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A NEW TYPE OF HYPER K-SUBALGEBRAS

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Abstract. In this paper, the concept of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy hyper K-subalgebras and fuzzy hyper K-subalgebras with thresholds are introduced, and related properties and characterizations are discussed.

1. Introduction

The hyperstructure theory (called also multialgebras) is introduced in 1934 by Marty [26] at the 8th congress of Scandinavian Mathematicians. Around the 40's, several authors worked on hypergroups, especially in France and in the United States, but also in Italy, Russia, Japan and Iran. Hyperstructures have many applications to several sectors of both pure and applied sciences. Jun et al. [22] introduced and studied hyperBCK-algebra which is a generalization of a BCK-algebra. Borzooei et al. constructed the hyper K-algebras, and studied (weak) implicative hyper K-ideals in hyper K-algebras (see [3, 4, 5]). Fuzzy sets and hyperstructures introduced by Zadeh and Marty, respectively, are now used in the real world, both from a theoretical point of view and for their many applications. The relations between fuzzy sets and hyperstructures have been already considered by several authors (for instance see [6, 7, 8, 9, 10, 11, 12, 17, 19, 20, 24, 25, 31]). Murali [27] proposed a definition of a fuzzy point belonging to fuzzy subset under a natural equivalence on fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [28], played a vital role to generate some different types of fuzzy subsets. Using the "belongs to" relation ($\in$) and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy set, the fuzzy set theory applied to many algebraic structures (for instance see [1, 2, 13, 14, 15, 16, 18, 21, 29, 30]). As a generalization of the notion of fuzzy hyper K-subalgebras, Kang [23] introduced the concept of fuzzy hyper K-subalgebras of type (α, β) where $\alpha, \beta \in \{\in$, q, $\in \vee\, q$, $\in \wedge\, q\}$ and $\alpha \neq\, \in \wedge\, q$. He investigated relations between each types, and discussed many related properties. In particular, he dealt with the notion of ($\in$, $\in \vee\, q$)-fuzzy hyper K-subalgebras, and considered characterizations

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of ($\in$, $\in\vee q$)-fuzzy hyper K-subalgebras. He provided conditions for an ($\in$, $\in\vee q$)-fuzzy hyper K-subalgebra to be an ($\in$, $\in$)-fuzzy hyper K-subalgebra, and made an ($\in$, $\in\vee q$)-fuzzy hyper K-subalgebra by using a collection of hyper K-subalgebras. He finally discussed the implication-based fuzzy hyper K-subalgebras. As a continuation of the paper [23], we introduce the notion of a fuzzy hyper K-subalgebra of type ($\overline{\in}$, $\overline{\in}\vee\overline{q}$) and with thresholds. We provide relations between fuzzy hyper K-subalgebras and ($\overline{\in}$, $\overline{\in}\vee\overline{q}$)-fuzzy hyper K-subalgebras. We describe characterizations of ($\overline{\in}$, $\overline{\in}\vee\overline{q}$)-fuzzy hyper K-subalgebras and fuzzy hyper K-subalgebras with thresholds. Using a special set, we provide conditions for a fuzzy set to be an ($\overline{\in}$, $\overline{\in}\vee\overline{q}$)-fuzzy hyper K-subalgebra and a fuzzy hyper K-subalgebra with thresholds. We also give conditions for an ($\overline{\in}$, $\overline{\in}\vee\overline{q}$)-fuzzy hyper K-subalgebra and an ($\in$, $\in\vee q$)-fuzzy hyper K-subalgebra to be a fuzzy hyper K-subalgebra with thresholds.

2. Preliminaries

In [5], Borzoei et al. established the notion of hyper I-algebras/hyper K-algebras as follows: By a *hyper I-algebra* we mean a non-empty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfying the following axioms:

(HI1) $(x\circ z)\circ(y\circ z) < x\circ y$,
(HI2) $(x\circ y)\circ z = (x\circ z)\circ y$,
(HI3) $x < x$,
(HI4) $x < y$ and $y < x$ imply $x = y$

for all $x, y, z \in H$, where $x < y$ is defined by $0 \in x\circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A$ and $\exists b \in B$ such that $a < b$. If a hyper I-algebra $(H, \circ, 0)$ satisfies

(HI5) $0 < x$ for all $x \in H$,

then $(H, \circ, 0)$ is called a *hyper K-algebra.*

Let $(H, \circ, 0)$ be a hyper K-algebra. Then for all $x, y, z \in H$ and for all nonempty subsets A and B of H the following hold (see [4] and [5, Proposition 3.6]):

(a1) $x \in x\circ 0$,
(a2) $x\circ y < x$,
(a3) $A\circ B < A$,
(a4) $A\circ A < A$,
(a5) $0 \in x\circ(x\circ 0)$,
(a6) $x < x\circ 0$,
(a7) $A < A\circ 0$,
(a8) $A < A\circ B$ if $0 \in B$.

Definition 2.1 ([5]). Let $(H, \circ, 0)$ be a hyper K-algebra and let S be a subset of H containing 0. If S is a hyper K-algebra with respect to the hyperoperation "$\circ$" on H, we say that S is a *hyper K-subalgebra* of H.

Lemma 2.2 ([5]). *Let S be a nonempty subset of a hyper K-algebra $(H, \circ, 0)$. Then S is a hyper K-subalgebra of H if and only if $x \circ y \subseteq S$ for all $x, y \in S$.*

A fuzzy subset μ of a set X of the form

$$\mu(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set μ in a set X, Pu and Liu [28] introduced the symbol $x_t \alpha \mu$, where $\alpha \in \{\in, q, \in\vee q, \in\wedge q\}$. To say that $x_t \in \mu$ (resp. $x_t\, q\, \mu$), we mean $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$), and in this case, x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set μ. To say that $x_t \in\vee q\ \mu$ (resp. $x_t \in\wedge q\ \mu$), we mean $x_t \in \mu$ or $x_t\, q\, \mu$ (resp. $x_t \in \mu$ and $x_t\, q\, \mu$). For $\alpha \in \{\in, q, \in\vee q, \in\wedge q\}$, we say that $x_t\, \overline{\alpha}\, \mu$ if $x_t\, \alpha\, \mu$ does not hold.

3. Fuzzy hyper K-subalgebras of type $(\overline{\in}, \overline{\in} \vee \overline{q})$

In what follows, let H denote a hyper K-algebra unless otherwise specified.

Definition 3.1 ([17]). A fuzzy set μ in H is called a *fuzzy hyper K-subalgebra* of H if it satisfies:

$$(3.1) \qquad (\forall x, y \in H) \left(\inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), \mu(y)\} \right).$$

Definition 3.2 ([23]). A fuzzy set μ in H is called a *fuzzy hyper K-subalgebra in H of type* $(\in, \in\vee q)$, or briefly, an $(\in, \in\vee q)$*-fuzzy hyper K-subalgebra* of H if for all $x, y \in H$ and $t_1, t_2 \in (0, 1]$,

$$(3.2) \qquad x_{t_1} \in \mu,\ y_{t_2} \in \mu \implies z_{\min\{t_1, t_2\}} \in\vee q\, \mu \quad \text{for all } z \in x \circ y.$$

Lemma 3.3 ([23]). *A fuzzy set μ in a hyper K-algebra H is an $(\in, \in\vee q)$-fuzzy hyper K-subalgebra of H if and only if it satisfies*

$$(3.3) \qquad (\forall x, y \in H) \left(\inf_{z \in x \circ y} \mu(z) \geq \min\{\mu(x), \mu(y), 0.5\} \right).$$

Theorem 3.4. *Let $\{\mu_i \mid i \in \Lambda\}$ be a family of $(\in, \in\vee q)$-fuzzy hyper K-subalgebras of H. Then $\mu := \bigcap_{i \in \Lambda} \mu_i$ is an $(\in, \in\vee q)$-fuzzy hyper K-subalgebras of H.*

Proof. Suppose that $\{\mu_i \mid i \in \Lambda\}$ is a family of $(\in, \in\vee q)$-fuzzy hyper K-subalgebras of H. By Lemma 3.3, $\inf_{z \in x \circ y} \mu_i(z) \geq \min\{\mu_i(x), \mu_i(y), 0.5\}$ for all

TABLE 1. Hyper operation for H

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{3\}$	$\{1,2\}$
2	$\{2\}$	$\{0\}$	$\{0\}$	$\{2\}$
3	$\{3\}$	$\{0\}$	$\{1,2,3\}$	$\{0,3\}$

$i \in \Lambda$ and $x, y \in H$. Hence

$$\begin{aligned}\inf_{z\in x\circ y}\mu(z) &= \inf_{z\in x\circ y}\left(\bigcap_{i\in\Lambda}\mu_i(z)\right) = \inf_{z\in x\circ y}\left(\inf_{i\in\Lambda}\mu_i(z)\right) = \inf_{i\in\Lambda}\left(\inf_{z\in x\circ y}\mu_i(z)\right)\\ &\geq \inf_{i\in\Lambda}\left(\min\{\mu_i(x),\mu_i(y),0.5\}\right) = \min\left\{\inf_{i\in\Lambda}\mu_i(x),\inf_{i\in\Lambda}\mu_i(y),0.5\right\}\\ &= \min\left\{\bigcap_{i\in\Lambda}\mu_i(x),\bigcap_{i\in\Lambda}\mu_i(y),0.5\right\} = \min\{\mu(x),\mu(y),0.5\}.\end{aligned}$$

Therefore, μ is an $(\in, \in\vee q)$-fuzzy hyper K-subalgebra of H. □

The following example shows that the union of $(\in, \in\vee q)$-fuzzy hyper K-subalgebras of H may not be an $(\in, \in\vee q)$-fuzzy hyper K-subalgebra of H.

Example 3.5. Let $H = \{0, 1, 2, 3\}$ and define a hyper operation $\circ$ on H by Table 1. Then $(H, \circ, 0)$ is a hyper K-algebra. Define fuzzy sets μ_1 and μ_2 in H as follows:

$$\mu_1 : H \to [0,1],\ h \mapsto \begin{cases} 0.9 & \text{if } h = 0,\\ 0.2 & \text{if } h = 1,\\ 0.2 & \text{if } h = 2,\\ 0.4 & \text{if } h = 3,\end{cases}$$

$$\mu_2 : H \to [0,1],\ h \mapsto \begin{cases} 0.9 & \text{if } h = 0,\\ 0.3 & \text{if } h = 1,\\ 0.1 & \text{if } h = 2,\\ 0.1 & \text{if } h = 3.\end{cases}$$

By routine calculations, we know that μ_1 and μ_2 are $(\in, \in\vee q)$-fuzzy hyper K-subalgebras of H. Note that $\mu := \mu_1 \cup \mu_2$ is given by

$$\mu : H \to [0,1],\ h \mapsto \begin{cases} 0.9 & \text{if } h = 0,\\ 0.3 & \text{if } h = 1,\\ 0.2 & \text{if } h = 2,\\ 0.4 & \text{if } h = 3,\end{cases}$$

which is not an $(\in, \in\vee q)$-fuzzy hyper K-subalgebra of H since

$$\inf_{z\in 1\circ 3}\mu(z) = \mu(2) = 0.2 \ngeq 0.3 = \min\{\mu(1),\mu(3),0.5\}.$$

TABLE 2. Hyper operation for H

$\circ$	0	a	b	x	y
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{b\}$	$\{0,b\}$	$\{0,b\}$	$\{b\}$	$\{b\}$
x	$\{x\}$	$\{x\}$	$\{x\}$	$\{0,x\}$	$\{x\}$
y	$\{y\}$	$\{y\}$	$\{y\}$	$\{y\}$	$\{0\}$

It is well known that a fuzzy set μ in H is a fuzzy hyper K-subalgebra of H if and only if the non-empty level subset $U(\mu;t)$, $t \in (0,1]$, of μ is a hyper K-subalgebra of H. Note that for a fuzzy set μ in H, the non-empty level subset $U(\mu;t)$, $t \in (0,0.5]$, of μ is a hyper K-subalgebra of H if and only if μ is an $(\in, \in\vee q)$-fuzzy hyper K-subalgebra of H (see [23, Theorem 4.5]).

Since it is natural to consider the number $t \in (0.5,1]$ for which $U(\mu;t)$ is a hyper K-subalgebra of H, we consider a new kind of a fuzzy hyper K-subalgebra as follows.

Definition 3.6. A fuzzy set μ in H is called a *fuzzy hyper K-subalgebra* in H of type $(\overline{\in}, \overline{\in}\vee\overline{q})$ (briefly, $(\overline{\in}, \overline{\in}\vee\overline{q})$*-fuzzy hyper K-subalgebra*) if for all $x, y \in H$ and $t_1, t_2 \in (0,1]$,

$$(3.4)\qquad (\exists\, z \in x \circ y)\ \big(z_{\min\{t_1,t_2\}}\,\overline{\in}\,\mu\big) \implies x_{t_1}\,\overline{\in}\vee\overline{q}\,\mu \ \text{ or } \ y_{t_2}\,\overline{\in}\vee\overline{q}\,\mu.$$

Example 3.7. Let $H = \{0, a, b, x, y\}$ and define a hyper operation $\circ$ on H by Table 2. Then $(H, \circ, 0)$ is a hyper K-algebra. Define a fuzzy set μ in H as follows:

$$\mu : H \to [0,1],\ h \mapsto \begin{cases} 0.7 & \text{if } h = 0,\\ 0.6 & \text{if } h = a,\\ 0.5 & \text{if } h = b,\\ 0.2 & \text{if } h = x,\\ 0.3 & \text{if } h = y. \end{cases}$$

By routine calculations, we know that μ is an $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy hyper K-subalgebra of H.

Obviously, every fuzzy hyper K-subalgebra is an $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy hyper K-subalgebra, but the converse may not be true as seen in the following example.

Example 3.8. Let $H = \{0, 1, 2, 3\}$ be the hyper K-algebra which is described in Example 3.5. Define a fuzzy set μ in H as follows:

$$\mu : H \to [0,1],\ h \mapsto \begin{cases} 0.9 & \text{if } h = 0,\\ 0.7 & \text{if } h = 1,\\ 0.4 & \text{if } h = 2,\\ 0.3 & \text{if } h = 3. \end{cases}$$

By routine calculations, we know that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy hyper K-subalgebras of H. But μ is not a fuzzy hyper K-subalgebra of H since $\inf_{z \in 1 \circ 2} \mu(z) = \mu(3) = 0.3 \not\geq 0.4 = \min\{\mu(1), \mu(2)\}$.

Theorem 3.9. *A fuzzy set μ in H is a fuzzy hyper K-subalgebra in H of type $(\overline{\in}, \overline{\in} \vee \overline{q})$ if and only if it satisfies the following condition*

$$(3.5) \qquad (\forall x, y \in H) \left(\max\Big\{\inf_{z \in x \circ y} \mu(z), 0.5\Big\} \geq \min\{\mu(x), \mu(y)\}\right).$$

Proof. Let μ be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy hyper K-subalgebra of H. If there exist $x, y \in H$ such that

$$\max\Big\{\inf_{z \in x \circ y} \mu(z), 0.5\Big\} < \min\{\mu(x), \mu(y)\},$$

then $\max\Big\{\inf_{z \in x \circ y} \mu(z), 0.5\Big\} < t \leq \min\{\mu(x), \mu(y)\}$ for some $t \in (0, 1]$. Thus $0.5 < t \leq 1$, $\inf_{z \in x \circ y} \mu(z) < t$, $x_t \in \mu$ and $y_t \in \mu$. From $\inf_{z \in x \circ y} \mu(z) < t$, it follows that there exists $w \in x \circ y$ such that $\mu(w) < t$. Hence $w_{\min\{t,t\}} = w_t \overline{\in} \mu$. Now $\mu(x) + t \geq 2t > 1$ and $\mu(y) + t \geq 2t > 1$, that is, $x_t \, q \, \mu$ and $y_t \, q \, \mu$. Hence $x_t \in \wedge \, q \, \mu$ and $y_t \in \wedge \, q \, \mu$, or equivalently, $x_t \overline{\overline{\in} \vee \overline{q}} \, \mu$ and $y_t \overline{\overline{\in} \vee \overline{q}} \, \mu$. This is a contradiction. Therefore (3.5) is valid.

Conversely assume that $\max\Big\{\inf_{z \in x \circ y} \mu(z), 0.5\Big\} \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in H$. Let $x, y \in H$ and $t_1, t_2 \in (0, 1]$ be such that $z_{\min\{t_1, t_2\}} \overline{\in} \mu$ for some $z \in x \circ y$. Then $\mu(z) < \min\{t_1, t_2\}$ for some $z \in x \circ y$, and so

$$\inf_{a \in x \circ y} \mu(a) < \min\{t_1, t_2\}.$$

If $\inf_{a \in x \circ y} \mu(a) \geq \min\{\mu(x), \mu(y)\}$, then

$$\min\{\mu(x), \mu(y)\} < \min\{t_1, t_2\}$$

which implies that $\mu(x) < t_1$ or $\mu(y) < t_2$, that is, $x_{t_1} \overline{\in} \mu$ or $y_{t_2} \overline{\in} \mu$. If $\inf_{a \in x \circ y} \mu(a) < \min\{\mu(x), \mu(y)\}$, then $\min\{\mu(x), \mu(y)\} \leq 0.5$. Assume that $x_{t_1} \in \mu$ and $y_{t_2} \in \mu$. Then $t_1 \leq \mu(x) \leq 0.5$ or $t_2 \leq \mu(y) \leq 0.5$, and thus $\mu(x) + t_1 \leq 2\mu(x) \leq 1$ or $\mu(y) + t_2 \leq 2\mu(y) \leq 1$. Therefore $x_{t_1} \overline{q} \, \mu$ or $y_{t_2} \overline{q} \, \mu$. Consequently $x_{t_1} \overline{\in} \vee \overline{q} \, \mu$ or $y_{t_2} \overline{\in} \vee \overline{q} \, \mu$. Hence μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy hyper K-subalgebra of H. $\square$

Theorem 3.10. *For a fuzzy set μ in H, the following are equivalent:*

(1) *μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy hyper K-subalgebra of H.*
(2) $(\forall t \in (0.5, 1])$ $(U(\mu; t) \neq \emptyset \Rightarrow U(\mu; t)$ *is a hyper K-subalgebra of H*).

Proof. Assume that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy hyper K-subalgebra of H. Let $t \in (0.5, 1]$ be such that $U(\mu; t) \neq \emptyset$. Let $x, y \in U(\mu; t)$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. Using (3.5), we have

$$\max\Big\{\inf_{z \in x \circ y} \mu(z), 0.5\Big\} \geq \min\{\mu(x), \mu(y)\} \geq t > 0.5.$$

It follows that $\mu(w) \geq \inf_{z \in x \circ y} \mu(z) \geq t$ for all $w \in x \circ y$ so that $w \in U(\mu; t)$. Hence $x \circ y \subseteq U(\mu; t)$, and therefore $U(\mu; t)$ is a hyper K-subalgebra of H for all $t \in (0.5, 1]$.

Conversely, let μ be a fuzzy set in H such that $U(\mu; t) \neq \emptyset$ is a hyper K-subalgebra of H for all $t \in (0.5, 1]$. If there exist $a, b \in H$ such that

$$\max\Big\{\inf_{c \in a \circ b} \mu(c), 0.5\Big\} < \min\{\mu(a), \mu(b)\} = r,$$

then $r \in (0.5, 1]$, $a, b \in U(\mu; r)$ and $\inf_{c \in a \circ b} \mu(c) < r$. The last inequality implies that $\mu(d) < r$ for some $d \in a \circ b$, and so $d \notin U(\mu; r)$. Thus $a \circ b \nsubseteq U(\mu; r)$, a contradiction. Consequently,

$$\max\Big\{\inf_{z \in x \circ y} \mu(z), 0.5\Big\} \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y \in H$. Using Theorem 3.9, we conclude that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy hyper K-subalgebra of H. □

Theorem 3.11. *Let $\{\mu_i \mid i \in \Lambda\}$ be a family of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy hyper K-subalgebras of H. Then $\mu := \bigcap_{i \in \Lambda} \mu_i$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy hyper K-subalgebra of H.*

Proof. Suppose that $\{\mu_i \mid i \in \Lambda\}$ is a family of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy hyper K-subalgebras of H. Note that

$$\begin{aligned}
\max\left\{\inf_{z \in x \circ y} \mu(z), 0.5\right\} &= \max\left\{\inf_{z \in x \circ y}\left(\bigcap_{i \in \Lambda} \mu_i(z)\right), 0.5\right\} \\
&= \max\left\{\inf_{z \in x \circ y}\left(\inf_{i \in \Lambda} \mu_i(z)\right), 0.5\right\} \\
&= \max\left\{\inf_{i \in \Lambda}\left(\inf_{z \in x \circ y} \mu_i(z)\right), 0.5\right\}.
\end{aligned}$$

If $\inf_{i \in \Lambda}\left(\inf_{z \in x \circ y} \mu_i(z)\right) \geq 0.5$, then $0.5 \leq \inf_{i \in \Lambda}\left(\inf_{z \in x \circ y} \mu_i(z)\right) \leq \inf_{z \in x \circ y} \mu_i(z)$ for all $i \in \Lambda$. Thus $\max\left\{\inf_{z \in x \circ y} \mu_i(z), 0.5\right\} = \inf_{z \in x \circ y} \mu_i(z)$ for all $i \in \Lambda$, and so

$$\begin{aligned}
\max\left\{\inf_{i \in \Lambda}\left(\inf_{z \in x \circ y} \mu_i(z)\right), 0.5\right\} &= \inf_{i \in \Lambda}\left(\inf_{z \in x \circ y} \mu_i(z)\right) \\
&= \inf_{i \in \Lambda}\left(\max\left\{\inf_{z \in x \circ y} \mu_i(z), 0.5\right\}\right) \\
&\geq \inf_{i \in \Lambda}\left(\min\left\{\mu_i(x), \mu_i(y)\right\}\right) \\
&= \min\left\{\inf_{i \in \Lambda} \mu_i(x), \inf_{i \in \Lambda} \mu_i(y)\right\} \\
&= \min\left\{\mu(x), \mu(y)\right\}.
\end{aligned}$$

If $\inf_{i\in\Lambda}\left(\inf_{z\in x\circ y}\mu_i(z)\right) < 0.5$, then $\inf_{z\in x\circ y}\mu_j(z) < 0.5$ for some $j \in \Lambda$. Then

$$\begin{aligned}\max\left\{\inf_{i\in\Lambda}\left(\inf_{z\in x\circ y}\mu_i(z)\right), 0.5\right\} &= 0.5 = \max\left\{\inf_{z\in x\circ y}\mu_j(z), 0.5\right\} \\ &\geq \min\left\{\mu_j(x), \mu_j(y)\right\} \\ &\geq \min\left\{\inf_{i\in\Lambda}\mu_i(x), \inf_{i\in\Lambda}\mu_i(y)\right\} \\ &= \min\left\{\mu(x), \mu(y)\right\}.\end{aligned}$$

Therefore μ is an $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy hyper K-subalgebra of H. □

The following example shows that the union of $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy hyper K-subalgebras may not be an $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy hyper K-subalgebra of H.

Example 3.12. Consider the hyper K-algebra $H = \{0, 1, 2, 3\}$ which is given in Example 3.5. Define fuzzy sets μ_1 and μ_2 in H as follows:

$$\mu_1 : H \to [0,1],\ h \mapsto \begin{cases} 0.9 & \text{if } h = 0, \\ 0.7 & \text{if } h = 1, \\ 0.4 & \text{if } h = 2, \\ 0.3 & \text{if } h = 3, \end{cases}$$

$$\mu_2 : H \to [0,1],\ h \mapsto \begin{cases} 0.9 & \text{if } h = 0, \\ 0.4 & \text{if } h = 1, \\ 0.8 & \text{if } h = 2, \\ 0.3 & \text{if } h = 3. \end{cases}$$

By routine calculations, we know that μ_1 and μ_2 are $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy hyper K-subalgebras of H and $\mu := \mu_1 \cup \mu_2$ is given by

$$\mu : H \to [0,1],\ h \mapsto \begin{cases} 0.9 & \text{if } h = 0, \\ 0.7 & \text{if } h = 1, \\ 0.8 & \text{if } h = 2, \\ 0.3 & \text{if } h = 3. \end{cases}$$

But $\mu := \mu_1 \cup \mu_2$ is not an $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy hyper K-subalgebra of H because $\max\left\{\inf_{z\in 1\circ 2}\mu(z), 0.5\right\} = \max\{\mu(3), 0.5\} = \max\{0.3, 0.5\} = 0.5 \ngeq 0.7 = \min\{0.7, 0.8\} = \min\{\mu(1), \mu(2)\}$.

Proposition 3.13. *Let μ be a fuzzy set in H such that $\mu(x) > 0.5$ for some $x \in H$. If μ is an $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy hyper K-subalgebra of H, then $\mu(0) \geq \mu(x)$ for all $x \in H$.*

Proof. Let μ be an $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy hyper K-subalgebra of H such that $\mu(x) > 0.5$ for some $x \in H$. We first show that $\mu(0) \geq 0.5$. Assume that

$\mu(0) < 0.5$. Since $0 \in x \circ x$ for all $x \in H$, we have

$$\begin{aligned}\mu(0) = \max\{\mu(0), 0.5\} &\geq \max\left\{\inf_{z \in x \circ x} \mu(z), 0.5\right\} \\ &\geq \min\{\mu(x), \mu(x)\} = \mu(x)\end{aligned}$$

for all $x \in H$. This is a contradiction, and so $\mu(0) \geq 0.5$. Now suppose that there exists $a \in H$ such that $\mu(0) < \mu(a)$. Then $\mu(0) < t \leq \mu(a)$ for some $t \in (0.5, 1]$, which implies that $0 \notin U(\mu; t)$. Hence $U(\mu; t)$ is not a hyper K-subalgebra of H and therefore μ is not an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy hyper K-subalgebra of H by Theorem 3.10. This is a contradiction. Hence $\mu(0) \geq \mu(x)$ for all $x \in H$. □

For any fuzzy set μ in H and any $t \in (0, 1]$, we consider the following set:

$$Q(\mu; t) := \{x \in H \mid x_t \, q \, \mu\}.$$

Theorem 3.14. *If μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy hyper K-subalgebra of H, then the set $Q(\mu; t)(\neq \emptyset)$ is a hyper K-subalgebra of H for all $t \in (0, 0.5]$.*

Proof. For any $t \in (0, 0.5]$, let $x, y \in Q(\mu; t)$. Then $x_t \, q \, \mu$ and $y_t \, q \, \mu$, that is, $\mu(x) + t > 1$ and $\mu(y) + t > 1$. Using Theorem 3.9, we have

$$\max\left\{\inf_{z \in x \circ y} \mu(z), 0.5\right\} \geq \min\{\mu(x), \mu(y)\} > 1 - t \geq 0.5 \geq t.$$

Thus $\inf_{z \in x \circ y} \mu(z) > 1 - t$, and so $\mu(z) > 1 - t$ for all $z \in x \circ y$. Hence $z_t \, q \, \mu$, i.e., $z \in Q(\mu; t)$ for all $z \in x \circ y$. Therefore $Q(\mu; t)$ is a hyper K-subalgebra of H for all $t \in (0, 0.5]$. □

Theorem 3.15. *Let H satisfy $|x \circ y| < \infty$ for all $x, y \in H$. Let μ be a fuzzy set in H such that the set $Q(\mu; t)$ is nonempty and is a hyper K-subalgebra of H for all $t \in (0, 0.5]$. Then μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy hyper K-subalgebra of H.*

Proof. Assume that $\max\left\{\inf_{c \in a \circ b} \mu(c), 0.5\right\} < \min\{\mu(a), \mu(b)\}$ for some $a, b \in H$. Let $r := \max\left\{\inf_{c \in a \circ b} \mu(c), 0.5\right\}$. Then $1 < \mu(a) + 1 - r$ and $1 < \mu(b) + 1 - r$. i.e., $a, b \in Q(\mu; 1 - r)$. If $\inf_{c \in a \circ b} \mu(c) < 0.5$, then $r = 0.5 > \inf_{c \in a \circ b} \mu(c)$ and so $1 \geq \mu(d) + 1 - r$ for some $d \in a \circ b$. Hence $d \notin Q(\mu; 1 - r)$ for some $d \in a \circ b$, and thus $a \circ b \nsubseteq Q(\mu; 1 - r)$, a contradiction. If $\inf_{c \in a \circ b} \mu(c) \geq 0.5$, then $r = \inf_{c \in a \circ b} \mu(c)$. Since $|x \circ y| < \infty$ for all $x, y \in H$, we have $r = \mu(d)$ for some $d \in a \circ b$. Then $\mu(d) + 1 - r = 1$ for some $d \in a \circ b$. Therefore $d \notin Q(\mu; 1 - r)$ for some $d \in a \circ b$, i.e., $a \circ b \nsubseteq Q(\mu; 1 - r)$, which is a contradiction. Hence $\max\left\{\inf_{z \in x \circ y} \mu(z), 0.5\right\} \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in H$. By Theorem 3.9, we conclude that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy hyper K-subalgebra of H. □

Corollary 3.16. *Let H be a finite hyper K-algebra. For a fuzzy set μ in H, the following are equivalent*:

(1) *μ is an $(\overline{\in},\ \overline{\in}\vee\overline{q})$-fuzzy hyper K-subalgebra of H.*
(2) $(\forall t \in (0, 0.5])$ $(Q(\mu; t) \neq \emptyset \ \Rightarrow\ Q(\mu; t)$ *is a hyper K-subalgebra of H*).

Proof. Straightforward. □

Theorem 3.17 ([23]). *If μ is an $(\in,\ \in\vee q)$-fuzzy hyper K-subalgebra of a hyper K-algebra H, then the set $Q(\mu; t)$ is a hyper K-subalgebra of H for all $t \in (0.5, 1]$.*

Theorem 3.18. *Let H satisfy $|x \circ y| < \infty$ for all $x, y \in H$. Let μ be a fuzzy set in H such that the set $Q(\mu; t)$ is nonempty and is a hyper K-subalgebra of H for all $t \in (0.5, 1]$. Then μ is an $(\in, \in\vee q)$-fuzzy hyper K-subalgebra of H.*

Proof. Assume $\inf\limits_{c\in a\circ b} \mu(c) < \min\{\mu(a), \mu(b), 0.5\}$ for some $a, b \in H$. Then

$$1 < \mu(a) + 1 - \inf_{c\in a\circ b} \mu(c) \text{ and } 1 < \mu(b) + 1 - \inf_{c\in a\circ b} \mu(c),$$

i.e., $a, b \in Q\left(\mu; 1 - \inf\limits_{c\in a\circ b} \mu(c)\right)$. Since $|x \circ y| < \infty$ for all $x, y \in H$, we get $\inf\limits_{c\in a\circ b} \mu(c) = \mu(d)$ for some $d \in a \circ b$. Then $\mu(d) + 1 - \inf\limits_{c\in a\circ b} \mu(c) = 1$ for some $d \in a \circ b$, and so $d \notin Q\left(\mu; 1 - \inf\limits_{c\in a\circ b} \mu(c)\right)$. Thus $a \circ b \nsubseteq Q\left(\mu; 1 - \inf\limits_{c\in a\circ b} \mu(c)\right)$, which is a contradiction. Hence $\inf\limits_{z\in x\circ y} \mu(z) \geq \min\{\mu(x), \mu(y), 0.5\}$ for all $x, y \in H$. By Lemma 3.3, we conclude that μ is an $(\in, \in\vee q)$-fuzzy hyper K-subalgebra of H. □

Corollary 3.19. *Let H be a finite hyper K-algebra. For a fuzzy set μ in H, the following are equivalent*:

(1) *μ is an $(\in, \in\vee q)$-fuzzy hyper K-subalgebra of H.*
(2) $(\forall t \in (0.5, 1])$ $(Q(\mu; t) \neq \emptyset \ \Rightarrow\ Q(\mu; t)$ *is a hyper K-subalgebra of H*).

Proof. Straightforward. □

For a fuzzy set μ in H, we consider the following set:

$$\Gamma := \{t \in (0, 1] \mid U(\mu; t) \neq \emptyset \ \Rightarrow\ U(\mu; t) \text{ is a hyper } K\text{-subalgebra of } H\}.$$

Then

(1) If $\Gamma = (0, 1]$, then μ is a fuzzy hyper K-subalgebra of H.
(2) If $\Gamma = (0, 0.5]$, then μ is an $(\in, \in\vee q)$-fuzzy hyper K-subalgebra of H.
(3) If $\Gamma = (0.5, 1]$, then μ is an $(\overline{\in},\ \overline{\in}\vee\overline{q})$-fuzzy hyper K-subalgebra of H.

Now we have the following question:

Question. If $\Gamma = (t, s)$ where $t < s$ in $(0, 1]$, then what kind of a fuzzy hyper K-subalgebra is μ? and what is the relation between them?

To discuss this question, we introduce the following definition.

Definition 3.20. Let $t, s \in (0,1]$ and $t < s$. A fuzzy set μ in H is called a *fuzzy hyper K-subalgebra with thresholds* (t, s) of H if it satisfies:

$$(3.6) \qquad (\forall x, y \in H) \left(\max\left\{ \inf_{z \in x \circ y} \mu(z), t \right\} \geq \min\{\mu(x), \mu(y), s\} \right).$$

Example 3.21. Consider a hyper K-algebra $H = \{0, a, b, x, y\}$ which is given in Example 3.7. Let μ be a fuzzy set in H defined by

$$\mu = \begin{pmatrix} 0 & a & b & x & y \\ 0.6 & 0.85 & 0.98 & 0.5 & 0.2 \end{pmatrix}.$$

Then μ is a fuzzy hyper K-subalgebra with thresholds $(0.2, 0.6)$ of H. But μ is a not a fuzzy hyper K-subalgebra with thresholds $(0.2, 0.7)$ of H since

$$\max\left\{ \inf_{z \in a \circ a} \mu(z), 0.2 \right\} = 0.6 < 0.7 = \min\{\mu(a), \mu(a), 0.7\}.$$

Example 3.22. Let $H = \{0, 1, 2, 3\}$ be the hyper K-algebra which is described in Example 3.5. Define a fuzzy set μ in H as follows:

$$\mu : H \to [0,1], \; h \mapsto \begin{cases} 0.9 & \text{if } h = 0, \\ 0.1 & \text{if } h = 1, \\ 0.3 & \text{if } h = 2, \\ 0.7 & \text{if } h = 3. \end{cases}$$

By routine calculations, we know that μ is a fuzzy hyper K-subalgebra with thresholds $(0.4, 0.6)$ of H. But μ is not a fuzzy hyper K-subalgebra with thresholds $(0.2, 0.6)$ of H since $\max\left\{ \inf_{z \in 3 \circ 2} \mu(z), 0.2 \right\} = \max\{\mu(1), 0.2\} = 0.2$ $\ngeq 0.3 = \min\{\mu(3), \mu(2), 0.6\}$.

Theorem 3.23. *Let $t, s \in (0.5, 1]$, $t < s$. If μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy hyper K-subalgebra of H, then μ is a fuzzy hyper K-subalgebra with thresholds (t, s) of H.*

Proof. Assume that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy hyper K-subalgebra of H. Then $\max\left\{ \inf_{z \in x \circ y} \mu(z), 0.5 \right\} \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in H$. Then

$$\begin{aligned} \max\left\{ \inf_{z \in x \circ y} \mu(z), t \right\} &\geq \max\left\{ \inf_{z \in x \circ y} \mu(z), 0.5 \right\} \\ &\geq \min\{\mu(x), \mu(y)\} \geq \min\{\mu(x), \mu(y), s\} \end{aligned}$$

for all $x, y \in H$. Hence μ is a fuzzy hyper K-subalgebra with thresholds (t, s). □

Theorem 3.24. *Let $t, s \in (0, 0.5]$, $t < s$. If μ is an $(\in, \in \vee q)$-fuzzy hyper K-subalgebra of H, then μ is a fuzzy hyper K-subalgebra with thresholds (t, s) of H.*

Proof. Assume that μ is an $(\in, \in\vee q)$-fuzzy hyper K-subalgebra of H. Then $\inf_{z\in x\circ y}\mu(z) \geq \min\{\mu(x), \mu(y), 0.5\}$ for all $x, y \in H$. Then

$$\begin{aligned}\max\left\{\inf_{z\in x\circ y}\mu(z), t\right\} &\geq \inf_{z\in x\circ y}\mu(z)\\ &\geq \min\{\mu(x), \mu(y), 0.5\} \geq \min\{\mu(x), \mu(y), s\}\end{aligned}$$

for all $x, y \in H$. Hence μ is a fuzzy hyper K-subalgebra with thresholds (t, s). □

Theorem 3.25. *Let μ be a fuzzy hyper K-subalgebra.*

(1) *If $t < s_1 < s_2$ in $(0, 1]$, then every fuzzy hyper K-subalgebra with thresholds (t, s_1) is a fuzzy hyper K-subalgebra with thresholds (t, s_2).*
(2) *If $t_1 < t_2 < s$ in $(0, 1]$, then every fuzzy hyper K-subalgebra with thresholds (t_1, s) is a fuzzy hyper K-subalgebra with thresholds (t_2, s).*
(3) *If $t_1 < t_2 < s_1 < s_2$ in $(0, 1]$, then every fuzzy hyper K-subalgebra with thresholds (t_1, s_2) is a fuzzy hyper K-subalgebra with thresholds (t_2, s_1).*

Proof. Straightforward. □

Examples 3.21 and 3.22 show that the converse of Theorem 3.25 may not be true.

We provide a characterization of a fuzzy hyper K-subalgebra with thresholds.

Theorem 3.26. *Let $t, s \in (0, 1]$ and $t < s$. For a fuzzy set μ in H, the following assertions are equivalent:*

(1) *μ is a fuzzy hyper K-subalgebra with thresholds (t, s) of H.*
(2) *$(\forall r \in (t, s])\ (U(\mu; r) \neq \emptyset \Rightarrow U(\mu; r)$ is a hyper K-subalgebra of $H)$.*

Proof. Assume that μ is a fuzzy hyper K-subalgebra with thresholds (t, s) of H. Let $r \in (t, s]$ be such that $U(\mu; r) \neq \emptyset$. If $x, y \in U(\mu; r)$, then $\mu(x) \geq r$ and $\mu(y) \geq r$. It follows from (3.6) that

$$\max\left\{\inf_{z\in x\circ y}\mu(z), t\right\} \geq \min\{\mu(x), \mu(y), s\} \geq \min\{r, s\} = r.$$

Since $r > t$, it follows that $\mu(w) \geq \inf_{z\in x\circ y}\mu(z) \geq r$ for all $w \in x \circ y$ so that $w \in U(\mu; r)$. Hence $x \circ y \subseteq U(\mu; r)$, and therefore $U(\mu; r)$ is a hyper K-subalgebra of H for all $r \in (t, s]$.

Conversely, let μ be a fuzzy set in H such that $U(\mu; r) \neq \emptyset$ is a hyper K-subalgebra of H for all $r \in (t, s]$. If there exist $a, b \in H$ such that

$$\max\left\{\inf_{z\in a\circ b}\mu(z), t\right\} < \min\{\mu(a), \mu(b), s\} = k,$$

then $k \in (t, s]$, $a, b \in U(\mu; k)$ and $\inf_{z\in a\circ b}\mu(z) < k$. The last inequality implies that $\mu(c) < k$ for some $c \in a \circ b$, and so $c \notin U(\mu; k)$. Thus $a \circ b \nsubseteq U(\mu; k)$, a contradiction. Consequently,

$$\max\Big\{\inf_{z\in x\circ y}\mu(z),t\Big\}\geq\min\{\mu(x),\mu(y),s\}$$

for all $x,y\in H$. Therefore μ is a fuzzy hyper K-subalgebra with thresholds (t,s) of H. □

Note that every fuzzy hyper K-subalgebra is a fuzzy hyper K-subalgebra with some thresholds, but the converse may not be true as seen in the following example.

Example 3.27. Let $H=\{0,1,2,3\}$ be the hyper K-algebra which is described in Example 3.5. Define a fuzzy set μ in H as follows:

$$\mu:H\to[0,1],\ h\mapsto\begin{cases}0.9 & \text{if } h=0,\\ 0.2 & \text{if } h=1,\\ 0.1 & \text{if } h=2,\\ 0.3 & \text{if } h=3.\end{cases}$$

By routine calculations, we know that μ is a fuzzy hyper K-subalgebra with thresholds $(0.4,0.6)$ of H. But μ is not a fuzzy hyper K-subalgebra of H since $\inf\limits_{z\in 1\circ 3}\mu(z)=\mu(2)=0.1\ngeq 0.2=\min\{\mu(1),\mu(3)\}$.

The following examples show that there exist $t,s\in(0,1]$ with $t<s$ such that μ is a fuzzy hyper K-subalgebra with thresholds (t,s) which is neither an $(\in, \in\vee q)$-fuzzy hyper K-subalgebra nor an $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy hyper K-subalgebra.

Example 3.28. Consider the hyper K-algebra $H=\{0,a,b,x,y\}$ which is given in Example 3.7. Define a fuzzy set μ in H as follows:

$$\mu:H\to[0,1],\ h\mapsto\begin{cases}0.4 & \text{if } h=0,\\ 0.85 & \text{if } h=a,\\ 0.98 & \text{if } h=b,\\ 0.5 & \text{if } h=x,\\ 0.2 & \text{if } h=y.\end{cases}$$

By routine calculations, we know that μ is a fuzzy hyper K-subalgebra with thresholds $(0.2,0.4)$ of H. But μ is not an $(\in, \in\vee q)$-fuzzy hyper K-subalgebra of H since

$$\inf_{z\in a\circ a}\mu(z)=\mu(0)=0.4\ngeq 0.5=\min\{\mu(a),\mu(a),0.5\}.$$

Also μ is not a $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy hyper K-subalgebra of H since

$$\max\left\{\inf_{z\in a\circ a}\mu(z),0.5\right\}=\max\{\mu(0),0.5\}=0.5\ngeq 0.85=\min\{\mu(a),\mu(a)\}.$$

Example 3.29. Consider the hyper K-algebra $H=\{0,1,2,3\}$ which is given in Example 3.5.

(1) Define a fuzzy set μ in H as follows:

$$\mu : H \to [0,1],\ h \mapsto \begin{cases} 0.9 & \text{if } h = 0, \\ 0.8 & \text{if } h = 1, \\ 0.1 & \text{if } h = 2, \\ 0.6 & \text{if } h = 3. \end{cases}$$

By routine calculations, we know that μ is a fuzzy hyper K-subalgebra with thresholds $(0.7, 0.85)$ of H. But μ is not an $(\in, \in\vee q)$-fuzzy hyper K-subalgebra since

$$\inf_{z\in 1\circ 3} \mu(z) = \mu(2) = 0.1 \ngeq 0.5 = \min\{\mu(1), \mu(3), 0.5\}.$$

Also μ is not an $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy hyper K-subalgebra since

$$\max\left\{\inf_{z\in 1\circ 3} \mu(z), 0.5\right\} = \max\{\mu(2), 0.5\} = 0.5 \ngeq 0.6 = \min\{\mu(1), \mu(3)\}.$$

(2) Define a fuzzy set μ in H as follows:

$$\mu : H \to [0,1],\ h \mapsto \begin{cases} 0.6 & \text{if } h = 0, \\ 0.8 & \text{if } h = 1, \\ 0.3 & \text{if } h = 2, \\ 0.1 & \text{if } h = 3. \end{cases}$$

By routine calculations, we know that μ is a fuzzy hyper K-subalgebra with thresholds $(0.4, 0.6)$ of H. But μ is not an $(\in, \in\vee q)$-fuzzy hyper K-subalgebra since

$$\inf_{z\in 1\circ 2} \mu(z) = \mu(3) = 0.1 \ngeq 0.3 = \min\{\mu(1), \mu(2), 0.5\}.$$

Also μ is not an $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy hyper K-subalgebra since

$$\max\left\{\inf_{z\in 1\circ 1} \mu(z), 0.5\right\} = \max\{\mu(0), 0.5\} = 0.6 \ngeq 0.8 = \min\{\mu(1), \mu(1)\}.$$

Theorem 3.30. *Let μ be a fuzzy set in H and $t, s \in (0,1]$ with $t < s$. Then*

(1) *μ is a fuzzy hyper K-subalgebra of H if and only if μ is a fuzzy hyper K-subalgebra of H with thresholds $(0,1)$.*
(2) *μ is an $(\in, \in\vee q)$-fuzzy hyper K-subalgebra of H if and only if μ is a fuzzy hyper K-subalgebra of H with thresholds $(0, 0.5)$.*
(3) *μ is an $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy hyper K-subalgebra of H if and only if μ is a fuzzy hyper K-subalgebra of H with thresholds $(0.5, 1)$.*

Proof. Straightforward. □

Theorem 3.31. *Let $t, s \in (0,1]$ and $t < s$. If μ is a fuzzy hyper K-subalgebra with thresholds (t,s) of H, then $Q(\mu; r)(\neq \emptyset)$ is a hyper K-subalgebra of H for all $r \in (1-s, 1-t]$.*

Proof. Assume that μ is a fuzzy hyper K-subalgebra with thresholds (t, s) of H, and $Q(\mu; r) \neq \emptyset$ for all $r \in (1-s, 1-t]$. Let $x, y \in Q(\mu; r)$. Then $x_r\, q\, \mu$ and $y_r\, q\, \mu$, i.e., $\mu(x) > 1-r$ and $\mu(y) > 1-r$. Since μ is a fuzzy hyper K-subalgebra with thresholds (t, s) of H and $s > 1-r$,

$$\max\left\{\inf_{z \in x \circ y} \mu(z), t\right\} \geq \min\{\mu(x), \mu(y), s\} > 1-r.$$

Since $1-r \geq t$, $\inf_{z \in x \circ y} \mu(z) > 1-r$. Then $\mu(z) > 1-r$ for all $z \in x \circ y$, i.e., $z \in Q(\mu; r)$ for all $z \in x \circ y$. Then $x \circ y \subseteq Q(\mu; r)$. Hence $Q(\mu; r)$ is a hyper K-subalgebra of H for all $r \in (1-s, 1-t]$. □

Theorem 3.32. *Let $t, s \in (0, 1]$ and $t < s$. Let $\mid x \circ y \mid < \infty$ for all $x, y \in H$. If $Q(\mu; r)(\neq \emptyset)$ is a hyper K-subalgebra of H for all $r \in (1-s, 1-t]$, then μ is a fuzzy hyper K-subalgebra with thresholds (t, s) of H.*

Proof. Assume that $\max\left\{\inf_{c \in a \circ b} \mu(c), t\right\} < \min\{\mu(a), \mu(b), s\}$ for some $a, b \in H$. Then

$$\max\left\{\inf_{c \in a \circ b} \mu(c), t\right\} \leq k < \min\{\mu(a), \mu(b), s\}$$

for some $k \in [t, s)$. Then $1 < \mu(a) + 1 - k$ and $1 < \mu(b) + 1 - k$ (i.e., $a, b \in Q(\mu; 1-k)$) and $\inf_{c \in a \circ b} \mu(c) \leq k$. Since $\mid x \circ y \mid < \infty$ for all $x, y \in H$, there exists $w \in a \circ b$ such that $\mu(w) \leq k$. Then $\mu(w) + 1 - k \leq 1$ for some $w \in a \circ b$, i.e., $w \notin Q(\mu; 1-k)$ for some $w \in a \circ b$. Then $a \circ b \nsubseteq Q(\mu; 1-k)$, which is a contradiction. Thus

$$\max\left\{\inf_{z \in x \circ y} \mu(z), t\right\} \geq \min\{\mu(x), \mu(y), s\}$$

for all $x, y \in H$. Hence μ is a fuzzy hyper K-subalgebra with thresholds (t, s) of H. □

Corollary 3.33. *Let H be a finite hyper K-algebra. Let $t, s \in (0, 1]$ and $t < s$. For a fuzzy set μ in H, the following are equivalent*:

(1) *μ is a fuzzy hyper K-subalgebra with thresholds (t, s) of H.*
(2) $(\forall r \in (1-s, 1-t])$ *$(Q(\mu; t)(\neq \emptyset)$ is a hyper K-subalgebra of $H)$.*

Proof. Straightforward. □

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