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AN EXPLICIT FORMULA FOR THE NUMBER OF SUBGROUPS OF A FINITE ABELIAN p-GROUP UP TO RANK 3

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Abstract. In this paper we give an explicit formula for the total number of subgroups of a finite abelian p-group up to rank three.

1. Introduction

Given a finite abelian group what is the total number of subgroups? This problem can be reduced to that of finding the number of subgroups of a finite abelian p-group because every finite abelian group is the direct product of its Sylow subgroups. Several different versions of the formula for the number of certain type subgroups of a given finite abelian p-group have been known (for example see [2, 3, 4, 6]). But in general these formulas do not lead us to an explicit formula for the total number of subgroups, which is well explained in [1]. As a result of this direction, G. Călugăreanu [1] and later J. Petrillo [5] have given an explicit formula for the total number of subgroups of a finite abelian p-group of rank two by using Goursat's Theorem. In this paper we reprove their result by finding its recurrence relation and as a new result we give an explicit formula for the total number of subgroups of a finite abelian p-group of rank three by a similar method.

2. The total number of subgroups of a finite abelian p-group up to rank 3

The following is the main result of this paper, which will be proved in the next section.

Theorem 2.1. *Let*

$$\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell} = \left\langle a, b, c \mid a^{p^m} = b^{p^n} = c^{p^\ell} = e, [a,b] = [a,c] = [b,c] = e \right\rangle$$

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be an abelian p-group of order $p^{m+n+\ell}$ where m, n and ℓ are non-negative integers such that $m \geq n \geq \ell$ and p is a prime number. Then the total number $S(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell})$ of subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$ is

$$\begin{aligned} S(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}) = & \sum_{t=1}^{\ell} t \Big[(m+n+\ell-3t+4) p^{2(t-1)} \\ & \qquad + (m+n+\ell-3t+2) p^{2(t-1)+1} \Big] \\ & + \sum_{k=\ell}^{n} (\ell+1)(m+n+1-2k) p^{k+\ell}, \end{aligned} \tag{1}$$

where the first iterated sum is 0 when $\ell = 0$.

We now evaluate Eq. (1) more specifically. Since

$$\begin{aligned} (p^2-1)\sum_{t=1}^{\ell} tp^{2t} &= p^2 \sum_{t=1}^{\ell} tp^{2t} - \sum_{t=1}^{\ell} tp^{2t} \\ &= \left(\ell p^{2\ell+2} + p^2 \sum_{t=1}^{\ell-1} tp^{2t} \right) - \left(p^2 + p^2 \sum_{t=1}^{\ell-1} (t+1) p^{2t} \right) \\ &= \ell p^{2\ell+2} - p^2 \sum_{t=0}^{\ell-1} p^{2t} = \ell p^{2\ell+2} - p^2 \frac{p^{2\ell}-1}{p^2-1}, \end{aligned}$$

we have

$$\sum_{t=1}^{\ell} tp^{2t} = \frac{\ell p^{2\ell+2}}{p^2-1} - p^2 \frac{p^{2\ell}-1}{(p^2-1)^2}. \tag{2}$$

Since

$$\begin{aligned} & (p^2-1)\sum_{t=1}^{\ell} t^2 p^{2t} \\ = {} & p^2 \sum_{t=1}^{\ell} t^2 p^{2t} - \sum_{t=1}^{\ell} t^2 p^{2t} \\ = {} & \left(\ell^2 p^{2\ell+2} + p^2 \sum_{t=1}^{\ell-1} t^2 p^{2t} \right) - \left(p^2 + p^2 \sum_{t=1}^{\ell-1} (t+1)^2 p^{2t} \right) \\ = {} & \ell^2 p^{2\ell+2} - p^2 + p^2 \sum_{t=1}^{\ell-1} (-2t-1) p^{2t} \\ = {} & \ell^2 p^{2\ell+2} - p^2 \sum_{t=0}^{\ell-1} p^{2t} - 2p^2 \sum_{t=1}^{\ell-1} tp^{2t} \end{aligned}$$

$$= \ell^2 p^{2\ell+2} - p^2\frac{p^{2\ell}-1}{p^2-1} - 2p^2\left[\frac{(\ell-1)p^{2\ell}}{p^2-1} - p^2\frac{p^{2\ell-2}-1}{(p^2-1)^2}\right] \text{ by Eq. (2),}$$

we have

$$\sum_{t=1}^{\ell} t^2 p^{2t} = \frac{\ell^2 p^{2\ell+2}}{p^2-1} - p^2\frac{p^{2\ell}-1}{(p^2-1)^2} - 2p^2\left[\frac{(\ell-1)p^{2\ell}}{(p^2-1)^2} - p^2\frac{p^{2\ell-2}-1}{(p^2-1)^3}\right]. \tag{3}$$

Since

$$\begin{aligned}(p-1)\sum_{k=1}^{n} kp^k &= p\sum_{k=1}^{n} kp^k - \sum_{k=1}^{n} kp^k \\ &= \left(np^{n+1} + p\sum_{k=1}^{n-1} kp^k\right) - \left(p + p\sum_{k=1}^{n-1}(k+1)p^k\right) \\ &= np^{n+1} - p\sum_{k=0}^{n-1} p^k = np^{n+1} - p\frac{p^n-1}{p-1},\end{aligned}$$

we have

$$\sum_{k=1}^{n} kp^k = \frac{np^{n+1}}{p-1} - p\frac{p^n-1}{(p-1)^2}. \tag{4}$$

Using Eqs. (2)-(4) we get that

$$\begin{aligned}&\sum_{t=1}^{\ell} t\left[(m+n+\ell-3t+4)p^{2(t-1)} + (m+n+\ell-3t+2)p^{2(t-1)+1}\right] \\ &+ \sum_{k=\ell}^{n}(\ell+1)(m+n+1-2k)p^{k+\ell} \\ =\ &\sum_{t=1}^{\ell} t\left[(m+n+\ell-3t+4)p^{2(t-1)} + (m+n+\ell-3t+2)p^{2(t-1)+1}\right] \\ &+ \sum_{k=0}^{n}(\ell+1)(m+n+1-2k)p^{k+\ell} - \sum_{k=0}^{\ell-1}(\ell+1)(m+n+1-2k)p^{k+\ell} \\ =\ &\left[\frac{m+n+\ell+4}{p^2} + \frac{m+n+\ell+2}{p}\right]\sum_{t=1}^{\ell} tp^{2t} - 3\left[\frac{1}{p^2}+\frac{1}{p}\right]\sum_{t=1}^{\ell} t^2p^{2t} \\ &+ (\ell+1)(m+n+1)p^\ell\left[\sum_{k=0}^{n} p^k - \sum_{k=0}^{\ell-1} p^k\right] - 2(\ell+1)p^\ell\left[\sum_{k=0}^{n} kp^k - \sum_{k=0}^{\ell-1} kp^k\right] \\ =\ &\frac{(m+n-2\ell+4)\ell p^{2\ell} + (m+n-2\ell+2)\ell p^{2\ell+1}}{p^2-1} \\ &- \frac{(m+n-5\ell+7)p^{2\ell} + (m+n-5\ell+5)p^{2\ell+1} - (m+n+\ell-1)p - m - n - \ell - 1}{(p^2-1)^2}\end{aligned}$$

$$- \frac{6(p^{2\ell}+p^{2\ell+1}-p^3-p^2)}{(p^2-1)^3} + \frac{(\ell+1)((m-n+1)p^{n+\ell+1}-(m+n-2\ell+3)p^{2\ell})}{p-1} + \frac{2(\ell+1)(p^{n+\ell+1}-p^{2\ell})}{(p-1)^2}.$$

Therefore, we have proved the following.

Corollary 2.2. *Let*

$$\mathbb{Z}_{p^m}\times\mathbb{Z}_{p^n}\times\mathbb{Z}_{p^\ell} = \left\langle a,b,c \mid a^{p^m}=b^{p^n}=c^{p^\ell}=e, [a,b]=[a,c]=[b,c]=e\right\rangle$$

be an abelian p-group of order $p^{m+n+\ell}$ where m, n and ℓ are non-negative integers such that $m \geq n \geq \ell$ and p is a prime number. Then the total number $S(\mathbb{Z}_{p^m}\times\mathbb{Z}_{p^n}\times\mathbb{Z}_{p^\ell})$ of subgroups of $\mathbb{Z}_{p^m}\times\mathbb{Z}_{p^n}\times\mathbb{Z}_{p^\ell}$ is

$$\begin{aligned} S(\mathbb{Z}_{p^m}\times\mathbb{Z}_{p^n}\times\mathbb{Z}_{p^\ell}) &= \frac{(m+n-2\ell+4)\ell p^{2\ell}+(m+n-2\ell+2)\ell p^{2\ell+1}}{p^2-1} \\ &- \frac{(m+n-5\ell+7)p^{2\ell}+(m+n-5\ell+5)p^{2\ell+1}-(m+n+\ell-1)p-m-n-\ell-1}{(p^2-1)^2} \\ &- \frac{6(p^{2\ell}+p^{2\ell+1}-p^3-p^2)}{(p^2-1)^3} + \frac{(\ell+1)((m-n+1)p^{n+\ell+1}-(m+n-2\ell+3)p^{2\ell})}{p-1} \\ &+ \frac{2(\ell+1)(p^{n+\ell+1}-p^{2\ell})}{(p-1)^2}. \end{aligned}$$

3. The proof of Theorem 2.1

Given a finite group G let $\mathcal{S}(G)$ and $\mathcal{T}(G)$ be the set of subgroups of G and the set of proper subgroups of G, respectively. Let $S(G) := |\mathcal{S}(G)|$ and $T(G) := |\mathcal{T}(G)|$.

Throughout the section we assume that

$$\mathbb{Z}_{p^m}\times\mathbb{Z}_{p^n}\times\mathbb{Z}_{p^\ell} = \left\langle a,b,c \mid a^{p^m}=b^{p^n}=c^{p^\ell}=e, [a,b]=[a,c]=[b,c]=e\right\rangle$$

is an abelian p-group of order $p^{m+n+\ell}$ where m, n and ℓ are non-negative integers such that $m \geq n \geq \ell$ and p is a prime number. Let

$$b_{m,n,\ell} := S(\mathbb{Z}_{p^m}\times\mathbb{Z}_{p^n}\times\mathbb{Z}_{p^\ell}),$$

where $m \geq n \geq \ell$. For convenience of notation we set $b_{m,n} := b_{m,n,0}$ and $b_m := b_{m,0,0}$.

Clearly $b_m = S(\mathbb{Z}_{p^m}) = m+1$. In the following we consider the case for rank 2 and 3 separately.

3.1. The number of subgroups of $\mathbb{Z}_{p^m}\times\mathbb{Z}_{p^n}$

In elementary group theory, the following is well-known.

Lemma 3.1. *Assume that m and n are positive integers. The group $\mathbb{Z}_{p^m}\times\mathbb{Z}_{p^n}$ has $(p+1)$ index p subgroups $\langle a^p, b\rangle \cong \mathbb{Z}_{p^{m-1}}\times\mathbb{Z}_{p^n}$, $\left\langle b^p, a^i b\right\rangle \cong \mathbb{Z}_{p^m}\times\mathbb{Z}_{p^{n-1}}$, $i=1,2,\ldots,p-1$, and $\langle a, b^p\rangle \cong \mathbb{Z}_{p^m}\times\mathbb{Z}_{p^{n-1}}$.*

Lemma 3.2. (1) *If $m > n$, then*

$$b_{m,n} = b_{m-1,n} + p(b_{m,n-1} - b_{m-1,n-1}) + 1. \tag{5}$$

(2) *If $m = n$, then*

$$b_{m,m} = (p+1)b_{m,m-1} - pb_{m-1,m-1} + 1. \tag{6}$$

Proof. We only give the proof when $m > n$. The remaining can be proved similarly.

By Lemma 3.1 we have

$$\begin{aligned}\mathcal{T}(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}) = \mathcal{S}(\langle a^p, b\rangle \cong \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_{p^n}) &\bigcup_{i=1}^{p-1} \mathcal{S}(\langle b^p, a^i b\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}}) \\ &\bigcup \mathcal{S}(\langle a, b^p\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}}).\end{aligned}$$

Using the inclusion-exclusion principle we have

$$\begin{aligned} b_{m,n} - 1 &= b_{m-1,n} + pb_{m,n-1} - \binom{p+1}{2} b_{m-1,n-1} + \binom{p+1}{3} b_{m-1,n-1} \\ &\quad + \cdots + (-1)^{p+2}\binom{p+1}{p+1} b_{m-1,n-1} \\ &= b_{m-1,n} + pb_{m,n-1} - pb_{m-1,n-1}. \end{aligned}$$

Thus

$$b_{m,n} = b_{m-1,n} + p(b_{m,n-1} - b_{m-1,n-1}) + 1. \qquad \square$$

As commented in Introduction, the following is already proved in [1, 5]. We reprove it by using Lemma 3.2.

Lemma 3.3.

$$b_{m,n} = \sum_{k=0}^{n} (m + n + 1 - 2k)p^k. \tag{7}$$

Proof. We prove Eq. (7) by induction on n. Assume first that $n = 1$. Since $b_{m,0} = S(\mathbb{Z}_{p^m}) = m+1$ and $b_{0,0} = S(\langle e\rangle) = 1$, Eq. (5) with $n = 1$ gives us that

$$b_{m,1} = b_{m-1,1} + p + 1.$$

Thus

$$b_{m,1} = b_{1,1} + (m-1)(p+1).$$

Since $b_{1,1} = p + 3$ by Eq. (6) with $m = 1$, we have

$$b_{m,1} = p + 3 + (m-1)(p+1).$$

Hence Eq. (7) holds for $n = 1$.

Assume now that Eq. (7) holds from 1 to n and consider the case for $n+1$. By Eq. (5) replacing n by $n + 1$ we have

$$b_{m,n+1} = b_{m-1,n+1} + p(b_{m,n} - b_{m-1,n}) + 1.$$

Since

$$p(b_{m,n}-b_{m-1,n}) = p\left[\sum_{k=0}^{n}(m+n+1-2k)p^k - \sum_{k=0}^{n}(m-1+n+1-2k)p^k\right]$$
$$= \sum_{k=0}^{n} p^{k+1}$$

by induction hypothesis, we have

$$b_{m,n+1} = b_{m-1,n+1} + \sum_{k=0}^{n} p^{k+1} + 1 = b_{m-1,n+1} + \sum_{k=0}^{n+1} p^k,$$

which implies that

$$b_{m,n+1} = b_{n+1,n+1} + (m-n-1)\sum_{k=0}^{n+1} p^k.$$

Furthermore, since

$$b_{n+1,n+1} = (p+1)b_{n+1,n} - pb_{n,n} + 1$$
$$= (p+1)\sum_{k=0}^{n}(n+1+n+1-2k)p^k - p\sum_{k=0}^{n}(n+n+1-2k)p^k + 1$$
$$= \sum_{k=0}^{n+1}(2n+3-2k)p^k$$

by induction hypothesis, we have

$$b_{m,n+1} = \sum_{k=0}^{n+1}(2n+3-2k)p^k + (m-n-1)\sum_{k=0}^{n+1} p^k = \sum_{k=0}^{n+1}(m+n+2-2k)p^k.$$

Hence Eq. (7) holds for $n+1$. □

3.2. The number of subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$

Given a positive integer n let $\mathbb{Z}_n$ be the cyclic group of order n with the addictive operation. By $\mathbb{Z}_n^*$ we denote the multiplicative group, that is, the group consisting of all multiplicatively invertible elements of $\mathbb{Z}_n$.

In elementary group theory, the following is well-known.

Lemma 3.4. *Assume that m, n and ℓ are positive integers. The group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$ has (p^2+p+1) index p subgroups as follows.*

(1) $\langle a, b, c^p\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{\ell-1}}$,
(2) $\left\langle a^i b, b^p, c\right\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p^\ell}$; $i = 1, 2, \ldots, p-1$,
(3) $\left\langle a^i b, b^j c, c^p\right\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{\ell-1}}$; $i = 1, 2, \ldots, p-1$ *and* $j = 1, 2, \ldots,$ $p-1$,
(4) $\langle a^p, b, c\rangle \cong \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$,
(5) $\left\langle a^i c, b, c^p\right\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{\ell-1}}$; $i = 1, 2, \ldots, p-1$,

(6) $\langle a, b^p, c\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p^\ell}$,
(7) $\left\langle a, b^i c, c^p\right\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{\ell-1}}$; $i = 1, 2, \ldots, p-1$.

Note that every index p subgroup of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$ contains the subgroup $\langle a^p, b^p, c^p\rangle \cong \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p^{\ell-1}}$. In the next lemma we find all index p^2 subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$ containing $\langle a^p, b^p, c^p\rangle$.

Lemma 3.5. *Assume that m, n and ℓ are positive integers. There exist (p^2+p+1) index p^2 subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$ containing the subgroup $\langle a^p, b^p, c^p\rangle \cong \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p^{\ell-1}}$ as follows.*

(1) $\left\langle a^i b^j c, b^p, c^p\right\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p^{\ell-1}}$; $i = 1, 2, \ldots, p-1$ *and* $j = 0, 1, \ldots, p-1$,
(2) $\left\langle a^p, b^k c, c^p\right\rangle \cong \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{\ell-1}}$; $k = 1, 2, \ldots, p-1$,
(3) $\langle a^p, b^p, c\rangle \cong \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p^\ell}$,
(4) $\left\langle a^i b, b^p, c^p\right\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p^{\ell-1}}$; $i = 1, 2, \ldots, p-1$,
(5) $\langle a^p, b, c^p\rangle \cong \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{\ell-1}}$,
(6) $\langle a, b^p, c^p\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p^{\ell-1}}$.

Proof. Let K be an index p^2 subgroup of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$ containing the subgroup $\langle a^p, b^p, c^p\rangle$. Then we have $a^i b^j c^k \in K$ for some integers i, j and k such that $0 \le i, j, k \le p-1$ and $(i, j, k) \neq (0, 0, 0)$. We now divide the argument into two cases depending on $i = 0$ or not.

Case 1: $i \neq 0$. If $j = k = 0$, then $K \ge \left\langle a^p, b^p, c^p, a^i\right\rangle = \langle a, b^p, c^p\rangle$, and hence $K = \langle a, b^p, c^p\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p^{\ell-1}}$. If $j \neq 0$ and $k = 0$, then $K \ge \left\langle a^p, b^p, c^p, a^i b^j\right\rangle$. Since $j \in \mathbb{Z}^*_{p^n}$, there exists $j' \in \mathbb{Z}^*_{p^n}$ such that $jj' \equiv 1 \pmod{p^n}$. So $K \ge \left\langle a^p, b^p, c^p, a^i b^j\right\rangle = \left\langle a^p, b^p, c^p, (a^i b^j)^{j'} = a^{ij'} b\right\rangle$, and hence $K = \left\langle a^{ij'} b, b^p, c^p\right\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p^{\ell-1}}$. If $j = 0$ and $k \neq 0$, then $K \ge \left\langle a^p, b^p, c^p, a^i c^k\right\rangle$. Since $k \in \mathbb{Z}^*_{p^\ell}$, there exists $k' \in \mathbb{Z}^*_{p^\ell}$ such that $kk' \equiv 1 \pmod{p^\ell}$. So $K \ge \left\langle a^p, b^p, c^p, a^i c^k\right\rangle = \left\langle a^p, b^p, c^p, (a^i c^k)^{k'} = a^{ik'} c\right\rangle$, and hence $K = \left\langle a^{ik'} c, b^p, c^p\right\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p^{\ell-1}}$. If $j \neq 0$ and $k \neq 0$, then $K \ge \left\langle a^p, b^p, c^p, a^i b^j c^k\right\rangle$. Since $k \in \mathbb{Z}^*_{p^\ell}$, there exists $k' \in \mathbb{Z}^*_{p^\ell}$ such that $kk' \equiv 1 \pmod{p^\ell}$. So $K \ge \left\langle a^p, b^p, c^p, a^i b^j c^k\right\rangle = \left\langle a^p, b^p, c^p, (a^i b^j c^k)^{k'} = a^{ik'} b^{jk'} c\right\rangle$, and hence $K = \left\langle a^{ik'} b^{jk'} c, b^p, c^p\right\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p^{\ell-1}}$.

Case 2: $i = 0$. If $j = 0$, then $k \neq 0$ and $K \ge \left\langle a^p, b^p, c^p, c^k\right\rangle = \langle a^p, b^p, c\rangle$, and hence $K = \langle a^p, b^p, c\rangle \cong \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p^\ell}$. If $j \neq 0$ and $k = 0$, then $K \ge \left\langle a^p, b^p, c^p, b^j\right\rangle = \langle a^p, b, c^p\rangle$, and hence $K = \langle a^p, b, c^p\rangle \cong \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{\ell-1}}$. If $j \neq 0$ and $k \neq 0$, then $K \ge \left\langle a^p, b^p, c^p, b^j c^k\right\rangle$. Since $k \in \mathbb{Z}^*_{p^\ell}$, there exists $k' \in \mathbb{Z}^*_{p^\ell}$ such that $kk' \equiv 1 \pmod{p^\ell}$. So $K \ge \left\langle a^p, b^p, c^p, b^j c^k\right\rangle = \left\langle a^p, b^p, c^p, (b^j c^k)^{k'} = b^{jk'} c\right\rangle$, and hence $K = \left\langle a^p, b^{jk'} c, c^p\right\rangle \cong \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{\ell-1}}$. $\square$

Lemma 3.6. *If K is an index p^2 subgroup of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$ and K contains the subgroup $\langle a^p, b^p, c^p \rangle$, then there exist $(p+1)$ index p subgroup of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$ containing K.*

Proof. Assume that K is a subgroup satisfying the assumption. Then K is one of the (p^2+p+1) subgroups in Lemma 3.4. We only give the proof when $K = \left\langle a^i b^j c, b^p, c^p \right\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p^{\ell-1}}$ for each integers i and j such that $1 \le i \le p-1$ and $0 \le j \le p-1$. The remaining can be proved in a similar way. Set $x := a^i b^j c$, $y := b$ and $z := c$. Then $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell} = \langle x, y, z \rangle$ and $K = \langle x, y^p, z^p \rangle$. Let H be an index p subgroup of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$ containing K. Since $K = \langle x, y^p, z^p \rangle$ is an index p^2 subgroup of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}$, we have $y^{pk_1+i} z^{pk_2+j} \in H$ for some integers k_1, k_2, i and j such that $0 \le i, j \le p-1$ and $(i,j) \ne (0,0)$. If $i = 0$, then $1 \le j \le p-1$ and $H \ge \left\langle x, y^p, z^p, z^j \right\rangle = \langle x, y^p, z \rangle$, and hence $H = \langle x, y^p, z \rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p^\ell}$. Assume now that $i \ne 0$. Then $0 \le j \le p-1$ and $H \ge \left\langle x, y^p, z^p, y^i z^i \right\rangle$. If $j = 0$, then $H \ge \left\langle x, y^p, z^p, y^i \right\rangle = \langle x, y, z^p \rangle$, and hence $H = \langle x, y, z^p \rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{\ell-1}}$. If $j \ne 0$, then there exists $j' \in \mathbb{Z}^*_{p^\ell}$ such that $jj' \equiv 1 \pmod{p^\ell}$, and so $H \ge \left\langle x, y^p, z^p, y^{ij'} z \right\rangle = \left\langle x, y^{ij'} z, z^p \right\rangle$. Hence we have $H = \left\langle x, y^k z, z^p \right\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{\ell-1}}$, $k = 1, 2, \ldots, p-1$. □

Lemma 3.7. (1) *If $m > n > \ell$, then*

$$b_{m,n,\ell} = b_{m-1,n,\ell} + p(b_{m,n-1,\ell} - b_{m-1,n-1,\ell}) + p^2(b_{m,n,\ell-1} - b_{m-1,n,\ell-1}) - p^3(b_{m,n-1,\ell-1} - b_{m-1,n-1,\ell-1}) + 1. \tag{8}$$

(2) *If $m = n$ and $n > \ell$, then*

$$b_{m,m,\ell} = (1+p)b_{m,m-1,\ell} - pb_{m-1,m-1,\ell} + p^2 b_{m,m,\ell-1} - p^2(1+p)b_{m,m-1,\ell-1} + p^3 b_{m-1,m-1,\ell-1} + 1. \tag{9}$$

(3) *If $m > n$ and $n = \ell$, then*

$$b_{m,n,n} = b_{m-1,n,n} + p(1+p)(b_{m,n,n-1} - b_{m-1,n,n-1}) - p^3(b_{m,n-1,n-1} - b_{m-1,n-1,n-1}) + 1. \tag{10}$$

(4) *If $m = n = \ell$, then*

$$b_{m,m,m} = (1+p+p^2)b_{m,m,m-1} - p(1+p+p^2)b_{m,m-1,m-1} + p^3 b_{m-1,m-1,m-1} + 1. \tag{11}$$

Proof. We only give the proof of Eq. (8). The remaining can be proved by a similar way.

By Lemma 3.4 we have

$$\mathcal{T}(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell}) = \mathcal{S}(\langle a, b, c^p \rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{\ell-1}})$$

$$\bigcup_{i=1}^{p-1} \mathcal{S}(\langle a^i b, b^p, c\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p^\ell})$$
$$\bigcup_{1\le i,j\le p-1} \mathcal{S}(\langle a^i b, b^j c, c^p\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{\ell-1}})$$
$$\bigcup \mathcal{S}(\langle a^p, b, c\rangle \cong \mathbb{Z}_{p^{m-1}} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^\ell})$$
$$\bigcup_{i=1}^{p-1} \mathcal{S}(\langle a^i c, b, c^p\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{\ell-1}})$$
$$\bigcup \mathcal{S}(\langle a, b^p, c\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p^\ell})$$
$$\bigcup_{i=1}^{p-1} \mathcal{S}(\langle a, b^i c, c^p\rangle \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{\ell-1}}).$$

Using the inclusion-exclusion principle and Lemmas 3.5 and 3.6 we have

$$\begin{aligned}
&b_{m,n,\ell} - 1\\
&= p^2 b_{m,n,\ell-1} + p b_{m,n-1,\ell} + b_{m-1,n,\ell}\\
&\quad - \left[p^2\binom{p+1}{2} b_{m,n-1,\ell-1} + p\binom{p+1}{2} b_{m-1,n,\ell-1} + \binom{p+1}{2} b_{m-1,n-1,\ell}\right]\\
&\quad + \left[p^2\binom{p+1}{3} b_{m,n-1,\ell-1} + p\binom{p+1}{3} b_{m-1,n,\ell-1} + \binom{p+1}{3} b_{m-1,n-1,\ell}\right.\\
&\qquad \left. + \left[\binom{p^2+p+1}{3} - (p^2+p+1)\binom{p+1}{3}\right] b_{m-1,n-1,\ell-1}\right]\\
&\quad - \left[p^2\binom{p+1}{4} b_{m,n-1,\ell-1} + p\binom{p+1}{4} b_{m-1,n,\ell-1} + \binom{p+1}{4} b_{m-1,n-1,\ell}\right.\\
&\qquad \left. + \left[\binom{p^2+p+1}{4} - (p^2+p+1)\binom{p+1}{4}\right] b_{m-1,n-1,\ell-1}\right]\\
&\quad + \cdots + (-1)^{p+2}\left[p^2\binom{p+1}{p+1} b_{m,n-1,\ell-1} + p\binom{p+1}{p+1} b_{m-1,n,\ell-1}\right.\\
&\quad \left. + \binom{p+1}{p+1} b_{m-1,n-1,\ell} + \left[\binom{p^2+p+1}{p+1} - (p^2+p+1)\binom{p+1}{p+1}\right] b_{m-1,n-1,\ell-1}\right]\\
&\quad + (-1)^{p+3}\binom{p^2+p+1}{p+2} b_{m-1,n-1,\ell-1} + (-1)^{p+4}\binom{p^2+p+1}{p+3} b_{m-1,n-1,\ell-1}\\
&\quad + \cdots + (-1)^{p^2+p+2}\binom{p^2+p+1}{p^2+p+1} b_{m-1,n-1,\ell-1}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
b_{m,n,\ell} &= b_{m-1,n,\ell} + p b_{m,n-1,\ell} + p^2 b_{m,n,\ell-1}\\
&\quad - p b_{m-1,n-1,\ell} - p^2 b_{m-1,n,\ell-1} - p^3 b_{m,n-1,\ell-1} + p^3 b_{m-1,n-1,\ell-1} + 1. \qquad \square
\end{aligned}$$

We prove Eq. (1) by double induction on n and ℓ. In the following lemma we show that Eq. (1) holds for $\ell = 1$.

Lemma 3.8.

$$b_{m,n,1} = (m+n+2) + (m+n)p + \sum_{k=1}^{n} 2(m+n+1-2k)p^{k+1}. \tag{12}$$

Proof. We prove Eq. (12) by induction on n. Assume first that $n = 1$. Note that

$$\begin{aligned} b_{m,1,1} &= b_{m-1,1,1} + p(1+p)(b_{m,1,0} - b_{m-1,1,0}) \\ &\quad - p^3(b_{m,0,0} - b_{m-1,0,0}) + 1 \qquad \text{by Eq. (10)} \\ &= b_{m-1,1,1} + (p+p^2)(p+1) - p^3 + 1 \quad \text{by Lemma 3.3} \\ &= b_{m-1,1,1} + 2p^2 + p + 1. \end{aligned}$$

Thus we have

$$b_{m,1,1} = b_{1,1,1} + (m-1)(2p^2+p+1).$$

Since

$$\begin{aligned} b_{1,1,1} &= (1+p+p^2)b_{1,1,0} - p(1+p+p^2)b_{1,0,0} + p^3 b_{0,0,0} + 1 \quad \text{by Eq. (11)} \\ &= (1+p+p^2)(p+3) - p(1+p+p^2)2 + p^3 + 1 \quad \text{by Lemma 3.3} \\ &= 4 + 2p + 2p^2, \end{aligned}$$

we have

$$b_{m,1,1} = 4 + 2p + 2p^2 + (m-1)(2p^2+p+1) = m + 3 + (m+1)p + 2mp^2.$$

Thus Eq. (12) holds for $n = 1$.

Assume now that Eq. (12) holds from 1 to n and consider the case for $n+1$. By Eq. (8) with $(m, n, \ell) = (m, n+1, 1)$ we have

$$\begin{aligned} b_{m,n+1,1} &= b_{m-1,n+1,1} + p(b_{m,n,1} - b_{m-1,n,1}) + p^2(b_{m,n+1,0} - b_{m-1,n+1,0}) \\ &\quad - p^3(b_{m,n,0} - b_{m-1,n,0}) + 1. \end{aligned}$$

Note that

$$\begin{aligned} b_{m,n,1} - b_{m-1,n,1} &= (m+n+2) + (m+n)p + \sum_{k=1}^{n}(2m+2n+2-4k)p^{k+1} \\ &\quad - \left[(m+n+1) + (m+n-1)p + \sum_{k=1}^{n}(2m+2n-4k)p^{k+1}\right] \\ &= 1 + p + 2\sum_{k=1}^{n} p^{k+1} \end{aligned}$$

by induction hypothesis and

$$b_{m,n+1,0} - b_{m-1,n+1,0} = \sum_{k=0}^{n+1} p^k, \quad b_{m,n,0} - b_{m-1,n,0} = \sum_{k=0}^{n} p^k$$

by Lemma 3.3. Thus

$$\begin{aligned} b_{m,n+1,1} - b_{m-1,n+1,1} &= 1 + p + p^2 + 2\sum_{k=1}^{n} p^{k+2} + \sum_{k=0}^{n+1} p^{k+2} - \sum_{k=0}^{n} p^{k+3} \\ &= 1 + p + 2p^2 + 2\sum_{k=1}^{n} p^{k+2}, \end{aligned}$$

which implies that

$$b_{m,n+1,1} = b_{n+1,n+1,1} + (m-n-1)\left[1 + p + 2p^2 + 2\sum_{k=1}^{n} p^{k+2}\right]. \tag{13}$$

On the other hand, Eq. (9) with $(m, \ell) = (n+1, 1)$ gives us that

$$\begin{aligned} b_{n+1,n+1,1} = {} & (1+p)b_{n+1,n,1} - pb_{n,n,1} + p^2 b_{n+1,n+1,0} \\ & - (p^3 + p^2)b_{n+1,n,0} + p^3 b_{n,n,0} + 1. \end{aligned}$$

Since

$$\begin{aligned} b_{n+1,n,1} &= 2n + 3 + (2n+1)p + \sum_{k=1}^{n}(4n + 4 - 4k)p^{k+1}, \\ b_{n,n,1} &= 2n + 2 + 2np + \sum_{k=1}^{n}(4n + 2 - 4k)p^{k+1} \end{aligned}$$

by induction hypothesis and

$$\begin{aligned} b_{n+1,n+1,0} &= \sum_{k=0}^{n+1}(2n + 3 - 2k)p^k, \quad b_{n+1,n,0} = \sum_{k=0}^{n}(2n + 2 - 2k)p^k, \\ b_{n,n,0} &= \sum_{k=0}^{n}(2n + 1 - 2k)p^k \end{aligned}$$

by Lemma 3.3, we have

$$\begin{aligned} b_{n+1,n+1,1} = {} & (1+p)\left[2n + 3 + (2n+1)p + \sum_{k=1}^{n}(4n + 4 - 4k)p^{k+1}\right] \\ & - p\left[2n + 2 + 2np + \sum_{k=1}^{n}(4n + 2 - 4k)p^{k+1}\right] \\ & + p^2\sum_{k=0}^{n+1}(2n + 3 - 2k)p^k \end{aligned}$$

$$
\begin{aligned}
&- (p^3+p^2)\sum_{k=0}^{n}(2n+2-2k)p^k + p^3\sum_{k=0}^{n}(2n+1-2k)p^k + 1 \\
&= 2n+4+(2n+2)p+\sum_{k=1}^{n+1}(4n+6-4k)p^{k+1}.
\end{aligned}
$$

Hence, together with Eq. (13) we have

$$
\begin{aligned}
b_{m,n+1,1} &= 2n+4+(2n+2)p+\sum_{k=1}^{n+1}(4n+6-4k)p^{k+1} \\
&\quad + (m-n-1)\left[1+p+2p^2+2\sum_{k=1}^{n}p^{k+2}\right] \\
&= (m+n+3)+(m+n+1)p+\sum_{k=1}^{n+1}(4n+6-4k)p^{k+1} \\
&\quad + (2m-2n-2)p^2 + 2(m-n-1)\sum_{k=1}^{n}p^{k+2} \\
&= (m+n+3)+(m+n+1)p+\sum_{k=1}^{n+1}2(m+n+2-2k)p^{k+1}.
\end{aligned}
$$

Therefore, Eq. (12) holds for $n+1$. $\square$

Assume now that Eq. (1) holds from 1 to ℓ and consider the case for $\ell+1$. Eq. (10) with $(m,n)=(m,\ell+1)$ gives us that

$$
\begin{aligned}
b_{m,\ell+1,\ell+1} = b_{m-1,\ell+1,\ell+1} + (p^2+p)(b_{m,\ell+1,\ell} - b_{m-1,\ell+1,\ell}) \\
- p^3(b_{m,\ell,\ell} - b_{m-1,\ell,\ell}) + 1.
\end{aligned}
$$

By induction hypothesis we know that

$$
\begin{aligned}
&b_{m,\ell+1,\ell} - b_{m-1,\ell+1,\ell} \\
= &\sum_{t=1}^{\ell} t\left[(m+2\ell-3t+5)p^{2t-2} + (m+2\ell-3t+3)p^{2t-1}\right] \\
&+ \sum_{k=\ell}^{\ell+1}(\ell+1)(m+\ell+2-2k)p^{k+\ell} - \sum_{t=1}^{\ell} t\left[(m+2\ell-3t+4)p^{2t-2}\right. \\
&\left.+(m+2\ell-3t+2)p^{2t-1}\right] - \sum_{k=\ell}^{\ell+1}(\ell+1)(m+\ell+1-2k)p^{k+\ell} \\
= &\sum_{t=1}^{\ell} t\left[p^{2t-2}+p^{2t-1}\right] + (\ell+1)p^{2\ell} + (\ell+1)p^{2\ell+1}
\end{aligned}
$$

and

$$\begin{aligned}
b_{m,\ell,\ell} - b_{m-1,\ell,\ell} = &\sum_{t=1}^{\ell} t\left[(m+2\ell-3t+4)p^{2t-2} + (m+2\ell-3t+2)p^{2t-1}\right] \\
&+ \sum_{k=\ell}^{\ell}(\ell+1)(m+\ell+1-2k)p^{k+\ell} \\
&- \sum_{t=1}^{\ell} t\left[(m+2\ell-3t+3)p^{2t-2} + (m+2\ell-3t+1)p^{2t-1}\right] \\
&- \sum_{k=\ell}^{\ell}(\ell+1)(m+\ell-2k)p^{k+\ell} \\
= &\sum_{t=1}^{\ell} t\left[p^{2t-2} + p^{2t-1}\right] + (\ell+1)p^{2\ell}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&b_{m,\ell+1,\ell+1} - b_{m-1,\ell+1,\ell+1} \\
&= (p^2+p)\left[\sum_{t=1}^{\ell} t\left[p^{2t-2}+p^{2t-1}\right] + (\ell+1)p^{2\ell} + (\ell+1)p^{2\ell+1}\right] \\
&\quad - p^3\left[\sum_{t=1}^{\ell} t\left[p^{2t-2}+p^{2t-1}\right] + (\ell+1)p^{2\ell}\right] + 1 \\
&= \sum_{t=1}^{\ell} t\left[p^{2t-1} + 2p^{2t} - p^{2t+2}\right] + 2(\ell+1)p^{2\ell+2} + (\ell+1)p^{2\ell+1} + 1 \\
&= \sum_{t=1}^{\ell+1} t\left[p^{2t-2}+p^{2t-1}\right] + (\ell+2)p^{2\ell+2},
\end{aligned}$$

which implies that

$$\begin{aligned}
b_{m,\ell+1,\ell+1} = {}& b_{\ell+1,\ell+1,\ell+1} \\
&+ (m-\ell-1)\left[\sum_{t=1}^{\ell+1} t\left[p^{2t-2}+p^{2t-1}\right] + (\ell+2)p^{2\ell+2}\right].
\end{aligned} \tag{14}$$

On the other hand, Eq. (11) with $m = \ell+1$ gives us that

$$b_{\ell+1,\ell+1,\ell+1} = (p^2+p+1)b_{\ell+1,\ell+1,\ell} - (p^3+p^2+p)b_{\ell+1,\ell,\ell} + p^3 b_{\ell,\ell,\ell} + 1.$$

Since

$$b_{\ell+1,\ell+1,\ell} = \sum_{t=1}^{\ell} t\left[(3\ell-3t+6)p^{2t-2} + (3\ell-3t+4)p^{2t-1}\right]$$

$$+\sum_{k=\ell}^{\ell+1}(\ell+1)(2\ell+3-2k)p^{k+\ell},$$

$$\begin{aligned}b_{\ell+1,\ell,\ell}= &\sum_{t=1}^{\ell} t\left[(3\ell-3t+5)p^{2t-2}+(3\ell-3t+3)p^{2t-1}\right]\\ &+\sum_{k=\ell}^{\ell}(\ell+1)(2\ell+2-2k)p^{k+\ell}\end{aligned}$$

and

$$\begin{aligned}b_{\ell,\ell,\ell}= &\sum_{t=1}^{\ell} t\left[(3\ell-3t+4)p^{2t-2}+(3\ell-3t+2)p^{2t-1}\right]\\ &+\sum_{k=\ell}^{\ell}(\ell+1)(2\ell+1-2k)p^{k+\ell}\end{aligned}$$

by induction hypothesis, we have

$$\begin{aligned}&b_{\ell+1,\ell+1,\ell+1}\\ =&\ (p^2+p+1)\\ &\times\left[\sum_{t=1}^{\ell} t\left[(3\ell-3t+6)p^{2t-2}+(3\ell-3t+4)p^{2t-1}\right]\right.\\ &\qquad\left.+\sum_{k=\ell}^{\ell+1}(\ell+1)(2\ell+3-2k)p^{k+\ell}\right]\\ &-(p^3+p^2+p)\left[\sum_{t=1}^{\ell} t\left[(3\ell-3t+5)p^{2t-2}+(3\ell-3t+3)p^{2t-1}\right]\right.\\ &\qquad\left.+\sum_{k=\ell}^{\ell}(\ell+1)(2\ell+2-2k)p^{k+\ell}\right]\\ &+p^3\left[\sum_{t=1}^{\ell} t\left[(3\ell-3t+4)p^{2t-2}+(3\ell-3t+2)p^{2t-1}\right]\right.\\ &\qquad\left.+\sum_{k=\ell}^{\ell}(\ell+1)(2\ell+1-2k)p^{k+\ell}\right]+1\\ =&\sum_{t=1}^{\ell} t\left[-p^{2t+2}+2p^{2t}+(3\ell-3t+5)p^{2t-1}+(3\ell-3t+6)p^{2t-2}\right]\\ &+2(\ell+1)p^{2\ell+2}+2(\ell+1)p^{2\ell+1}+3(\ell+1)p^{2\ell}+1\end{aligned}$$

$$= \sum_{t=1}^{\ell+1} t\left[(3\ell-3t+7)p^{2t-2}+(3\ell-3t+5)p^{2t-1}\right]+(\ell+2)p^{2\ell+2}.$$

Hence, together with Eq. (14) we have

$$\begin{aligned} b_{m,\ell+1,\ell+1} &= \sum_{t=1}^{\ell+1} t\left[(3\ell-3t+7)p^{2t-2}+(3\ell-3t+5)p^{2t-1}\right]+(\ell+2)p^{2\ell+2} \\ &\quad (m-\ell-1)\left[\sum_{t=1}^{\ell+1} t\left[p^{2t-2}+p^{2t-1}\right]+(\ell+2)p^{2\ell+2}\right] \\ &= \sum_{t=1}^{\ell+1} t\left[(m+2\ell-3t+6)p^{2t-2}+(m+2\ell-3t+4)p^{2t-1}\right] \\ &\quad + \sum_{k=\ell+1}^{\ell+1}(\ell+2)(m+\ell+2-2k)p^{k+\ell+1}. \end{aligned}$$

Therefore, Eq. (1) holds for $n=\ell+1$.

Assume now that Eq. (1) holds from $\ell+1$ to n, and consider the case for $n+1$. Eq. (8) with $(m,n,\ell)=(m,n+1,\ell+1)$ gives us that

$$\begin{aligned} b_{m,n+1,\ell+1} &= b_{m-1,n+1,\ell+1}+p(b_{m,n,\ell+1}-b_{m-1,n,\ell+1}) \\ &\quad + p^2(b_{m,n+1,\ell}-b_{m-1,n+1,\ell})-p^3(b_{m,n,\ell}-b_{m-1,n,\ell})+1. \end{aligned}$$

By induction hypothesis we know that

$$\begin{aligned} & b_{m,n,\ell+1}-b_{m-1,n,\ell+1} \\ &= \sum_{t=1}^{\ell+1} t\left[(m+n+\ell-3t+5)p^{2t-2}+(m+n+\ell-3t+3)p^{2t-1}\right] \\ &\quad + \sum_{k=\ell+1}^{n}(\ell+2)(m+n+1-2k)p^{k+\ell+1} - \sum_{t=1}^{\ell+1} t\left[(m+n+\ell-3t+4)p^{2t-2}\right. \\ &\quad \left.+(m+n+\ell-3t+2)p^{2t-1}\right] - \sum_{k=\ell+1}^{n}(\ell+2)(m+n-2k)p^{k+\ell+1} \\ &= \sum_{t=1}^{\ell+1} t\left[p^{2t-2}+p^{2t-1}\right]+\sum_{k=\ell+1}^{n}(\ell+2)p^{k+\ell+1}, \end{aligned}$$

$$\begin{aligned} & b_{m,n+1,\ell}-b_{m-1,n+1,\ell} \\ &= \sum_{t=1}^{\ell} t\left[(m+n+\ell-3t+5)p^{2t-2}+(m+n+\ell-3t+3)p^{2t-1}\right] \\ &\quad + \sum_{k=\ell}^{n+1}(\ell+1)(m+n+2-2k)p^{k+\ell} - \sum_{t=1}^{\ell} t\left[(m+n+\ell-3t+4)p^{2t-2}\right. \end{aligned}$$

$$
\begin{aligned}
&+(m+n+\ell-3t+2)p^{2t-1}\big] - \sum_{k=\ell}^{n+1}(\ell+1)(m+n+1-2k)p^{k+\ell} \\
= &\sum_{t=1}^{\ell} t\left[p^{2t-2}+p^{2t-1}\right] + \sum_{k=\ell}^{n+1}(\ell+1)p^{k+\ell}
\end{aligned}
$$

and

$$
\begin{aligned}
&b_{m,n,\ell} - b_{m-1,n,\ell} \\
= &\sum_{t=1}^{\ell} t\left[(m+n+\ell-3t+4)p^{2t-2} + (m+n+\ell-3t+2)p^{2t-1}\right] \\
&+\sum_{k=\ell}^{n}(\ell+1)(m+n+1-2k)p^{k+\ell} - \sum_{t=1}^{\ell} t\big[(m+n+\ell-3t+3)p^{2t-2} \\
&+(m+n+\ell-3t+1)p^{2t-1}\big] - \sum_{k=\ell}^{n}(\ell+1)(m+n-2k)p^{k+\ell} \\
= &\sum_{t=1}^{\ell} t\left[p^{2t-2}+p^{2t-1}\right] + \sum_{k=\ell}^{n}(\ell+1)p^{k+\ell}.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&b_{m,n+1,\ell+1} - b_{m-1,n+1,\ell+1} \\
&= p\left[\sum_{t=1}^{\ell+1} t\left[p^{2t-2}+p^{2t-1}\right] + \sum_{k=\ell+1}^{n}(\ell+2)p^{k+\ell+1}\right] \\
&\quad + p^2\left[\sum_{t=1}^{\ell} t\left[p^{2t-2}+p^{2t-1}\right] + \sum_{k=\ell}^{n+1}(\ell+1)p^{k+\ell}\right] \\
&\quad - p^3\left[\sum_{t=1}^{\ell} t\left[p^{2t-2}+p^{2t-1}\right] + \sum_{k=\ell}^{n}(\ell+1)p^{k+\ell}\right] + 1 \\
&= \sum_{t=1}^{\ell+1} t\left[p^{2t-2}+p^{2t-1}\right] + \sum_{k=\ell+1}^{n+1}(\ell+2)p^{k+\ell+1},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&b_{m,n+1,\ell+1} \\
&= b_{n+1,n+1,\ell+1} \\
&\quad + (m-n-1)\left[\sum_{t=1}^{\ell+1} t\left[p^{2t-2}+p^{2t-1}\right] + \sum_{k=\ell+1}^{n+1}(\ell+2)p^{k+\ell+1}\right].
\end{aligned}
\tag{15}
$$

On the other hand, by Eq. (9) with $(m, \ell) = (n+1, \ell+1)$ we have

$$\begin{aligned} b_{n+1,n+1,\ell+1} = (p+1)b_{n+1,n,\ell+1} - pb_{n,n,\ell+1} + p^2 b_{n+1,n+1,\ell} \\ - (p^3+p^2)b_{n+1,n,\ell} + p^3 b_{n,n,\ell} + 1. \end{aligned}$$

Since

$$\begin{aligned} b_{n+1,n,\ell+1} = &\sum_{t=1}^{\ell+1} t\left[(2n+\ell-3t+6)p^{2t-2} + (2n+\ell-3t+4)p^{2t-1}\right] \\ &+ \sum_{k=\ell+1}^{n} (\ell+2)(2n+2-2k)p^{k+\ell+1}, \end{aligned}$$

$$\begin{aligned} b_{n,n,\ell+1} = &\sum_{t=1}^{\ell+1} t\left[(2n+\ell-3t+5)p^{2t-2} + (2n+\ell-3t+3)p^{2t-1}\right] \\ &+ \sum_{k=\ell+1}^{n} (\ell+2)(2n+1-2k)p^{k+\ell+1}, \end{aligned}$$

$$\begin{aligned} b_{n+1,n+1,\ell} = &\sum_{t=1}^{\ell} t\left[(2n+\ell-3t+6)p^{2t-2} + (2n+\ell-3t+4)p^{2t-1}\right] \\ &+ \sum_{k=\ell}^{n+1} (\ell+1)(2n+3-2k)p^{k+\ell}, \end{aligned}$$

$$\begin{aligned} b_{n+1,n,\ell} = &\sum_{t=1}^{\ell} t\left[(2n+\ell-3t+5)p^{2t-2} + (2n+\ell-3t+3)p^{2t-1}\right] \\ &+ \sum_{k=\ell}^{n} (\ell+1)(2n+2-2k)p^{k+\ell} \end{aligned}$$

and

$$\begin{aligned} b_{n,n,\ell} = &\sum_{t=1}^{\ell} t\left[(2n+\ell-3t+4)p^{2t-2} + (2n+\ell-3t+2)p^{2t-1}\right] \\ &+ \sum_{k=\ell}^{n} (\ell+1)(2n+1-2k)p^{k+\ell} \end{aligned}$$

by induction hypothesis, we have

$$\begin{aligned} &b_{n+1,n+1,\ell+1} \\ &= (p+1)\left[\sum_{t=1}^{\ell+1} t\left[(2n+\ell-3t+6)p^{2t-2} + (2n+\ell-3t+4)p^{2t-1}\right]\right. \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=\ell+1}^{n}(\ell+2)(2n+2-2k)p^{k+\ell+1}\Bigg] - p\left[\sum_{t=1}^{\ell+1} t\left[(2n+\ell-3t+5)p^{2t-2}\right.\right.\\
&\left.\left.+(2n+\ell-3t+3)p^{2t-1}\right] + \sum_{k=\ell+1}^{n}(\ell+2)(2n+1-2k)p^{k+\ell+1}\right]\\
&+p^2\left[\sum_{t=1}^{\ell} t\left[(2n+\ell-3t+6)p^{2t-2}+(2n+\ell-3t+4)p^{2t-1}\right]\right.\\
&\left.+\sum_{k=\ell}^{n+1}(\ell+1)(2n+3-2k)p^{k+\ell}\right] - (p^3+p^2)\left[\sum_{t=1}^{\ell} t\left[(2n+\ell-3t+5)p^{2t-2}\right.\right.\\
&\left.\left.+(2n+\ell-3t+3)p^{2t-1}\right] + \sum_{k=\ell}^{n}(\ell+1)(2n+2-2k)p^{k+\ell}\right]\\
&+p^3\left[\sum_{t=1}^{\ell} t\left[(2n+\ell-3t+4)p^{2t-2}+(2n+\ell-3t+2)p^{2t-1}\right]\right.\\
&\left.+\sum_{k=\ell}^{n}(\ell+1)(2n+1-2k)p^{k+\ell}\right]+1\\
=&\sum_{t=1}^{\ell+1} t\left[(2n+\ell-3t+6)p^{2t-2}+(2n+\ell-3t+5)p^{2t-1}+p^{2t}\right]\\
&+\sum_{t=1}^{\ell} t\left[p^{2t}-p^{2t+2}\right] + \sum_{k=\ell+1}^{n}(\ell+2)(2n+2-2k)p^{k+\ell+1}\\
&+\sum_{k=\ell+1}^{n}(\ell+2)p^{k+\ell+2} + \sum_{k=\ell}^{n+1}(\ell+1)(2n+3-2k)p^{k+\ell+2}\\
&-\sum_{k=\ell}^{n}(\ell+1)(2n+2-2k)p^{k+\ell+2} - \sum_{k=\ell}^{n}(\ell+1)p^{k+\ell+3}+1\\
=&\sum_{t=1}^{\ell+1} t\left[(2n+\ell-3t+7)p^{2t-2}+(2n+\ell-3t+5)p^{2t-1}\right]\\
&+\sum_{k=\ell+1}^{n+1}(\ell+2)(2n+3-2k)p^{k+\ell+1}.
\end{aligned}
$$

Hence, together with Eq. (15) we have

$$
b_{m,n+1,\ell+1} = \sum_{t=1}^{\ell+1} t\left[(2n+\ell-3t+7)p^{2t-2}+(2n+\ell-3t+5)p^{2t-1}\right]
$$

$$
\begin{aligned}
&+ \sum_{k=\ell+1}^{n+1} (\ell+2)(2n+3-2k)p^{k+\ell+1} \\
&+ (m-n-1)\left[\sum_{t=1}^{\ell+1} t\left[p^{2t-2}+p^{2t-1}\right] + \sum_{k=\ell+1}^{n+1}(\ell+2)p^{k+\ell+1}\right] \\
= &\sum_{t=1}^{\ell+1} t\left[(m+n+\ell-3t+6)p^{2t-2} + (m+n+\ell-3t+4)p^{2t-1}\right] \\
&+ \sum_{k=\ell+1}^{n+1} (\ell+2)(m+n+2-2k)p^{k+\ell+1}.
\end{aligned}
$$

Therefore, Eq. (1) holds for $n+1$. Consequently, we have proved Theorem 2.1.

In general, for the group $\mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}} \times \cdots \times \mathbb{Z}_{p^{k_\ell}}$, where $k_1, k_2, \ldots, k_\ell$ are positive integers, ℓ is a positive integer such that $\ell \geq 4$ and p is a prime number, it seems not easy to obtain an explicit formula for the total number of subgroups with the method used in this paper.

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