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ON PERMUTING n-DERIVATIONS IN NEAR-RINGS

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Abstract. In this paper, we introduce the notion of permuting n-derivations in near-ring N and investigate commutativity of addition and multiplication of N. Further, under certain constrants on a $n!$-torsion free prime near-ring N, it is shown that a permuting n-additive mapping D on N is zero if the trace d of D is zero. Finally, some more related results are also obtained.

1. Introduction

Throughout this paper N will denote a zero-symmetric left near ring. A near ring N is called zero symmetric if $0x = 0$ for all $x \in N$ (recall that in a left near ring $x0 = 0$ for all $x \in N$). N is called prime if $xNy = \{0\}$ implies $x = 0$ or $y = 0$. It is called semi prime if $xNx = \{0\}$ implies $x = 0$. Near-ring N is called n-torsion free if $nx = 0$ implies $x = 0$. The symbol Z will represent the multiplicative center of N, that is, $Z = \{x \in N \mid xy = yx \text{ for all } y \in N\}$. As usual, for $x, y \in N, [x, y]$ will denote the commutator $xy - yx$, while (x, y) will indicate the additive group commutator $x + y - x - y$. The symbol C will represent the set of all additive commutators of near ring N. For terminologies concerning near-rings we refer to G. Pilz [10].

An additive map $f : N \longrightarrow N$ is called a derivation if $f(xy) = f(x)y + xf(y)$ holds for all $x, y \in N$. The concepts of symmetric bi-derivation, permuting tri-derivation and permuting n-derivation have already been introduced in rings by G. Maksa, M. A. Öztürk and K. H. Park in [4, 5, 6], and [8], respectively. These concepts of symmetric bi-derivations and permuting tri-derivations have been studied in near-rings by M. A. Öztürk and K. H. Park in [7] and [9], respectively. In the present paper, motivated by these concepts, we define permuting n-derivations in near-rings and study some properties involved there. Some relations between permuting n-derivations and C, the set of all additive commutators in near-ring N have also been studied.

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A map $D : \underbrace{N \times N \times \cdots \times N}_{n\text{-times}} \longrightarrow N$ is said to be permuting if the equation $D(x_1, x_2, \ldots, x_n) = D(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$ holds for all $x_1, x_2, \ldots, x_n \in N$ and for every permutation $\pi \in S_n$, where S_n is the permutation group on $\{1, 2, \ldots, n\}$. A map $d : N \to N$ defined by $d(x) = D(x, x, \ldots, x)$ for all $x \in N$ where $D : \underbrace{N \times N \times \cdots \times N}_{n\text{-times}} \to N$ is a permuting map, is called the trace of D. A permuting n-additive (i.e., additive in each argument) mapping $D : \underbrace{N \times N \times \cdots \times N}_{n\text{-times}} \longrightarrow N$ is called a permuting n-derivation if $D(x_1 x_1^{'}, x_2, \ldots, x_n) = D(x_1, x_2, \ldots, x_n)x_1^{'} + x_1 D(x_1^{'}, x_2, \ldots, x_n)$ holds for all $x_1, x_1^{'}, \ldots, x_n \in N$. Of course, a permuting 1-derivation is a derivation and permuting 2-derivation is a symmetric bi-derivation. For an example of permuting n-derivation let $n \geq 1$ be a fixed positive integer, N a commutative near-ring. Then $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b, 0 \in N \right\}$ is a non-commutative near-ring with regard to matrix addition and matrix multiplication. Define $D : \underbrace{R \times R \times \ldots \times R}_{n\text{-times}} \longrightarrow R$ such that

$$D\left(\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} a_n & b_n \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & a_1 a_2 \cdots a_n \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that D is a permuting n-derivation of R.

Now let D be a permuting n-derivation of a near-ring N. Then it can be easily seen that $D(0, x_2, \ldots, x_n) = D(0 + 0, x_2, \ldots, x_n) = D(0, x_2, \ldots, x_n) + D(0, x_2, \ldots, x_n)$. Therefore $D(0, x_2, \ldots, x_n) = 0$ for all $x_2, \ldots, x_n \in N$. We also observe that $D(-x_1, x_2, \ldots, x_n) = -D(x_1, x_2, \ldots, x_n)$ for all $x_i \in N; i = 1, 2, \ldots, n$.

There has been a great deal of work concerning derivations, biderivations and triderivations in near-rings (see [1, 2, 3, 4, 9] where further references can be found). In this paper we study the commutativity of addition and multiplication of near-rings. Many well known results for derivations, bi-derivations and tri-derivations in near-rings have been generalized for permuting n-derivation. In fact, our results generalize and complement several well known theorems for near-rings.

2. Preliminary results

We begin with the following lemmas which are essential for developing the proofs of our main results . Proofs of Lemmas 2.1 and 2.2 can be seen in [2, Lemma 3] and [3, Lemma 1.2], respectively.

Lemma 2.1. *Let N be a prime near-ring.*

(i) *If $z \in Z \setminus \{0\}$, then z is not a zero divisor.*

(ii) *If $Z \setminus \{0\}$ contains an element z for which $z + z \in Z$, then $(N, +)$ is abelian.*

Lemma 2.2. *Let N be a prime near-ring. If $z \in Z \setminus \{0\}$ and x is an element of N such that $xz \in Z$ or $zx \in Z$, then $x \in Z$.*

Lemma 2.3. *Let N be a near-ring. Then D is a permuting n-derivation of N if and only if $D(x_1x_1^{'}, x_2, \ldots, x_n) = x_1D(x_1^{'}, x_2, \ldots, x_n) + D(x_1, x_2, \ldots, x_n)x_1^{'}$ for all $x_1, x_1{'}, x_2, \ldots, x_n \in N$.*

Proof. We have

$$\begin{aligned} & D(x_1(x_1^{'} + x_1^{'}), x_2, \ldots, x_n) \\ &= D(x_1, x_2, \ldots, x_n)(x_1^{'} + x_1^{'}) + x_1D(x_1^{'} + x_1^{'}, x_2, \ldots, x_n) \\ &= D(x_1, x_2, \ldots, x_n)x_1^{'} + D(x_1, x_2, \ldots, x_n)x_1^{'} \\ &\quad + x_1D(x_1^{'}, x_2, \ldots, x_n) + x_1D(x_1^{'}, x_2, \ldots, x_n) \end{aligned}$$

and

$$\begin{aligned} & D(x_1x_1^{'} + x_1x_1^{'}, x_2, \ldots, x_n) \\ &= D(x_1x_1^{'}, x_2, \ldots, x_n) + D(x_1x_1^{'}, x_2, \ldots, x_n) \\ &= D(x_1, x_2, \ldots, x_n)x_1^{'} + x_1D(x_1^{'}, x_2, \ldots, x_n) \\ &\quad + D(x_1, x_2, \ldots, x_n)x_1^{'} + x_1D(x_1^{'}, x_2, \ldots, x_n). \end{aligned}$$

Combining above two equalities we obtain that

$$\begin{aligned} & D(x_1, x_2, \ldots, x_n)x_1^{'} + x_1D(x_1^{'}, x_2, \ldots, x_n) \\ &= x_1D(x_1^{'}, x_2, \ldots, x_n) + D(x_1, x_2, \ldots, x_n)x_1^{'}. \end{aligned}$$

Therefore, $D(x_1x_1^{'}, x_2, \ldots, x_n) = x_1D(x_1^{'}, x_2, \ldots, x_n) + D(x_1, x_2, \ldots, x_n)x_1^{'}$.

Converse can be proved in a similar way. □

In a left near-ring N, right distributive law does not hold in general, however, we can prove the following partial distributive properties in N.

Lemma 2.4. *Let N be a near-ring. Let D be a permuting n-derivation of N and d be the trace of D. Then for every $x_1, x_1^{'}, \ldots, x_n, y \in N$,*

(i) $\{D(x_1, x_2, \ldots, x_n)x_1^{'} + x_1D(x_1^{'}, x_2, \ldots, x_n)\}y$
$= D(x_1, x_2 \ldots, x_n)x_1^{'}y + x_1D(x_1^{'}, x_2, \ldots, x_n)y,$

(ii) $\{x_1D(x_1^{'}, x_2, \ldots, x_n) + D(x_1, x_2, \ldots, x_n)x_1{'}\}y$
$= x_1D(x_1^{'}, x_2, \ldots, x_n)y + D(x_1, x_2, \ldots, x_n)x_1{'}y,$

(iii) $\{d(x)x_1 + xD(x, x, \ldots, x, x_1)\}y = d(x)x_1y + xD(x, x, \ldots, x, x_1)y,$

(iv) $\{xD(x, x, \ldots, x, x_1) + d(x)x_1\}y = xD(x, x, \ldots, x, x_1)y + d(x)x_1y.$

Proof. (i) For all $x_1, x_1{'}, x_1{''}, x_2, \ldots, x_n \in N$

$$\begin{aligned} & D((x_1x_1^{'})x_1^{''}, x_2, \ldots, x_n) \\ &= D(x_1x_1^{'}, x_2, \ldots, x_n)x_1^{''} + (x_1x_1^{'})D(x_1^{''}, x_2, \ldots, x_n) \\ &= \{D(x_1, x_2, \ldots, x_n)x_1^{'} + x_1D(x_1^{'}, x_2, \ldots, x_n)\}x_1^{''} + (x_1x_1^{'})D(x_1^{''}, x_2, \ldots, x_n). \end{aligned}$$

Also

$$
\begin{aligned}
& D(x_1(x_1^{'}x_1''), x_2, \dots, x_n) \\
&= D(x_1, x_2, \dots, x_n)x_1^{'}x_1'' + x_1 D(x_1^{'}x_1^{''}, x_2, \dots, x_n) \\
&= D(x_1, x_2, \dots, x_n)x_1^{'}x_1'' + x_1\{D(x_1^{'}, x_2, \dots, x_n)x_1^{''} + x_1^{'}D(x_1^{''}, x_2, \dots, x_n)\} \\
&= D(x_1, x_2, \dots, x_n)x_1^{'}x_1'' + x_1 D(x_1^{'}, x_2, \dots, x_n)x_1^{''} + x_1x_1^{'}D(x_1^{''}, x_2, \dots, x_n).
\end{aligned}
$$

Combining the above two relations , we get

$$
\begin{aligned}
&\{D(x_1, x_2, \dots, x_n)x_1^{'} + x_1 D(x_1^{'}, x_2, \dots, x_n)\}x_1^{''} \\
&= D(x_1, x_2, \dots, x_n)x_1^{'}x_1^{''} + x_1 D(x_1^{'}, x_2, \dots, x_n)x_1^{''}.
\end{aligned}
$$

Putting y in the place of $x_1^{''}$, we find that

$$
\begin{aligned}
&\{D(x_1, x_2, \dots, x_n)x_1^{'} + x_1 D(x_1^{'}, x_2, \dots, x_n)\}y \\
&= D(x_1, x_2, \dots, x_n)x_1^{'}y + x_1 D(x_1^{'}, x_2, \dots, x_n)y.
\end{aligned}
$$

(ii) It can be proved, in a similar, way as above, with the help of Lemma 2.3.

(iii) In the proof (i) above putting $x_1 = x_2 = x_3 = \cdots = x_n = x$, we get

$$\{d(x)x_1^{'} + xD(x_1^{'}, x, \dots, x)\}y = d(x)x_1^{'}y + xD(x_1^{'}, x \dots, x)y.$$

In particular for $x_1^{'} = x_1$ we get

$$\{d(x)x_1 + xD(x, x, \dots, x_1)\}y = d(x)x_1y + xD(x, x, \dots, x_1)y.$$

(iv) It can be proved in a similar way as above. □

Lemma 2.5. *Let N be prime near-ring and D be a non zero permuting n-derivation of N,*

(i) *If $D(N, N, \dots, N)x = \{0\}$ where $x \in N$, then $x = 0$,*
(ii) *If $xD(N, N, \dots, N) = \{0\}$ where $x \in N$, then $x = 0$.*

Proof. (i) Given that $D(x_1x_1^{'}, x_2, \dots, x_n)x = 0$ for all $x_1, x_1^{'}, \dots, x_n \in N$. This yields that $\{D(x_1, x_2, \dots, x_n)x_1^{'} + x_1D(x_1^{'}, x_2, \dots, x_n)\}x = 0$. By hypothesis and Lemma 2.4(i) we have $D(x_1, x_2, \dots, x_n)Nx = \{0\}$. But since N is a prime near ring and $D \neq 0$, we have $x = 0$.

(ii) It can be proved in a similar way. □

Lemma 2.6. *Let D be a nonzero permuting n-derivation of a prime near ring N. Then $D(C, C, \dots, C) \neq \{0\}$ where $C \neq \{0\}$.*

Proof. If possible assume $D(C, C, \dots, C) = \{0\}$, then $D(c_1, c_2, \dots, c_n) = 0$ for all $c_1, c_2, \dots, c_n \in C$. For all $r_1 \in N$ and $c_1 \in C$ we get $r_1c_1 \in C$. Also $D(r_1c_1, c_2, \dots, c_n) = 0$ implies $r_1D(c_1, c_2, \dots, c_n) + D(r_1, c_2, \dots, c_n)c_1 = 0$. Thus we get

$$D(r_1, c_2, \dots, c_n)c_1 = 0. \tag{2.1}$$

Replacing c_1 by xc_1 in equation (2.1) where $x \in N$ we find that

$$D(r_1, c_2, \ldots, c_n)Nc_1 = \{0\}.$$

Primeness of N yields,

$$D(r_1, c_2, \ldots, c_n) = 0. \tag{2.2}$$

Now putting $r_2c_2 \in C$ in place of c_2 where $r_2 \in N$ in the equation (2.2) and proceeding as above we have $D(r_1, r_2, c_3, \ldots, c_n) = 0$. Proceeding inductively we conclude that $D(r_1, r_2, \ldots, r_n) = 0$ for all $r_1, r_2, \ldots, r_n \in N$ leading to a contradiction. □

Lemma 2.7. *Let N be a $m!$-torsion free near-ring, where $(N, +)$ is an abelian group. Suppose $y_1, y_2, \ldots, y_m \in N$ satisfy $\alpha y_1 + \alpha^2 y_2 + \cdots + \alpha^m y_m = 0$ for $\alpha = 1, 2, \ldots, m$. Then $y_i = 0$ for all i.*

Proof. Let $A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^m \\ \vdots & \vdots & \vdots & \vdots \\ m & m^2 & \cdots & m^m \end{pmatrix}$ be any $m \times m$ matrix. Then by our assumption $A \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Now pre multiplying by Adj A yields Det$A \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$ $= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Since Det$A$, as a Vondermonde determinant, is equal to a product of positive integers, each of which is less than or equal to m and as N is a $m!$-torsion free near-ring, it follows immediately that $y_i = 0$ for all i. □

3. Main results

Recently M. A. Öztürk and Y. B. Jun [7, Lemma 3.1] proved that in a 2-torsion free near-ring which admits a symmetric bi-additive mapping D if the trace d of D is zero, then $D = 0$. Further, this result was generalized by K. H. Park and Y. S. Jung [9, Lemma 2.2] for permuting tri-additive mapping in 3!-torsion free near-ring in the year 2010. We have extended this result, as below, for permuting n-additive mapping in a $n!$-torsion free prime near-ring under some constraints.

Theorem 3.1. *Let N be $n!$-torsion free prime near-ring and D be a permuting n-additive mapping of N such that $D(N, N, \ldots, N) \subseteq Z$. If $d(x) = 0$ for all $x \in N$, then $D = 0$.*

Proof. If $D = 0$, then we have nothing to do, if not then D is a non zero permuting n-additive mapping of prime near-ring N such that $D(N, N, \ldots, N) \subseteq Z$. Hence there exist $x_1, x_2, \ldots, x_n \in N$, all nonzero such that $D(x_1, x_2, \ldots, x_n) \neq 0$ and $D(x_1, x_2, \ldots, x_n) \in Z$. Since $D(x_1 + x_1, x_2, \ldots, x_n) = D(x_1, x_2, \ldots, x_n) + D(x_1, x_2, \ldots, x_n) \in Z$, by Lemma 2.1(ii), $(N, +)$ is an abelian group. Hence the trace $d(x) = D(x, x, \ldots, x)$ of permuting n-additive mapping D can be expressed as;

$$d(x+y) = d(x) + d(y) + \sum_{k=1}^{n-1} \binom{n}{k} h_k(x,y), \tag{3.1}$$

where $x, y \in N$ and $h_k(x,y) = D(\underbrace{x, x, \ldots, x}_{(n-k)\text{-times}}, \underbrace{y, y, \ldots, y}_{k\text{-times}})$. In particular by our hypothesis $d(\mu x + x_n) = 0$ where $1 \leq \mu \leq n-1$. With the help of equation (3.1) we get

$$\begin{aligned} 0 &= d(\mu x) + d(x_n) + \sum_{k=1}^{n-1} \binom{n}{k} h_k(\mu x, x_n) \\ &= \sum_{k=1}^{n-1} \binom{n}{k} h_k(\mu x, x_n). \end{aligned}$$

This yields that

$$\mu y_1 + \mu^2 y_2 + \cdots + \mu^{n-2} y_{n-2} + \mu^{n-1} n D(x, x, \ldots, x, x_n) = 0,$$

where $y_1, y_2, \ldots, y_{n-2} \in N$. By our hypothesis and Lemma 2.7, we deduce that

$$D(x, x, \ldots, x, x_n) = 0 \tag{3.2}$$

for all $x, x_n \in N$. Let $\nu (1 \leq \nu \leq n-2)$ be any integer. By equation (3.2) we find that

$$D(\nu x + x_{n-1}, \nu x + x_{n-1}, \ldots, \nu x + x_{n-1}, x_n) = 0.$$

Expanding the above relation and using equation (3.2) again we obtain

$$\nu z_1 + \nu^2 z_2 + \ldots + \nu^{n-3} z_{n-3} + \nu^{n-2} \binom{n}{2} D(x, x, \ldots, x, x_{n-1}, x_n) = 0,$$

where $z_1, z_2, \ldots, z_{n-3} \in N$. By our hypothesis and Lemma 2.7, we conclude that $D(x, x, \ldots, x, x_{n-1}, x_n) = 0$ for all $x, x_{n-1}, x_n \in N$. Now if we continue the above process inductively, then we finally arrive at $D(x_1, x_2, \ldots, x_{n-1}, x_n) = 0$. This gives that $D = 0$, a contradiction. □

In the year 1987 H. E. Bell [3, Theorem 2] proved that if a 2-torsion free zero symmetric prime near-ring N admits a non zero derivation D for which $D(N) \subseteq Z$, then N is a commutative ring. Further, this result was generalized by K. H. Park [5, Theorem 3.1] in the year 2010 for permuting tri-derivation, who showed that if 3!-torsion free zero symmetric prime near-ring N admits a non zero permuting tri-derivation D for which $D(N, N, N) \subseteq Z$, then N is a commutative ring. The following result shows that 2-torsion free and 3!-torsion free restrictions in the above results used by Bell and Park are superfluous. In fact, for permuting n-derivation in a prime near-ring N we have obtained the following:

Theorem 3.2. *Let D be a non zero permuting n-derivation of prime near-ring N such that $D(N, N, \ldots, N) \subseteq Z$. Then N is a commutative ring.*

Proof. For all $x_1, x_1^{'}, \ldots, x_n \in N$, we have

$$D(x_1 x_1^{'}, x_2, \ldots, x_n) = D(x_1, x_2, \ldots, x_n)x_1^{'} + x_1 D(x_1^{'}, x_2, \ldots, x_n) \in Z. \tag{3.3}$$

Hence $x_1^{'}\{D(x_1, x_2, \ldots, x_n)x_1^{'} + x_1 D(x_1^{'}, x_2, \ldots, x_n)\} = \{D(x_1, x_2, \ldots, x_n)x_1^{'} + x_1 D(x_1^{'}, x_2, \ldots, x_n)\}x_1^{'}$. Using the hypothesis and Lemma 2.4(i) we get $x_1^{'} x_1 D(x_1^{'}, x_2, \ldots, x_n) = x_1 x_1^{'} D(x_1^{'}, x_2, \ldots, x_n)$. This yields that $D(x_1^{'}, x_2, \ldots, x_n)(x_1{'} x_1 - x_1 x_1^{'}) = 0$. Since Z has no zero divisors, for each fixed $x_1^{'} \in N$ either $(x_1^{'} x_1 - x_1 x_1^{'}) = 0$ or $D(x_1^{'}, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in N$. If first holds, then $x_1^{'} \in Z$ if not, i.e., $D(x_1^{'}, x_2, \ldots, x_n) = 0$, then equation(3.3) reduces to $D(x_1 x_1^{'}, x_2, \ldots, x_n) = D(x_1, x_2, \ldots, x_n)x_1^{'}$. Since $D \neq 0$ and $D(x_1, x_2, \ldots, x_n) \in Z$, by Lemma 2.2 $x_1^{'} \in Z$. Hence we conclude that $N \subseteq Z$. Thus we obtain that $N = Z$, i.e., N is a commutative near-ring. If $N = \{0\}$, then N is trivially a commutative ring. If $N \neq \{0\}$, then there exists $0 \neq x \in N$ and hence $x + x \in N = Z$. Now by Lemma 2.1(ii), we conclude that N is a commutative ring. □

Theorem 3.3. *Let N be a prime near-ring and D_1 and D_2 be any two non zero permuting n-derivations of N. If $[D_1(N, N, \ldots, N), D_2(N, N, \ldots, N)] = \{0\}$, then $(N, +)$ is an abelian group.*

Proof. If both z and $z + z$ commute element wise with $D_2(N, N, \ldots, N)$, then $zD_2(x_1, x_2, \ldots, x_n) = D_2(x_1, x_2, \ldots, x_n)z$ and $(z + z)D_2(x_1, x_2, \ldots, x_n) = D_2(x_1, x_2, \ldots, x_n)(z + z)$ for all $x_1, x_2, \ldots, x_n \in N$. In particular, $(z + z)D_2(x_1 + x_1^{'}, x_2, \ldots, x_n) = D_2(x_1 + x_1^{'}, x_2, \ldots, x_n)(z + z)$ for all $x_1, x_1^{'}, \ldots, x_n \in N$. From the previous equalities we get $zD_2(x_1 + x_1^{'} - x_1 - x_1^{'}, x_2, \ldots, x_n) = 0$, i.e., $zD_2((x_1, x_1^{'}), x_2, \ldots, x_n) = 0$. Putting $z = D_1(y_1, y_2, \ldots, y_n)$ we get $D_1(y_1, y_2, \ldots, y_n)D_2((x_1, x_1^{'}), x_2, \ldots, x_n) = 0$. By Lemma 2.5(i) we conclude that $D_2((x_1, x_1^{'}), x_2, \ldots, x_n) = 0$. Putting $w(x_1, x_1^{'})$ in place of additive commutator $(x_1, x_1^{'})$ where $w \in N$ we have $D_2(w(x_1, x_1^{'}), x_2, \ldots, x_n) = 0$, i.e., $D_2(w, x_2, \ldots, x_n)(x_1, x_1^{'}) + wD_2((x_1, x_1^{'}), x_2, \ldots, x_n) = 0$. Previous equality yields $D_2(w, x_2, \ldots, x_n)(x_1, x_1^{'}) = 0$. By Lemma 2.5(i) again we conclude that $(x_1, x_1^{'}) = 0$. Hence $(N, +)$ is an abelian group. □

Theorem 3.4. *Let N be a prime near-ring with non zero permuting n-derivations D_1 and D_2 such that*

$$D_1(x_1, x_2, \ldots, x_n)D_2(y_1, y_2, \ldots, y_n) = -D_2(x_1, x_2, \ldots, x_n)D_1(y_1, y_2, \ldots, y_n)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N$. Then $(N, +)$ is an abelian group.

Proof. By our hypothesis we have, $D_1(x_1, x_2, \ldots, x_n)D_2(y_1, y_2, \ldots, y_n) + D_2(x_1, x_2, \ldots, x_n)D_1(y_1, y_2, \ldots, y_n) = 0$ for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N$. Replacing y_1 by $y_1 + y_1^{'}$ in previous equation we get $D_1(x_1, x_2, \ldots, x_n)D_2(y_1 +$

$y_1^{'}, y_2, \ldots, y_n) + D_2(x_1, x_2, \ldots, x_n)D_1(y_1 + y_1^{'}, y_2, \ldots, y_n) = 0$. Using our hypothesis we get, $D_1(x_1, x_2, \ldots, x_n)D_2(y_1, y_2, \ldots, y_n) + D_1(x_1, x_2, \ldots, x_n)D_2(y_1^{'}, y_2, \ldots, y_n) + D_1(x_1, x_2, \ldots, x_n)D_2(-y_1, y_2, \ldots, y_n) + D_1(x_1, x_2, \ldots, x_n)D_2(-y_1^{'}, y_2, \ldots, y_n) = 0$, i.e., $D_1(x_1, x_2, \ldots, x_n)D_2((y_1, y_1^{'}), y_2, \ldots, y_n) = 0$. Now using Lemma 2.5(i) we conclude that $D_2((y_1, y_1^{'}), y_2, \ldots, y_n) = 0$. Putting $w(y_1, y_1^{'})$ in place of $(y_1, y_1^{'})$ where $w \in N$ in the previous equality and using Lemma 2.5(i); as used in the previous theorem, we conclude that $(N, +)$ is an abelian group. □

Corollary 3.1 ([1, Lemma 2.1]). *Let N be a prime near-ring with non zero derivations d_1 and d_2 such that $d_1(x)d_2(y) = -d_2(x)d_1(y)$ for all $x, y \in N$. Then $(N, +)$ is an abelian group.*

Theorem 3.5. *LetD be a non zero permuting n-derivation of prime near-ring N. If $D(C, N, N, \ldots, N) = \{0\}$, then $(N, +)$ is an abelian group.*

Proof. Since $D(c, r_2, \ldots, r_n) = 0$ for all $c \in C$ and for all $r_2, \ldots, r_n \in N$, $D(wc, r_2, \ldots, r_n) = 0$ where $w \in N$, i.e., $wD(c, r_2, \ldots, r_n) + D(w, r_2, \ldots, r_n)c = 0$. In turn we get $D(w, r_2, \ldots, r_n)c = 0$ but $D \neq 0$, and therefore by Lemma 2.5(i); $c = 0$. Hence $(N, +)$ is an abelian group. □

Theorem 3.6. *Let N be a semi prime near-ring and D be a permuting n-derivation of N. If $D(x_1, x_2, \ldots, x_n)y_1 = x_1D(y_1, y_2, \ldots, y_n)$ for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N$, then $D = 0$.*

Proof. We have

$$D(x_1, x_2, \ldots, x_n)y_1 = x_1D(y_1, y_2, \ldots, y_n). \tag{3.4}$$

Putting y_1z_1 in place of y_1 in the above equation; where $z_1 \in N$, we get

$$\begin{aligned} D(x_1, x_2, \ldots, x_n)y_1z_1 &= x_1D(y_1z_1, y_2, \ldots, y_n) \\ &= x_1D(y_1, y_2, \ldots, y_n)z_1 + x_1y_1D(z_1, y_2, \ldots, y_n). \end{aligned}$$

By equation (3.4) we get $D(x_1, x_2, \ldots, x_n)y_1z_1 = D(x_1, x_2, \ldots, x_n)y_1z_1 + x_1y_1D(z_1, y_2, \ldots, y_n)$. This yields that $x_1y_1D(z_1, y_2, \ldots, y_n) = 0$. Now replacing x_1 by $D(z_1, y_2, \ldots, y_n)$ we get $D(z_1, y_2, \ldots, y_n)ND(z_1, y_2, \ldots, y_n) = \{0\}$. But since N is a semi prime near-ring, we conclude that $D = 0$. □

Theorem 3.7. *Let N be any prime near-ring and D be any non-zero permuting n-derivation of N. If $K = \{a \in N \mid [D(N, N, \ldots, N), a] = \{0\}\}$, then*

(i) *$a \in K$ implies either $a \in Z$ or $d(a) = 0$,*
(ii) *$d(K) \subseteq Z$,*
(iii) *K is a semigroup under multiplication,*
(iv) *If there exists an element $a \in K$ for which $d(a) \neq 0$ and $D(a^2, a, \ldots, a) \in Z$, then $(N, +)$ is an abelian group.*

Proof. (i) We have

$$D(x_1, x_2, \ldots, x_n)a = aD(x_1, x_2, \ldots, x_n) \tag{3.5}$$

for all $x_1, x_2, \ldots, x_n \in N$. Putting ax_1 in place of x_1 in the above equation and using Lemma 2.4(i) we get $D(a, x_2, \ldots, x_n)x_1a + aD(x_1, x_2, \ldots, x_n)a = aD(a, x_2, \ldots, x_n)x_1 + aaD(x_1, x_2, \ldots, x_n)$. Using the equation (3.5), we get $D(a, x_2, \ldots, x_n)x_1a = aD(a, x_2, \ldots, x_n)x_1$. Now putting x_1y_1 for x_1 in the latter relation and using it again, we have $D(a, x_2, \ldots, x_n)x_1[y_1, a] = 0$ where $y_1 \in N$. This gives us $D(a, x_2, \ldots, x_n)N[a, y_1] = \{0\}$. Since N is a prime near-ring, either $[a, y_1] = 0$ for all $y_1 \in N$ or $D(a, x_2, \ldots, x_n) = 0$ for all $x_2, \ldots, x_n \in N$. If first holds, then $a \in Z$, if not then $D(a, x_2, \ldots, x_n) = 0$, and hence in particular, $D(a, a, \ldots, a) = 0$ or $d(a) = 0$.

(ii) From the above proof we observe that if $a \in K$, then either $a \in Z$ or $d(a) = 0$. But $d(a) = 0$ implies $d(a) \in Z$. If $d(a) \neq 0$, then we have $a \in Z$. In this case we have $D(xa, a \ldots, a) = D(ax, a, \ldots, a)$ for all $x \in N$. This yields that $xD(a, a, \ldots, a) + D(x, a, \ldots, a)a = D(a, a, \ldots, a)x + aD(x, a, \ldots, a)$. This reduces to $xD(a, a, \ldots, a) = D(a, a, \ldots, a)x$, which shows that $d(a) \in Z$ and thus $d(K) \subseteq Z$.

(iii) Let $a, b \in K$. Hence $abD(r_1, r_2, \ldots, r_n) = D(r_1, r_2, \ldots, r_n)ab$ holds trivially. Associativity of N shows that K is a semigroup.

(iv) Consider $D(a^2, a, \ldots, a) = aD(a, a, \ldots, a) + D(a, a, \ldots, a)a \in Z$. As $d(a) = D(a, a, \ldots, a) \neq 0$ implies that $a \in Z$ by (i). Hence $D(a^2, a, \ldots, a) = D(a, a, \ldots, a)(a + a)$. By above proof (ii) we find that $D(a, a, \ldots, a) \in Z \setminus \{0\}$ and hence using Lemma 2.2, $(a + a) \in Z$. By Lemma 2.1(ii) we conclude that $(N, +)$ is an abelian group. □

Theorem 3.8. *Let N be a prime near-ring which admits a non zero permuting n-derivation D such that $D(C, C, \ldots, C) \subseteq Z$. Then N is a commutative ring where $C \neq \{0\}$.*

Proof. For all $c_1, c_1^{'}, \ldots, c_n \in C$, we get

$$D(c_1c_1^{'}, c_2, \ldots, c_n) = D(c_1, c_2, \ldots, c_n)c_1^{'} + c_1D(c_1^{'}, c_2, \ldots, c_n) \in Z \tag{3.6}$$

and commuting this element with $c_1{'}$ we arrive at $D(c_1^{'}, c_2, \ldots, c_n)(c_1^{'}c_1 - c_1c_1^{'}) = 0$ for all $c_1, c_1^{'}, \ldots, c_n \in C$. Now by Lemma 2.1(i), we observe that for each $c_1^{'}$ either $c_1^{'}$ centralizes C or $D(c_1^{'}, c_2, \ldots, c_n) = 0$. If first case holds for each element of C, then C becomes commutative with respect to multiplication. On the other hand if second case holds, i.e., $D(c_1^{'}, c_2, \ldots, c_n) = 0$, then equation(3.6) takes the form

$$D(c_1c_1^{'}, c_2, \ldots, c_n) = D(c_1, c_2, \ldots, c_n)c_1^{'} \in Z \tag{3.7}$$

for all $c_1, c_2, \ldots, c_n \in C$. By Lemmas 2.2 and 2.6, we conclude that $c_1^{'} \in Z$. Hence in this case also we conclude that $c_1^{'}$ centralizes C. Hence in both cases we conclude that C is a commutative semi group with respect to multiplication. Now we separate the proof in two cases:

Case I: Let $C \cap Z \neq \{0\}$. Then in this case it follows that if C contains a non zero central element w, then we have $wxc = xcw = xwc = wcx$ for all $c \in C$ and for all $x \in N$. Hence we have $w(xc - cx) = 0$. By Lemma 2.1(i), we conclude that $c \in Z$, i.e., $C \subseteq Z$. For all $c \in C$ and for all $x, y \in N$ we have $xcy = yxc$ or $cxy = cyx$ since $xc \in C$. Lastly we get $c(xy - yx) = 0$. As $C \neq \{0\}$, by Lemma 2.1(i), N becomes a commutative near-ring, i.e., $N = Z$. If $N = \{0\}$, then N is trivially a commutative ring. If $N \neq \{0\}$, then there exists $t \in N \setminus \{0\}$. Hence $t + t \in N = Z$, and by Lemma 2.1(ii) we conclude that N is a commutative ring.

Case II: Let $C \cap Z = \{0\}$. For this case in the light of equation (3.7) we claim that $D(c_1, c_2, \ldots, c_n) \neq 0$ for all $c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n \in C$ and for all $c_i \in C \setminus \{0\}$. For each $c_i \in C$ and for all $c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n \in C$, $D(c_1, c_2, \ldots, c_i^2, \ldots, c_n) = D(c_1, c_2, \ldots, c_i, \ldots, c_n)(c_i + c_i)$ and hence by Lemmas 2.2 and 2.6, $2c_i \in Z$. Suppose that $2c_i \neq 0$ for all $c_i \in C \setminus \{0\}$. It is obvious that $xC = \{xc \mid c \in C\} = \{0\}$ implies $x = 0$. This shows that for each $x \in N \setminus \{0\}$, there exists $c_x \in C$ such that $xc_x \neq 0$. Since xc_x being an additive commutator also belongs to C, we have $2xc_x = x(2c_x)$ and by Lemma 2.2 we conclude that $x \in Z$. Hence $N = Z$, i.e., N is a commutative near-ring. If $N = \{0\}$, then N is trivially a commutative ring. If $N \neq \{0\}$, then there exists $p \in N \setminus \{0\}$ such that $p + p \in N = Z$. By Lemma 2.1(ii) we conclude that N is a commutative ring. The only remaining possibility is that $C \cap Z = \{0\}$ and there exists $c_i \in C \setminus \{0\}$ such that $2c_i = 0$ and we complete our proof by showing that this leads to a contradiction. Suppose that $c_i \in C \setminus \{0\}$ and $2c_i = 0$. We have $D(c_1, c_2, \ldots, c_i^3, \ldots, c_n) = 3c_i^2 D(c_1, c_2, \ldots, c_i, \ldots, c_n) \in Z$. Since $2c_i^2 D(c_1, c_2, \ldots, c_i, \ldots, c_n) = 0$, we get $c_i^2 D(c_1, c_2, \ldots, c_i, \ldots, c_n) \in Z$. This implies that $c_i^2 \in Z$ by Lemma 2.2. Since $C \cap Z = \{0\}$, $c_i^2 = 0$. Now $D(c_1, c_2, \ldots, xc_i, \ldots, c_n) = xD(c_1, c_2, \ldots, c_i, \ldots, c_n) + D(c_1, c_2, \ldots, x, \ldots, c_n)c_i \in Z$ for all $x \in N$ and $c_1, c_2, \ldots, c_n \in C$. Hence $c_i\{xD(c_1, c_2, \ldots, c_i, \ldots, c_n) + D(c_1, c_2, \ldots, x, \ldots, c_n)c_i\} = \{xD(c_1, c_2, \ldots, c_i, \ldots, c_n) + D(c_1, c_2, \ldots, x, \ldots, c_n) c_i\}c_i = xD(c_1, c_2, \ldots, c_i, \ldots, c_n)c_i$. Left multiplying by c_i we get $c_i^2\{xD(c_1, c_2, \ldots, c_i, \ldots, c_n) + D(c_1, c_2, \ldots, x, \ldots, c_n)c_i\} = c_i x D(c_1, c_2, \ldots, c_i, \ldots, c_n)c_i$. Finally we get $c_i x D(c_1, c_2, \ldots, c_i, \ldots, c_n)c_i = 0$. This implies that $c_i N D(c_1, c_2, \ldots, c_i, \ldots, c_n)c_i = \{0\}$, but primeness of N yields that $D(c_1, c_1, \ldots, c_i, \ldots, c_n) c_i = 0$. Since $D(c_1, c_2, \ldots, c_i, \ldots, c_n) \in Z \setminus \{0\}$, by Lemma 2.1(i), we conclude that $c_i = 0$, a contradiction. □

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