A NOTE ON THE LORENTZIAN LIMIT CURVE THEOREM

Jong-Gug Yun

Abstract. In this paper, we extend the familiar limit curve theorem in
[2] to a situation where each causal curve lies in a sequence of compact
interpolating spacetimes converging to a limit Lorentz space in the sense
of Lorentzian Gromov-Hausdorff distance.

1. Introduction

The familiar limit curve theorem in [2] guarantees the existence of limit
causal curves for a sequence $\{\gamma_n\}$ of causal curves having points of accumulation
in a given spacetime.

In general, the compactness of the space of causal curves between two points
of a spacetime plays an essential role in the study of global structure of the
spacetime. In fact, many of the singularity theorems in general relativity and a
recent proof of positive energy theorem as well as Lorentzian splitting theorem
depend on the compactness of the space of causal curves in the spacetime ([6]).

In 1996, R. Sorkin and E. Woolgar [7] introduced the concept of K-causal
relation and extended the standard compactness theorem of causal curves to a
situation where the differentiability of metric of the spacetime is only C^0. From
the viewpoint of modern physics, there are strong evidences that our spacetime
is not smooth but just a structure which looks like a smooth manifold on a scale
larger than the Plank length. So lower degree of smoothness of spacetimes is
of importance in the study of modern model of spacetime including theories of
quantum gravity.

Another approach to spacetimes with lower degree of smoothness was in-
troduced by J. Noldus using Lorentzian Gromov-Hausdorff theory ([1], [3], [4],
[5]).

In [3], a Lorentzian notion of Gromov-Hausdorff distance d_{GH} (GH) and the
generalized Gromov-Hausdorff uniformity (GGH) were first defined and some
properties of limit space M of GGH-Cauchy sequences of compact, interpolat-
ing spacetimes $\{M_\alpha\}$ were shown and proved.

In general, the limit space M needs not to be a smooth manifold, but belongs to a class of *Lorentz spaces*. Here, in connection with the compactness of causal curves, a natural question which arises is whether we could guarantee the existence of limit causal curve γ in the limit space M of a sequence of causal curves $\{\gamma_\alpha\}$, where each γ_α lies in M_α rather than a single spacetime M and M_α converges to M in the sense of Lorentzian Gromov-Hausdorff distance.

The main purpose of this paper is to answer this question in an affirmative way and investigate the type of the convergence of γ_α to γ.

In the next section, we introduce some definitions and notions which are necessary to state our main result.

2. Preliminaries and main results

We begin with the notion of Gromov-Hausdorff distance d_{GH} (GH) between two globally hyperbolic compact interpolating (i.e., with spacelike future and past boundaries) spacetimes.

Definition 2.1. We call (M, g) and (N, h) ϵ-close if and only if there exist mappings $\psi : M \to N$, $\xi : N \to M$, not necessarily continuous, such that

$$|d_h(\psi(p_1), \psi(p_2)) - d_g(p_1, p_2)| \le \epsilon \; \forall p_1, \; p_2 \in M,$$
$$|d_g(\xi(q_1), \xi(q_2)) - d_h(q_1, q_2)| \le \epsilon \; \forall q_1, \; q_2 \in N,$$

where d_h and d_g are the Lorentz distance defined from g and h, respectively.

The Gromov-Hausdorff distance $d_{GH}((M, g), (N, h))$ is defined as the infimum over all ϵ such that (M, g) and (N, h) are ϵ-close.

In [1], the author constructed a very useful metric D_M for any compact interpolating spacetime (M, g) such that every d_g isometry is a D_M isometry.

Definition 2.2. Let (M, g) be a compact interpolating spacetime. The strong metric D_M is defined as

$$D_M(p, q) = \max_{r \in M} |d_g(p, r) + d_g(r, p) - d_g(q, r) - d_g(r, q)|.$$

It was shown in [3] that the manifold, strong, and Alexandrov topology coincide for any compact interpolating spacetime.

We also introduce the following definition of *Generalized Lorentzian Gromov-Hausdorff uniformity* (GGH):

Definition 2.3. We call (M, g) and (N, h) (ϵ-δ) close if and only if there exist mappings $\psi : M \to N$, $\xi : N \to M$ such that

$$|d_h(\psi(p_1), \psi(p_2)) - d_g(p_1, p_2)| \le \epsilon \; \forall p_1, \; p_2 \in M,$$
$$|d_g(\xi(q_1), \xi(q_2)) - d_h(q_1, q_2)| \le \epsilon \; \forall q_1, \; q_2 \in N$$

and $D_M(p, \xi \circ \psi(p)) \le \delta$, $D_N(q, \psi \circ \xi(q)) \le \delta$ for all $p \in M$ and $q \in N$.

As stated before, the limit space M of GH (or GGH)-Cauchy sequence of compact interpolating spacetime need not to be a smooth manifold, but it belongs to a class of *Lorentz space* which can be defined as follows:

Definition 2.4. Let M be a set. Lorentz distance is a function $d : M \times M \to \mathbb{R}^+ \cup \{\infty\}$ which satisfies for all $x, y, z \in M$:
- $d(x, x) = 0$,
- $d(x, y) > 0$ implies $d(y, x) = 0$ (antisymmetry),
- if $d(x, y)d(y, z) > 0$, then $d(x, z) \le d(x, y) + d(y, z)$ (reverse triangle inequality).

Definition 2.5. A Lorentz space is a pair (M, d), where M is a set and d is a Lorentz distance on M such that (M, D_M) is a compact metric space with D_M the strong metric induced by d.

We also note that any limit Lorentz space (M, d) of a GH-Cauchy sequence (M_i, d_i) of Lorentz spaces can be regarded as the limit space of GGH-Cauchy sequence (M_i, d_i) as well (For details, see Theorem 26 in [4]).

In general, it is not easy problem how to define a natural causal relation on the limit Lorentz space and it is difficult especially on the *degenerate regions* of the limit space, which does not contain any set of the form $I^+(p) \cap I^-(q)$ for $p \ll q$. Here, we used the standard notation $\ll$ for the chronological relation between p and q and $I^+(p) = \{r \in M \mid p \ll r\}$ for the chronological future of p and $I^-(p) = \{r \in M \mid r \ll p\}$ for the chronological past of p. In fact, as was pointed in [4], it is not entirely clear whether defining a causal relation on such regions is physically meaningful or not, and what degenerate region would mean from the physical point of view.

To avoid these difficulties, a controlling mechanism was suggested in [4] to prohibit the limit space from containing degenerate regions as follows.

Let $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing, continuous function such that $\alpha(x) \le x$ for all $x \in \mathbb{R}^+$. We say that (M, g) has the C_α^+ or C_α^- property if and only if for any ϵ with $0 < \epsilon \le \text{tdiam}(M)(:= \max_{p,q \in M} d(p, q))$ we have that, respectively:
- $\alpha(\epsilon) \le \min_{p \in M^{\downarrow \epsilon}} [\max_{r \in \overline{B_{D_M}(p, \epsilon)}} d_g(p, r)] \le \epsilon$,
- $\alpha(\epsilon) \le \min_{p \in M^{\uparrow \epsilon}} [\max_{r \in \overline{B_{D_M}(p, \epsilon)}} d_g(r, p)] \le \epsilon$, where $M^{\downarrow \epsilon} = \{p \in M \mid p \notin (\partial_F M)^\epsilon\}$ and $M^{\uparrow \epsilon} = \{p \in M \mid p \notin (\partial_P M)^\epsilon\}$.

Here, we denoted the *future* (resp. *past*) *boundary* of M by $\partial_F M$ (resp. $\partial_P M$) and used the notation $A^\epsilon = \{q \in M \mid \exists a \in A \text{ such that } D_M(a, q) < \epsilon\}$ for any $A \subset M$.

It was shown in [4] that the GGH-limit space (M, d) of a sequence $\{(M_i, g_i)\}$ satisfying the C_α^+ property has the C_α^+ property too (and likewise for C_α^-). That is, C_α^+, C_α^- properties are stable under GGH-convergence (See Theorem 21 in [4]).

Furthermore, if the limit Lorentz space (M, d) satisfies the C_α^+, C_α^- property, then a suitable causal structure on (M, d), which was first defined by R. Sorkin

and E. Woolgar can be introduced. We call it *K-causal relation* on M and introduce the definition of it here.

Recall first that the chronological relation $\ll$ on M can be defined by

$$p \ll q \text{ if and only if } d(p, q) > 0.$$

Then K-causal relation $\prec$ on M is defined using the topology of M as follows.

Definition 2.6. We say $p \prec q$ if and only if $q \in K^+(p)$, where $K^+(p)$ is the smallest, topologically closed, partial order in $M \times M$ containing $I^+(p)$ ($K^-(p)$ can be defined dually).

Remark 2.1. We note that the requirement of "partial order" in Definition 2.6 contains additional conditions (i.e., reflexive and antisymmetric) when compared with the original definition of K^+ in [7] where only transitivity is required).

Now we introduce the concept of causal curves in a Lorentz space M with a causal relation $\prec$ as in [4].

Definition 2.7. Let (M, d) be a Lorentz space. Assume $a < b$ and let $\gamma : [a, b] \to M$ be a continuous (with respect to the strong topology) mapping such that for all $a \leq t < s \leq b : \gamma(t) \prec \gamma(s)(\gamma(t) \ll \gamma(s))$; then γ is called a basic, causal (timelike) curve.

For any topological space X, the basic open sets $\mathcal{B}(A_0; A_1; \cdots; A_n)$ (A_i's are open in X) of the *Vietoris topology* on the space 2^X of nonempty closed subsets can be defined as follows:

$C \in \mathcal{B}(A_0; A_1; \cdots; A_n) \iff C \subseteq A_0$ and C meets A_i for $i = 1, \ldots, n$.

It is known in [7] that 2^X with the Vietoris topology is compact if X is compact. It is also shown that the Viertoris limit of a sequence of K-causal curves is a K-causal curve using only the topological arguments.

For any sequence of compact, interpolating spacetime $\{(M_\alpha, d_\alpha)\}$ converging to a Lorentz space (M, d) in the sense of GH-distance, we can extend each strong metric D_α on M_α and D_M on M to an admissible metric $\widetilde{D}$ on $\widetilde{M} = M \coprod (\coprod_\alpha M_\alpha)$ as in Theorem 17 of [4].

Keeping this extension of metric in mind, it makes sense to say that a sequence of causal curves $\{\gamma_\alpha\}$, where $\gamma_\alpha \subset M_\alpha$, converges to a causal curve Γ in M in the Vietoris topology on $2^{(\widetilde{M}, \widetilde{D})}$.

Now we are in a position to state our main result as follows.

Theorem 2.1. *Let $\{M_\alpha\}$ be a sequence of compact, interpolating spacetimes satisfy $C_\alpha^\pm$ properties and $\gamma_\alpha : [a, b] \to M_\alpha$ be a causal curve in M_α. Assume that $\{M_\alpha\}$ converges to a Lorentz space M with K-causal relation in the sense of GH-distance. Then there exists a subsequence $\{\gamma_\beta\}$ of $\{\gamma_\alpha\}$ such that $\{\gamma_\beta\}$ converges to a causal curve Γ in M in the Vietoris topology on $2^{(\widetilde{M}, \widetilde{D})}$, where $(\widetilde{M}, \widetilde{D})$ is defined as above.*

In Section 4, we shall also investigate the convergence of the lengths of the causal curves γ_α and obtain a result as we expected.

Theorem 2.2. *Let $\{\gamma_\beta\}$ and Γ be as in Theorem 2.1. Then we have*

$$\limsup_{\beta \to \infty} L[\gamma_\beta] \leq L[\Gamma].$$

3. Proof of Theorem 2.1

We first note that for each α, we have a map $\psi_\alpha : M_\alpha \to M$ which makes M_α and M $(\epsilon_\alpha, \epsilon_\alpha)$-close ($\epsilon_\alpha \to 0$ as $\alpha \to \infty$), since M can be regard as the GGH-limit space of $\{M_\alpha\}$.

Let $\Gamma_\alpha = \psi_\alpha \circ \gamma_\alpha$ and consider the closure $\overline{\Gamma}_\alpha$ of Γ_α in M for each α. Since $\{\overline{\Gamma}_\alpha\}$ is a sequence of closed subsets of M which is compact, we have a closed subset Γ in M such that a subsequence $\{\overline{\Gamma}_\beta\}$ of $\{\overline{\Gamma}_\alpha\}$ converges to Γ in the Vietoris topology in M.

Now we claim that Γ is a basic, causal curve in M by showing that it satisfies the hypotheses of the following theorem.

Theorem 3.1 ([7]). *Let Γ be a set provided with both a linear order and a topology such that*
• with respect to the topology it is compact and connected and contains a countable dense subset,
• with respect to the order it has both a minimal and a maximal element, and
• (with respect to both) it has the property that $\ll x, y \gg = \{r \in M \mid x \prec r \prec y\}$ is topologically closed for all x, $y \in \Gamma$.
Then Γ is isomorphic to the interval $[0, 1] \subset \mathbb{R}$ by a simultaneous order and topological isomorphism.

Note that we can apply the same arguments as in Theorem 20 in [7] to verify that Γ has a countable dense subset, minimal and maximal elements and every interval in Γ is topologically closed. Since Γ is compact clearly, we need only to show that it is connected and linearly ordered to verify that Γ satisfies the hypotheses in Theorem 3.1.

We first show that Γ is connected as follows. Note that the number of components of Γ should be finite by the compactness of Γ, so we may let $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_n\}$ be the finite components of Γ.

For simplicity, we assume $n = 2$ (we can argue similarly for the general case) and we first claim Γ_1 is closed in M. This can easily be seen by noting that every accumulation point of Γ_1 should be in $\Gamma = \Gamma_1 \coprod \Gamma_2$ since Γ is closed and it cannot be in Γ_2. Consequently, every accumulation point of Γ_1 should be in Γ_1 itself and this says that Γ_1 is closed. Γ_2 is closed by the same arguments. Now we can choose $x_1 \in \Gamma_1$ and $x_2 \in \Gamma_2$ such that $D_M(x_1, x_2) = \min\{D_M(x, y) \mid x \in \Gamma_1, y \in \Gamma_2\} > 0$ which is possible, since Γ_1, Γ_2 are compact and disjoint.

Letting $\epsilon = D_M(x_1, x_2) > 0$, we know that $V_1 := N(\Gamma_1, \frac{\epsilon}{3}) = \{y \in M \mid D_M(\Gamma_1, y) < \frac{\epsilon}{3}\}$ and $V_2 := N(\Gamma_2, \frac{\epsilon}{3}) = \{y \in M \mid D_M(\Gamma_2, y) < \frac{\epsilon}{3}\}$ are disjoint open sets containing Γ_1 and Γ_2 respectively and satisfy the following:

$$(3.1) \qquad\qquad D_M(x, y) > \frac{\epsilon}{3} \text{ for all } x \in V_1, \ y \in V_2.$$

Since $\{\overline{\Gamma}_\beta\}$ converges to Γ in the Vietoris topology, we may assume that $\{\overline{\Gamma}_\beta\} \subset V_1 \coprod V_2$ for all β (by taking a subsequence if necessary).

Now we let $U_1 = \{x \in \gamma_\beta|_{[a,b]} \mid \psi_\beta(x) \in V_1\}$, $U_2 = \{x \in \gamma_\beta|_{[a,b]} \mid \psi_\beta(x) \in V_2\}$ and claim that $\inf\{D_\alpha(x, y) \mid x \in U_1, \ y \in U_2\} = 0$.

Suppose that $\tau = \inf\{D_\alpha(x, y) \mid x \in U_1, \ y \in U_2\} > 0$ and consider $\mathcal{U}$, a family of open balls with centers on $\gamma_\beta|_{[a,b]}$ and radius $\frac{\tau}{3}$. That is, $\mathcal{U} = \{B_\beta(z, \frac{\tau}{3}) \mid z \in \gamma_\beta|_{[a,b]}\}$.

By the compactness of $\gamma_\beta|_{[a,b]}$, we have a family of finite open balls $\{B_\beta(z_1, \frac{\tau}{3}), \dots, B_\beta(z_n, \frac{\tau}{3})\}$ and each $B_\beta(z_i, \frac{\tau}{3})$ contains an element in U_1 or U_2 exclusively. But this gives a disconnection of $\gamma_\beta|_{[a,b]}$ which is a contradiction.

We now note from the proof of Theorem 16 in [4] that

$$|D_M(\psi_\beta(x), \psi_\beta(y)) - D_\beta(x, y)| < 4\epsilon_\beta \ \forall x, y \in \gamma_\beta|_{[a,b]}.$$

So arbitrary closeness between $x_1 \in U_1 \subset \gamma_\beta|_{[a,b]}$ and $x_2 \in U_2 \subset \gamma_\beta|_{[a,b]}$ implies arbitrary closeness between $\psi_\beta(x_1) \in V_1$ and $\psi_\beta(x_2) \in V_2$ in M.

But this contradicts to (3.1) and we conclude that Γ is connected.

Next, we show that Γ is a linearly ordered set. For this purpose, we need to show that for any $p, q \in \Gamma$, they are related by $\prec$.

In case that p or q is the minimal or maximal element, we have nothing to prove and assume that the two points are neither minimal nor maximal. Then we can say that neither point is in the future or past boundary of M. For, if q is in the future boundary $\partial_F M$, then we have $I^+(q) = \emptyset$ and $K^+(q) = \{q\}$ by the definition of K-causal relation, which implies that q is the maximal element. The same argument holds for the past boundary of M.

Now since $\{\overline{\Gamma}_\beta\}$ converges to Γ in the Vietoris topology, we may consider sequences $\{\psi_\beta(p_\beta)\}$ and $\{\psi_\beta(q_\beta)\}$, where $p_\beta, q_\beta \in \gamma_\beta$ and $\psi_\beta(p_\beta) \to p$, $\psi_\beta(q_\beta) \to q$ in (M, D_M) (passing to a subsequence if necessary). We may also assume that q_β is in the causal future of p_β without loss of generality.

From now on, we claim that there is a sequence $\{r_{\beta_k}\}$, where $\{\beta_k\} \subset \{\beta\}$ such that $\{\psi_{\beta_k}(r_{\beta_k})\} \to q$ in (M, D_M) and $\psi_{\beta_k}(p_{\beta_k}) \ll \psi_{\beta_k}(r_{\beta_k})$ for each β_k.

Note first that for sufficiently large $N_1 \in \mathbb{N}$, we have $B_M(q, \frac{1}{N_1}) \cap \partial_F M = \phi$, since $q \notin \partial_F M$. Then by the C_α^+ property, there exists an $r' \in \overline{B_M(q, \frac{1}{2N_1})}$ such that $d(q, r') = \alpha(\frac{1}{2N_1}) > 0$. Let $\{r_\beta\}$ be a sequence in $(\widetilde{M}, \widetilde{D})$ such that $r_\beta \in M_\beta \ (\forall \beta)$ and $\{r_\beta\} \to r'$ in $(\widetilde{M}, \widetilde{D})$. Then there exists a $\beta^{1,1} > 0$ such that $\widetilde{D}(r_\beta, r') < \frac{1}{N_1}$ for all β with $\beta \geq \beta^{1,1}$.

On the other hand, since $d_\beta(q_\beta, r_\beta)$ converges to $d(q, r') = \alpha(\frac{1}{2N_1}) > 0$ as $\beta \to \infty$ (Refer to the proof of Theorem 17 in [4] for details), we know that there exists a $\beta^{1,2}$ such that $q_\beta \ll r_\beta$ for all β with $\beta \geq \beta^{1,2}$.

Thus, by the reverse triangle inequality, we have

$$d_\beta(p_\beta, r_\beta) \geq d_\beta(q_\beta, r_\beta) \sim \alpha(\frac{1}{2N_1}) \text{ for all } \beta \text{ with } \beta \geq \beta^{1,2}.$$

Now we obtain

$$
\begin{aligned}
d(\psi_\beta(p_\beta), \psi_\beta(r_\beta)) &\geq& d(p_\beta, r_\beta) - \epsilon_\beta \\
&\geq& d_\beta(q_\beta, r_\beta) - \epsilon_\beta \\
&\sim& \alpha(\frac{1}{2N_1}) - \epsilon_\beta > 0 \text{ for all sufficiently large } \beta.
\end{aligned}
$$

Consequently, there exists a $\beta^{1,3} > 0$ such that $\psi_\beta(p_\beta) \ll \psi_\beta(r_\beta)$ for all β with $\beta \geq \beta^{1,3}$. Now we let $\beta_1 = \max\{\beta^{1,1}, \beta^{1,2}, \beta^{1,3}\}$ and note that $\widetilde{D}(r_{\beta_1}, r') < \frac{1}{N_1}$, $r' \in \overline{B_M(q, \frac{1}{2N_1})}$ and $\psi_{\beta_1}(p_{\beta_1}) \ll \psi_{\beta_1}(r_{\beta_1})$.

Let $N_k = N_1 + (k-1)$ $(k = 2, 3, \ldots)$ and proceed the same process to obtain $r^k \in \overline{B_M(q, \frac{1}{2N_k})}$ and β_k such that $\psi_{\beta_k}(p_{\beta_k}) \ll \psi_{\beta_k}(r_{\beta_k})$ $(k = 2, 3, \ldots)$ and $\widetilde{D}(r^k, r_{\beta_k}) < \frac{1}{N_k} = \frac{1}{N_1 + (k-1)}$.

We now have

$$\widetilde{D}(q, \psi_{\beta_k}(r_{\beta_k})) \leq \widetilde{D}(q, r^k) + \widetilde{D}(r^k, r_{\beta_k}) + \widetilde{D}(r_{\beta_k}, \psi_{\beta_k}(r_{\beta_k})),$$

where the three terms of the right hand side of the above inequality converge to zero as $k \to \infty$ by the construction of r_k, r_{β_k} and the inequality $\widetilde{D}(x_\alpha, \psi_\alpha(x_\alpha)) < \epsilon_\alpha + \frac{3\epsilon_\alpha}{2} = \frac{5\epsilon_\alpha}{2}$ for all $x_\alpha \in M_\alpha$ (See p. 70 in [4] for reference).

Thus we obtain that $\psi_\beta(r_{\beta_k})$ converges to q in (M, D_M). Now $p \prec q$ follows from the closeness of K^+, which proves that Γ is a linearly ordered set.

Now we claim that $\{\gamma_\beta\}$ converges to Γ in the Vietoris topology on $(\widetilde{M}, \widetilde{D})$. Recall that our claim is equivalent to show that $\{\gamma_\beta\}$ converges to Γ in the both "upper" and "lower" topology on $(\widetilde{M}, \widetilde{D})$ ([7]).

In the lower topology, $\{\gamma_\beta\}$ converges to Γ if and only if the $\{\gamma_\beta\}$ eventually "meet" every open set $U \subseteq \widetilde{M}$ which meets Γ. In the upper topology (also called the C^0 topology), $\{\gamma_\beta\}$ converges to Γ if and only if the $\{\gamma_\beta\}$ are eventually "included" in every open set $U \subseteq \widetilde{M}$ which includes Γ itself (For details, refer to p. 32 in [7]).

We start with the convergence in the lower topology on $(\widetilde{M}, \widetilde{D})$.

Let $U \subseteq \widetilde{M}$ be any open set which meets Γ at p and consider an ϵ-ball $B_{\widetilde{M}}(p, \epsilon)$ with $(p \in) B_{\widetilde{M}}(p, \epsilon) \subset U$. Since $\{\overline{\Gamma}_\beta\}$ converges to Γ in the Vietoris topology on (M, D_M), we can easily see that there exists a $\beta_1 > 0$ such that $B_M(p, \frac{\epsilon}{2}) \cap \Gamma_\beta \neq \emptyset$ for all β with $\beta \geq \beta_1$, where we note $B_M(p, \frac{\epsilon}{2}) = B_{\widetilde{M}}(p, \frac{\epsilon}{2}) \cap M$.

We denote an element in $B_M(p, \frac{\epsilon}{2}) \cap \Gamma_\beta$ by $\psi_\beta(p_\beta)$, where $p_\beta \in \gamma_\beta$ for $\beta \geq \beta_1$

Since $\{M_\beta\}$ converges to M in the GGH-sense, we also know that there exists a $\beta_2 > 0$ such that $\widetilde{D}(p_\beta, \psi_\beta(p_\beta)) < \frac{\epsilon}{2}$ for all $\beta \geq \beta_2$.

We now have $\widetilde{D}(p_\beta, p) \leq \widetilde{D}(p_\beta, \psi_\beta(p_\beta)) + \widetilde{D}(\psi_\beta(p_\beta), p) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for all β with $\beta \geq \max\{\beta_1, \beta_2\}$. This means that $B_{\widetilde{M}}(p, \epsilon) \cap \gamma_\beta \neq \emptyset$ for all β with $\beta \geq \max\{\beta_1, \beta_2\}$, which implies that $U \cap \gamma_\beta \neq \emptyset$ for all β with $\beta \geq \max\{\beta_1, \beta_2\}$ since $B_{\widetilde{M}}(p, \epsilon) \subset U$. Thus we conclude that $\{\gamma_\beta\}$ converges to Γ in the lower topology on $(\widetilde{M}, \widetilde{D})$ (In the terminology in [2], Γ is the limit curve of $\{\gamma_\alpha\}$).

We next show that $\{\gamma_\beta\}$ converges to Γ in the upper topology (C^0-topology) on $(\widetilde{M}, \widetilde{D})$. Let $U \subseteq \widetilde{M}$ be any open set which contains Γ and for each point $p \in \Gamma$ consider open balls $B_{\widetilde{M}}(p, \delta(p))$, $B_{\widetilde{M}}(p, \frac{\delta(p)}{2})$ for some $\delta(p) > 0$ such that $(p \in) B_{\widetilde{M}}(p, \frac{\delta(p)}{2}) \subset B_{\widetilde{M}}(p, \delta(p)) \subset U$.

Since $\{B_M(p, \frac{\delta(p)}{2}) \mid p \in \Gamma\}$ is an open covering of Γ in M and Γ is compact, we have finite $p_1, \ldots, p_n$ on Γ such that $\Gamma \subset \cup_{i=1}^n B_M(p_i, \frac{\delta(p)}{2}) \subset U$.

By the convergence of $\{\overline{\Gamma}_\beta\}$ to Γ in the Vietoris topology on (M, D_M) we know that there exists a $\beta_3 > 0$ such that $\overline{\Gamma}_\beta \subset \cup_{i=1}^n B_M(p_i, \frac{\delta(p)}{2})$ for all β with $\beta \geq \beta_3$.

Let $\delta_0 = \frac{1}{2} \min\{\delta(p_1), \ldots, \delta(p_n)\}$ and note that there exists a $\beta_4 > 0$ such that $\widetilde{D}(p_\beta, \psi_\beta(p_\beta)) < \delta_0$ for all $p_\beta \in \gamma_\beta$ and $\beta \geq \beta_4$.

Now, for any $p_\beta \in \gamma_\beta$ where $\beta \geq \max\{\beta_3, \beta_4\}$, take $p_i (1 \leq i \leq n)$ such that $\psi_\beta(p_\beta) \in B_M(p_i, \frac{\delta(p_i)}{2})$, which is possible since $\psi_\beta(p_\beta) \in \Gamma_\beta \subset \overline{\Gamma}_\beta \subset \cup_{i=1}^n B_M(p_i, \frac{\delta(p_i)}{2})$ for all $\beta \geq \beta_3$.

We then have the following inequalities, for all $\beta \geq \max\{\beta_3, \beta_4\}$,

$$\widetilde{D}(p_\beta, p_i) \leq \widetilde{D}(p_\beta, \psi_\beta(p_\beta)) + \widetilde{D}(\psi_\beta(p_\beta), p_i)$$

$$\leq \delta_0 + \frac{\delta(p_i)}{2} \leq \delta(p_i).$$

Thus if $\beta \geq \max\{\beta_3, \beta_4\}$, then every p_β on γ_β is contained in some ball $B_{\widetilde{M}}(p_i, \delta(p_i))$ in $(\widetilde{M}, \widetilde{D})$. This means that $\gamma_\beta \subset \cup_{i=1}^n B_{\widetilde{M}}(p_i, \delta(p_i)) \subset U$ for all $\beta \geq \max\{\beta_3, \beta_4\}$ in $(\widetilde{M}, \widetilde{D})$. So we conclude that $\{\gamma_\beta\}$ converges to Γ in the upper topology. Consequently, $\{\gamma_\beta\}$ converges to Γ in the Vietoris topology on $(\widetilde{M}, \widetilde{D})$ and we complete the proof of Theorem 2.1.

4. Proof of Theorem 2.2

The main point of the proof of Theorem 2.2 is the following Lemma 4.1, which can be easily shown by the similar arguments in the proof of Lemma 22 in [7].

Lemma 4.1. $\{\gamma_\beta(a)\}$ (resp. $\{\gamma_\beta(b)\}$) converges to the initial (resp. final) point of Γ in $(\widetilde{M}, \widetilde{D})$.

Proof. Since we know $\widetilde{D}(\gamma_\beta(a), \psi_\beta(\gamma_\beta(a))) \to 0$ as $\beta \to \infty$ from p. 70 in [4], it suffices to show that $\psi_\beta(\gamma_\beta(a))$ converges to the initial point of Γ in (M, D_M) (The same is true for the final point).

Note that we may assume without loss of generality that $\{\psi_\beta(\gamma_\beta(a))\}$ converges to a point Γ_0 in M, since (M, D_M) is compact.

We now claim that Γ_0 is the initial point of Γ. Because Γ is the Vietoris-limit of $\{\overline{\psi_\beta(\gamma_\beta)}\}$ and Γ is a closed set, it is easy to check that Γ_0 should be contained in Γ itself.

For any p in Γ and $\{\psi_\beta(q_\beta)\}$ converging to q in (M, D_M), we know that q_β is in the causal future of $\gamma_\beta(a)$ since γ_β is a causal curve and $q_\beta \in \gamma_\beta$. Recall that Γ is linearly ordered, so we have either $\Gamma_0 \prec q$ or $q \prec \Gamma$. But we can apply the same arguments as in the proof of Theorem 2.1 to show that $\Gamma_0 \prec q$ is true noting that q_β is in the causal future of $p_\beta := \gamma_\beta(a)$ and $\psi_\beta(p_\beta)$ converges to Γ_0 in $(\widetilde{M}, \widetilde{D})$ (In case that q is in the future boundary of M it is the final point of Γ automatically as stated previously in the proof of Theorem 2.1, so there is nothing to prove).

Thus we conclude that Γ_0 is the initial point of Γ. $\qquad\square$

Now for any partition Δ of $[a,b]$, $\Delta = \{t_i \mid a = t_0 < t_1 < \cdots < t_n = b\}$ and for each $\Gamma(t_i)$ $(i = 0, 1, \ldots, n)$, consider a sequence $\{p_\beta^i\}$ in $\widetilde{M}$ such that $p_\beta^i \in \gamma_\beta$ and $p_\beta^i \to \Gamma(t_i)$ as $\beta \to \infty$. Note that we can put $p_\beta^0 = \gamma_\beta(a)$, $p_\beta^n = \gamma_\beta(b)$ by the above Lemma 4.1.

Then we have

$$\sum_{i=1}^{n} d(\Gamma(t_{i-1}), \Gamma(t_i)) = \lim_{\beta \to \infty} \sum_{i=1}^{n} d_\beta(p_\beta^{i-1}, p_\beta^i) \geq \limsup_{\beta \to \infty} L[\gamma_\beta].$$

Since $L[\Gamma] = \inf_\Delta \{\sum_{i=1}^n d(\Gamma(t_{i-1}, t_i)\}$, we conclude that $L[\Gamma] \geq \limsup_{\beta \to \infty} L[\gamma_\beta]$.

This completes the proof of Theorem 2.2.

References

[1] L. Bombelli and J. Noldus, *The moduli space of isometry classes of globally hyperbolic spacetimes*, Classical Quantum Gravity **21** (2004), no. 18, 4429–4453.

[2] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambidge University Press, 1973.

[3] J. Noldus, *A Lorentzian Gromov-Hausdorff notion of distance*, Classical Quantum Gravity **21** (2004), no. 4, 839–850.

[4] ———, *Lorentzian Gromov Hausdorff theory as a tool for quantum gravity kinematics*, PhD thesis, Gent University, 2004.

[5] ———, *The limit space of a Cauchy sequence of globally hyperbolic spacetimes*, Classical Quantum Gravity 21 (2004), no. 4, 851–874.

[6] R. Penrose, R. D. Sorkin, and E. Woolgar, *A positive mass theorem based on the focusing and retardation of null geodesics*, gr-qc/9301015.

[7] R. Sorkin and E. Woolgar, *A causal order for spacetimes with C^0 Lorentzian metrics: proof of compactness of the space of causal curves*, Classical Quantum Gravity **13** (1996), no. 7, 1971–1993.

Department of Mathematics Education
Korea National University of Education
Chungbuk 363-791, Korea
E-mail address: jgyun69@knue.ac.kr