $\mathcal{C}=(f_\mathcal{C},g_\mathcal{C})$ is a coupled $\mathcal{N}$-subalgebra of X. □

The converse of Theorem 3.13 may not be true in general as seen in the following example.

Example 3.14. Let $X=\{0,a,b,c\}$ be a d^*-algebra ([1]), which is not a BCK/BCI-algebra, with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
c	c	c	a	0

Let $\mathcal{C}=(f_\mathcal{C},g_\mathcal{C})$ be a coupled $\mathcal{N}$-structure in X defined by

$$\mathcal{C}=\{\langle 0;-0.7,-0.2\rangle,\langle a;-0.5,-0.4\rangle,\langle b;-0.5,-0.4\rangle,\langle c;-0.2,-0.6\rangle\}.$$

It is easily verified that $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ is a a coupled $\mathcal{N}$-subalgebra, but not a coupled $\mathcal{N}$-ideal of X, since

$$f_\mathcal{C}(c) = -0.2 \not\leq -0.5 = \bigvee \{f_\mathcal{C}(c * b), f_\mathcal{C}(b)\}$$

and/or

$$g_\mathcal{C}(c) = -0.6 \not\geq -0.4 = \bigvee \{g_\mathcal{C}(c * b), g_\mathcal{C}(b)\}.$$

Proposition 3.15. *Every coupled positive implicative $\mathcal{N}$-ideal $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ of an edge d-algebra X is a coupled $\mathcal{N}$-ideal of X.*

Proof. Put $z := 0$ in (c83). □

Proposition 3.16. *Let $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ be a coupled positive implicative $\mathcal{N}$-ideal of an edge d-algebra X.*

(i) *If $x \leq y$ for any $x, y \in X$, then $f_\mathcal{C}(x) \leq f_\mathcal{C}(y)$, $g_\mathcal{C}(x) \geq g_\mathcal{C}(y)$.*
(ii) *If $f_\mathcal{C}(x * y) = f_\mathcal{C}(0)$ for any $x, y \in X$, then $f_\mathcal{C}(x) \leq f_\mathcal{C}(y)$.*
(iii) *If $g_\mathcal{C}(x * y) = g_\mathcal{C}(0)$ for any $x, y \in X$, then $g_\mathcal{C}(x) \geq g_\mathcal{C}(y)$.*

Proof. (i) Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$. Hence

$$\begin{aligned} f_\mathcal{C}(x) = f_\mathcal{C}(x * 0) &\leq \bigvee\{f_\mathcal{C}((x * y) * 0), f_\mathcal{C}(y * 0)\} \\ &= \bigvee\{f_\mathcal{C}(x * y), f_\mathcal{C}(y)\} \\ &= \bigvee\{f_\mathcal{C}(0), f_\mathcal{C}(y)\} = f_\mathcal{C}(y) \end{aligned}$$

and

$$\begin{aligned} g_\mathcal{C}(x) = g_\mathcal{C}(x * 0) &\geq \bigwedge\{g_\mathcal{C}((x * y) * 0), g_\mathcal{C}(y * 0)\} \\ &= \bigwedge\{g_\mathcal{C}(x * y), g_\mathcal{C}(y)\} \\ &= \bigwedge\{g_\mathcal{C}(0), g_\mathcal{C}(y)\} = g_\mathcal{C}(y). \end{aligned}$$

(ii) For any $x, y \in X$, we have

$$\begin{aligned} f_\mathcal{C}(x) = f_\mathcal{C}(x * 0) &\leq \bigvee\{f_\mathcal{C}((x * y) * 0), f_\mathcal{C}(y * 0)\} \\ &= \bigvee\{f_\mathcal{C}(x * y), f_\mathcal{C}(y)\} \\ &= \bigvee\{f_\mathcal{C}(0), f_\mathcal{C}(y)\} \\ &= f_\mathcal{C}(y). \end{aligned}$$

(iii) For any $x, y \in X$, we have

$$\begin{aligned} g_\mathcal{C}(x) = g_\mathcal{C}(x * 0) &\geq \bigwedge\{g_\mathcal{C}((x * y) * 0), g_\mathcal{C}(y * 0)\} \\ &= \bigwedge\{g_\mathcal{C}(x * y), g_\mathcal{C}(y)\} \\ &= \bigwedge\{g_\mathcal{C}(0), g_\mathcal{C}(y)\} \\ &= g_\mathcal{C}(y). \end{aligned}$$

□

Proposition 3.17. *Let $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ be a coupled positive implicative $\mathcal{N}$-ideal of an edge d^*-algebra X. Then the following hold*:

(i) $(\forall x, y \in X)(f_\mathcal{C}(x * y) \leq f_\mathcal{C}(x), g_\mathcal{C}(x * y) \geq g_\mathcal{C}(x))$.

(ii) $(\forall x, y \in X)(f_\mathcal{C}(x * y) \leq \bigvee\{f_\mathcal{C}(x), f_\mathcal{C}(y)\}, g_\mathcal{C}(x * y) \geq \bigwedge\{g_\mathcal{C}(x), g_\mathcal{C}(y)\})$.

(iii) $(\forall x, y, z \in X)(f_\mathcal{C}(x * (y * z)) \leq \bigvee\{f_\mathcal{C}(x), f_\mathcal{C}(y), f_\mathcal{C}(z)\}, g_\mathcal{C}(x * (y * z)) \geq \bigwedge\{g_\mathcal{C}(x), g_\mathcal{C}(y), g_\mathcal{C}(z)\})$.

Proof. (i) Since X is a d^*-algebra, we have $(x*y)*x = 0$ for any $x, y \in X$. Hence $x * y \leq x$ for any $x, y \in X$. Using Proposition 3.8(ii), we have $f_\mathcal{C}(x * y) \leq f_\mathcal{C}(x)$ and $g_\mathcal{C}(x * y) \geq g_\mathcal{C}(x)$ for any $x, y \in X$.

(ii) It is easily verified from Theorem 3.13 and Proposition 3.15.

(iii) For any $x, y, z \in X$, we have

$$\begin{aligned} f_\mathcal{C}(x * (y * z)) &\leq \bigvee\{f_\mathcal{C}(x), f_\mathcal{C}(y * z)\} \\ &\leq \bigvee\{f_\mathcal{C}(x), f_\mathcal{C}(y), f_\mathcal{C}(z)\} \end{aligned}$$

and

$$\begin{aligned} g_\mathcal{C}(x * (y * z)) &\geq \bigwedge\{g_\mathcal{C}(x), g_\mathcal{C}(y * z)\} \\ &\geq \bigwedge\{g_\mathcal{C}(x), g_\mathcal{C}(y), g_\mathcal{C}(z)\}. \end{aligned}$$ □

For any element a of a d-algebra X, let

$$X_a := \{x \in X \mid f_\mathcal{C}(x) \leq f_\mathcal{C}(a),\ g_\mathcal{C}(x) \geq g_\mathcal{C}(a)\}.$$

Obviously, X_a is a non-empty subset of X.

Proposition 3.18. *Let a be any element of a d-algebra X. If $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ is a coupled* (*positive implicative*) *$\mathcal{N}$-ideal of X, then the set X_a is a* (*positive implicative*) *ideal of X.*

Proof. Obviously, $0 \in X_a$. Let $x, y \in X$ be such that $x * y \in X_a$ and $y \in X_a$. Then $f_\mathcal{C}(x * y) \leq f_\mathcal{C}(a), g_\mathcal{C}(x * y) \geq g_\mathcal{C}(a), f_\mathcal{C}(y) \leq f_\mathcal{C}(a)$ and $g_\mathcal{C}(y) \geq g_\mathcal{C}(a)$. It follows from (c82) that $f_\mathcal{C}(x) \leq \bigvee\{f_\mathcal{C}(x * y), f_\mathcal{C}(y)\} \leq f_\mathcal{C}(a)$ and $g_\mathcal{C}(x) \geq \bigwedge\{g_\mathcal{C}(x * y), g_\mathcal{C}(y)\} \geq g_\mathcal{C}(a)$ so that $x \in X_a$. Therefore X_a is an ideal of X.

Let $x, y, z \in X$ be such that $(x * y) * z \in X_a$ and $y * z \in X_a$. Then $f_\mathcal{C}((x * y) * z) \leq f_\mathcal{C}(a), g_\mathcal{C}((x * y) * z) \geq g_\mathcal{C}(a), f_\mathcal{C}(y * z) \leq f_\mathcal{C}(a)$ and $g_\mathcal{C}(y * z) \geq g_\mathcal{C}(a)$. It follows from (c83) that $f_\mathcal{C}(x*z) \leq \bigvee\{f_\mathcal{C}((x*y)*z), f_\mathcal{C}(y*z)\} \leq f_\mathcal{C}(a)$ and $g_\mathcal{C}(x*z) \geq \bigwedge\{g_\mathcal{C}((x*y)*z), g_\mathcal{C}(y*z)\} \geq g_\mathcal{C}(a)$ so that $x*z \in X_a$. Therefore X_a is a positive implicative ideal of X. □

Proposition 3.19. *Let a be any element of a d-algebra X and let $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ be a coupled $\mathcal{N}$-structure in X. Then*

(i) *if X_a is an ideal of X, then $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ satisfies the following assertion*:

$$(3.2) \quad (\forall x, y, z \in X) \left(\begin{array}{l} f_\mathcal{C}(x) \geq \bigvee\{f_\mathcal{C}(y * z), f_\mathcal{C}(z)\} \ \Rightarrow\ f_\mathcal{C}(x) \geq f_\mathcal{C}(y) \\ g_\mathcal{C}(x) \leq \bigwedge\{g_\mathcal{C}(y * z), g_\mathcal{C}(z)\} \ \Rightarrow\ g_\mathcal{C}(x) \leq g_\mathcal{C}(y) \end{array} \right).$$

(ii) *if* $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ *satisfies* (3.2) *and*

$$(\forall x \in X)\,(f_{\mathcal{C}}(0) \leq f_{\mathcal{C}}(x),\ g_{\mathcal{C}}(0) \geq g_{\mathcal{C}}(x)), \tag{3.3}$$

then X_a *is an ideal of* X.

Proof. (i) Assume that X_a is an ideal of X for all $a \in X$. Let $x, y, z \in X$ be such that $f_{\mathcal{C}}(x) \geq \bigvee \{f_{\mathcal{C}}(y * z), f_{\mathcal{C}}(z)\}$ and $g_{\mathcal{C}}(x) \leq \bigwedge \{g_{\mathcal{C}}(y * z), g_{\mathcal{C}}(z)\}$. Then $y * z \in X_x$ and $z \in X_x$. Since X_x is an ideal of X, it follows that $y \in X_x$ so that $f_{\mathcal{C}}(y) \leq f_{\mathcal{C}}(x)$ and $g_{\mathcal{C}}(y) \geq g_{\mathcal{C}}(x)$.

(ii) Suppose that $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ satisfies two conditions (3.2) and (3.3). Let $x, y \in X$ be such that $x*y \in X_a$ and $y \in X_a$. Then $f_{\mathcal{C}}(x*y) \leq f_{\mathcal{C}}(a)$, $g_{\mathcal{C}}(x*y) \geq g_{\mathcal{C}}(a)$, $f_{\mathcal{C}}(y) \leq f_{\mathcal{C}}(a)$ and $g_{\mathcal{C}}(y) \geq g_{\mathcal{C}}(a)$. Hence $f_{\mathcal{C}}(a) \geq \bigvee \{f_{\mathcal{C}}(x * y), f_{\mathcal{C}}(y)\}$ and $g_{\mathcal{C}}(a) \leq \bigwedge \{g_{\mathcal{C}}(x * y), g_{\mathcal{C}}(y)\}$, which imply from (3.2) that $f_{\mathcal{C}}(a) \geq f_{\mathcal{C}}(x)$ and $g_{\mathcal{C}}(a) \leq g_{\mathcal{C}}(x)$. Thus $x \in X_a$. Obviously, $0 \in X_a$. Therefore X_a is an ideal of X. □

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Sun Shin Ahn
Department of Mathematics Education
Dongguk University
Seoul 100-715, Korea
E-mail address: sunshine@dongguk.edu

Jung Mi Ko
Department of Mathematics
Gangneung-Wonju National University
Gangneung 210-702, Korea
E-mail address: jmko@gwnu.ac.kr