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# JORDAN DERIVATIONS ON PRIME RINGS AND THEIR APPLICATIONS IN BANACH ALGEBRAS, I

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Abstract. The purpose of this paper is to prove that the noncommutative version of the Singer-Wermer Conjecture is affirmative under certain conditions. Let $A$ be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \to A$ such that $D(x)^3[D(x), x] \in \mathrm{rad}(A)$ for all $x \in A$. In this case, we show that $D(A) \subseteq \mathrm{rad}(A)$.

## 1. Introduction

Throughout, $R$ represents an associative ring and $A$ will be a complex Banach algebra. We write $[x, y]$ for the commutator $xy - yx$ for $x, y$ in a ring. Let $\mathrm{rad}(R)$ denote the (*Jacobson*) *radical* of a ring $R$. And a ring $R$ is said to be (*Jacobson*) *semisimple* if its Jacobson radical $\mathrm{rad}(R)$ is zero.

A ring $R$ is called *n-torsion free* if $nx = 0$ implies $x = 0$. Recall that $R$ is *prime* if $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is *semiprime* if $aRa = (0)$ implies $a = 0$. On the other hand, let $X$ be an element of a normed algebra. Then for every $a \in X$ the *spectral radius* of $a$, denoted by $r(a)$, is defined by $r(a) = \inf\{||a^n||^{\frac{1}{n}} : n \in \mathbb{N}\}$. It is well-known that the following theorem holds: if $a$ is an element of a normed algebra, then $r(a) = \lim_{n\to\infty} ||a^n||^{\frac{1}{n}}$ (see Bonsall and Duncan [1]).

An additive mapping $D$ from $R$ to $R$ is called a *derivation* if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. And an additive mapping $D$ from $R$ to $R$ is called a *Jordan derivation* if $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$.

Johnson and Sinclair [5] have proved that any linear derivation on a semisimple Banach algebra is continuous. A result of Singer and Wermer [12] states that every continuous linear derivation on a commutative Banach algebra maps the algebra into its radical. From these two results, we can conclude that there are no nonzero linear derivations on a commutative semisimple Banach algebra. Thomas [13] has proved that any linear derivation on a commutative Banach algebra maps the algebra into its radical.

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A noncommutative version of Singer and Wermer's Conjecture states that every continuous linear derivation on a noncommutative Banach algebra maps the algebra into its radical.

Vukman [15] has proved the following: Let $R$ be a 2-torsion free prime ring. If $D : R \longrightarrow R$ is a derivation such that $[D(x), x]D(x) = 0$ for all $x \in R$, then $D = 0$.

Moreover, using the above result, he has proved that the following holds: Let $A$ be a noncommutative semisimple Banach algebra. Suppose that $[D(x), x]$ $D(x) = 0$ holds for all $x \in A$. In this case, $D = 0$.

Kim [6] has showed that the following result holds: Let $R$ be a 3!-torsion free semiprime ring. Suppose there exists a Jordan derivation $D : R \to R$ such that

$$[D(x), x]D(x)[D(x), x] = 0$$

for all $x \in R$. In this case, we have $[D(x), x]^5 = 0$ for all $x \in R$.

And, Kim [7] has showed that the following result holds: Let $A$ be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \to A$ such that $D(x)[D(x), x]D(x) \in \mathrm{rad}(A)$ for all $x \in A$. In this case, we have $D(A) \subseteq \mathrm{rad}(A)$.

In this paper, our aim is to prove the following results in the ring theory in order to apply it to the Banach algebra theory:

Let $R$ be a 7!-torsion free prime ring. Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that

$$D(x)^3[D(x), x] = 0$$

for all $x \in R$. In this case, we obtain $D(x) = 0$ for all $x \in R$.

Let $A$ be a noncommutative Banach Algebra. Suppose there exists a continuous linear Jordan derivation $D : A \longrightarrow A$ such that

$$D(x)^3[D(x), x] \in \mathrm{rad}(A)$$

for all $x \in A$. In this case, we obtain $D(A) \subseteq \mathrm{rad}(A)$ for all $x \in A$.

## 2. Preliminaries

In this section, we review the basic results in prime and semiprime rings.

The following lemma is due to Chung and Luh [4].

**Lemma 2.1.** *Let $R$ be a $n$!-torsion free ring. Suppose there exist elements $y_1, y_2, \ldots, y_{n-1}, y_n$ in $R$ such that $\sum_{k=1}^{n} t^k y_k = 0$ for all $t = 1, 2, \ldots, n$. Then we have $y_k = 0$ for every positive integer $k$ with $1 \leq k \leq n$.*

The following theorem is due to Brešar [3].

**Theorem 2.2.** *Let $R$ be a 2-torsion free semiprime ring and let $D : R \longrightarrow R$ be a Jordan derivation. In this case, $D$ is a derivation.*

The following theorem is due to Chung and Luh [4].

**Theorem 2.3.** *Let $R$ be a semiprime ring with a derivation $D$. Suppose there exists a positive integer $n$ such that $(Dx)^n = 0$ for all $x \in R$ and suppose $R$ is $(n-1)!$-torsion free. Then $D = 0$.*

## 3. Main results

We need the following notations. After this, by $S_m$ we denote the set $\{k \in \mathbb{N} \mid 1 \leq k \leq m\}$ where $m$ is a positive integer. When $R$ is a ring, we shall denote the maps $B : R \times R \longrightarrow R,\ f, g : R \longrightarrow R$ by $B(x,y) \equiv [D(x), y] + [D(y), x], f(x) \equiv [D(x), x],\ g(x) \equiv [f(x), x]$ for all $x, y \in R$, respectively. And we have the basic properties:

$$B(x,y) = B(y,x),\ B(x,yz) = B(x,y)z + yB(x,z) + D(y)[z,x] + [y,x]D(z),$$
$$B(x,x) = 2f(x),\ B(x,x^2) = 2(f(x)x + xf(x)),\ x,y,z \in R.$$

**Lemma 3.1.** *Let $R$ be a 2-torsion free noncommutative prime ring. Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that*

$$[D(x), x] = 0$$

*for all $x \in R$. Then we have $D(x) = 0$ for all $x \in R$.*

*Proof.* From Theorem 2.2, we see that $D$ is a derivation on $R$. Let

(1) $$f(x) = [D(x), x] = 0, \quad x \in R.$$

Substituting $x + y$ for $x$ in (1), we have

(2) $$f(x+y) \equiv f(x) + B(x,y) + [D(y), y] = 0,\ x, y \in R.$$

From (1) and (2), we obtain

(3) $$B(x,y) = 0, \quad x, y \in R.$$

Replacing $yx$ for $y$ in (3), we have

(4) $$B(x,y)x + 2yf(x) + [y,x]D(x) = 0, \quad x, y \in R.$$

Combining (1), (3) with (4), we get

(5) $$[y,x]D(x) = 0,\ x, y \in R.$$

Substituting $zy$ for $y$ in (5), we have

(6) $$[z,x]yD(x) + z[y,x]D(x) = 0, \quad x, y, z \in R.$$

Combining (5) with (6), we obtain

(7) $$[z,x]yD(x) = 0, \quad x, y, z \in R.$$

Substituting $x + u$ for $x$ in (5),

(8) $$[z,x]yD(x) + [z,u]yD(x) + [z,x]yD(u) + [z,u]yD(u) = 0,\ u, x, y, z \in R.$$

From (7) and (8),

(9) $$[z,u]yD(x) + [z,x]yD(u) = 0,\ u, x, y, z \in R.$$

Writing $yD(x)v[z,u]y$ for $y$ in (9), we have

$$\begin{aligned}&[z,u]yD(x)v[z,u]yD(x)+[z,x]yD(x)v[z,u]yD(u)\\ &=0,\ u,v,x,y,z\in R.\end{aligned} \tag{10}$$

Combining (7) with (10), we get

$$[z,u]yD(x)v[z,u]yD(x)=0, \quad u,v,x,y,z\in R. \tag{11}$$

From (11) and the (semi)primeness of $R$,

$$[z,u]yD(x)=0, \quad u,x,y,z\in R. \tag{12}$$

By the primeness and noncommutativity of $R$, (12) gives

$$D(x)=0, \quad x\in R. \qquad \square$$

**Lemma 3.2.** *Let $R$ be a 2-torsion free noncommutative semiprime ring. Suppose there exists a Jordan derivation $D:R\longrightarrow R$ such that*

$$[[D(x),x],x]=0$$

*for all $x\in R$. Then we have $[D(x),x]=0$ for all $x\in R$.*

*Proof.* From Theorem 2.2, we see that $D$ is a derivation on $R$.

Let

$$g(x)=[[D(x),x],x]=0, \quad x\in R. \tag{13}$$

Substituting $x+ty$ for $x$ in (13), we have

$$\begin{aligned}g(x+ty)\equiv g(x)&+t([f(x),y]+[B(x,y),x])\\ &+t^2A(x,y)+t^3g(y)=0,\ x,y\in R,t\in S_2,\end{aligned} \tag{14}$$

where $A(x,y)$ denotes the term satisfying the identity (14).
From (13) and (14), we obtain

$$t([f(x),y]+[B(x,y),x])+t^2A(x,y)=0,\ x,y\in R,t\in S_2. \tag{15}$$

Since $R$ is 2-torsion free, by Lemma 2.1, (15) yields

$$[f(x),y]+[B(x,y),x]=0, \quad x,y\in R. \tag{16}$$

Replacing $yx$ for $y$ in (16), we have

$$\begin{aligned}&[f(x),y]x+yg(x)+[B(x,y),x]x\\ &+3yg(x)+3[y,x]f(x)+[[y,x],x]D(x)=0, \quad x,y\in R.\end{aligned} \tag{17}$$

Right multiplication of (16) by $x$ leads to

$$[f(x),y]x+[B(x,y),x]x=0, \quad x,y\in R. \tag{18}$$

Comparing (13), (17) and (18), we get

$$3yg(x)+3[y,x]f(x)+[[y,x],x]D(x)=0, \quad x,y\in R. \tag{19}$$

From (13) and (19),

$$3[y,x]f(x)+[[y,x],x]D(x)=0, \quad x,y\in R. \tag{20}$$

From (20), we have

$$3[y,x]g(x)+[[y,x],x]f(x)=0, \quad x,y\in R. \tag{21}$$

From (13) with (21),

$$[[y,x],x]f(x)=0, \quad x,y\in R. \tag{22}$$

Writing $D(x)y$ for $y$ in (20), we have

$$\begin{aligned}&3D(x)[y,x]f(x)+3f(x)yf(x)+D(x)[[y,x],x]D(x)\\ &+2f(x)[y,x]D(x)+g(x)yD(x)=0, \quad x,y\in R.\end{aligned} \tag{23}$$

Left multiplication of (20) by $D(x)$, we obtain

$$3D(x)[y,x]f(x)+D(x)[[y,x],x]D(x)=0, \quad x,y\in R. \tag{24}$$

From (13), (23) and (24), we get

$$3f(x)yf(x)+2f(x)[y,x]D(x)=0, \quad x,y\in R. \tag{25}$$

Replacing $yD(x)w$ for $y$ in (25), we have

$$\begin{aligned}&3f(x)yD(x)wf(x)+2f(x)[y,x]D(x)wD(x)\\ &+2f(x)yf(x)wD(x)+2f(x)yD(x)[w,x]D(x)=0, \quad w,x,y\in R.\end{aligned} \tag{26}$$

From (25) and (26), we get

$$\begin{aligned}&3f(x)yD(x)wf(x)-f(x)yf(x)wD(x)\\ &+2f(x)yD(x)[w,x]D(x)=0, \quad w,x,y\in R.\end{aligned} \tag{27}$$

Substituting $wx$ for $w$ in (27), we have

$$\begin{aligned}&3f(x)yD(x)wxf(x)-f(x)yf(x)wxD(x)\\ &+2f(x)yD(x)[w,x]xD(x)=0, \quad w,x,y\in R.\end{aligned} \tag{28}$$

Right multiplication of (27) by $x$ leads to

$$\begin{aligned}&3f(x)yD(x)wf(x)x-f(x)yf(x)wD(x)x\\ &+2f(x)yD(x)[w,x]D(x)x=0, \quad w,x,y\in R.\end{aligned} \tag{29}$$

From (28) and (29), we get

$$\begin{aligned}&3f(x)yD(x)wg(x)-f(x)yf(x)wf(x)\\ &+2f(x)yD(x)[w,x]f(x)=0, \quad w,x,y\in R.\end{aligned} \tag{30}$$

Comparing (13) and (30), we get

$$f(x)y(f(x)wf(x)-2yD(x)[w,x]f(x))=0, \quad w,x,y\in R. \tag{31}$$

From (31), we obtain

$$\begin{aligned}&(f(x)wf(x)-2D(x)[w,x]f(x))y(f(x)wf(x)\\ &-2D(x)[w,x]f(x)=0, \quad w,x,y\in R.\end{aligned} \tag{32}$$

Since $R$ is semiprime, (32) yields

$$f(x)yf(x)-2D(x)[y,x]f(x)=0, \quad x,y\in R. \tag{33}$$

Putting $xy$ instead of $y$ in (16), we have

$$x([f(x), y] + [B(x, y), x]) + 3g(x)y + 3f(x)[y, x] + D(x)[[y, x], x] = 0, \quad x, y \in R. \tag{34}$$

Comparing (13), (16) and (34), we get

$$3f(x)[y, x] + D(x)[[y, x], x] = 0, \quad x, y \in R. \tag{35}$$

Writing $yD(x)$ for $y$ in (35), we have

$$3f(x)[y, x]D(x) + 3f(x)yf(x) + D(x)[[y, x], x]D(x) +2D(x)[y, x]f(x) + D(x)yg(x) = 0, \quad x, y \in R. \tag{36}$$

From (35) and (36),

$$3f(x)yf(x) + 2D(x)[y, x]f(x) + D(x)yg(x) = 0, \quad x, y \in R. \tag{37}$$

From (13) and (37), we have

$$3f(x)yf(x) + 2D(x)[y, x]f(x) = 0, \quad x, y \in R. \tag{38}$$

Combining (33) with (38),

$$4f(x)yf(x) = 0, \quad x, y \in R. \tag{39}$$

Since $R$ is 2-torsionfree, (39) gives

$$f(x)yf(x) = 0, \quad x, y \in R. \tag{40}$$

Since $R$ is semiprime, (40) yields

$$f(x) = [D(x), x] = 0, \quad x \in R.$$

□

**Lemma 3.3.** *Let $R$ be a $7!$-torsion free noncommutative prime ring. Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that*

$$D(x)^5 y[[D(x), x], x] = 0$$

*for all $x, y \in R$. Then we have $D(x) = 0$ for all $x \in R$.*

*Proof.* From Theorem 2.2, we see that $D$ is a derivation on $R$. Let

$$D(x)^5 y[[D(x), x], x] = 0, \quad x, y \in R. \tag{41}$$

Substituting $x + tz$ for $x$ in (41), we have

$$\begin{aligned} &D(x + tz)^5 yg(x + tz) \\ &= D(x + tz)^5 y[[D(x + tz), x + tz], x + tz] \\ &\equiv D(x)^5 yg(x) + t\{(D(z)D(x)^4 + D(x)D(z)D(x)^3 \\ &\quad +D(x)^2 D(z)D(x)^2 + D(x)^3 D(z)D(x) + D(x)^4 D(z))yg(x) \\ &\quad +D(x)^5 y([[D(z), x], x] + [[D(x), z], x] + [f(x), z])\} \\ &\quad +t^2 C_1(x, y, z) + t^3 C_2(x, y, z) + t^4 C_3(x, y, z) + t^5 C_4(x, y, z) \\ &\quad +t^6 C_5(x, y, z) + t^7 C_6(x, y, z) + t^8 D(z)^5 yg(z) \\ &= 0, \ x, y, z \in R, t \in S_7, \end{aligned} \tag{42}$$

where $C_i(x,y,z)$ $(1 \leq i \leq 6)$ denotes the term satisfying the identity (42).

From (41) and (42), we obtain

$$\begin{aligned} & t\{(D(z)D(x)^4 + D(x)D(z)D(x)^3 \\ & +D(x)^2D(z)D(x)^2 + D(x)^3D(z)D(x) + D(x)^4D(z))yg(x) \\ & +D(x)^5y([[D(z),x],x] + [[D(x),z],x] + [f(x),z])\} \\ & +t^2C_1(x,y,z) + t^3C_2(x,y,z) + t^4C_3(x,y,z) + t^5C_4(x,y,z) \\ & +t^6C_5(x,y,z) + t^7C_6(x,y,z) = 0,\ x,y,z \in R, t \in S_7. \end{aligned} \tag{43}$$

Since $R$ is 7!-torsion free, by Lemma 2.1, (43) yields

$$\begin{aligned} & \{D(z)D(x)^4 + D(x)D(z)D(x)^3 + D(x)^2D(z)D(x)^2 \\ & +D(x)^3D(z)D(x) + D(x)^4D(z)\}yg(x) \\ & +D(x)^5y\{[[D(z),x],x] + [[D(x),z],x] + [f(x),z]\} = 0,\ x,y,z \in R. \end{aligned} \tag{44}$$

Replacing $yg(x)u$ for $y$ in (44),

$$\begin{aligned} & (D(z)D(x)^4 + D(x)D(z)D(x)^3 + D(x)^2D(z)D(x)^2 \\ & +D(x)^3D(z)D(x) + D(x)^4D(z))yg(x)ug(x) \\ & +D(x)^5yg(x)u([[D(z),x],x] + [[D(x),z],x] + [f(x),z]) = 0,\ u,x,y,z \in R. \end{aligned} \tag{45}$$

Combining (41) with (45), we get

$$\begin{aligned} & (D(z)D(x)^4 + D(x)D(z)D(x)^3 + D(x)^2D(z)D(x)^2 \\ & +D(x)^3D(z)D(x) + D(x)^4D(z))yg(x)ug(x) = 0,\ u,x,y,z \in R. \end{aligned} \tag{46}$$

Putting $u(D(z)D(x)^4+D(x)D(z)D(x)^3+D(x)^2D(z)D(x)^2+D(x)^3D(z)D(x)+D(x)^4D(z))y$ instead of $u$ in (46), we obtain

$$\begin{aligned} & (D(z)D(x)^4 + D(x)D(z)D(x)^3 + D(x)^2D(z)D(x)^2 \\ & +D(x)^3D(z)D(x) + D(x)^4D(z))yg(x)u(D(z)D(x)^4 \\ & +D(x)D(z)D(x)^3 + D(x)^2D(z)D(x)^2 + D(x)^3D(z)D(x) \\ & +D(x)^4D(z))yg(x) = 0,\ u,x,y,z \in R. \end{aligned} \tag{47}$$

Since $R$ is semiprime, (47) yields

$$\begin{aligned} & (D(z)D(x)^4 + D(x)D(z)D(x)^3 + D(x)^2D(z)D(x)^2 \\ & +D(x)^3D(z)D(x) + D(x)^4D(z))yg(x) = 0,\ x,y,z \in R. \end{aligned} \tag{48}$$

By using the same process of relations so obtained from (41) to (48) under the 5!-torsionfreeness repeatedly, we arrive at

$$\begin{aligned} & (D(z)D(v)D(w)D(p)D(q) + D(v)D(z)D(w)D(p)D(q) + \cdots \\ & +D(q)D(p)D(w)D(v)D(z))yg(x) = 0,\ u,v,w,p,q,x,y,z \in R. \end{aligned} \tag{49}$$

Let $u = v = w = p = q = z$ in (49).

$$120D(z)^5yg(x) = 0,\ x,y,z \in R. \tag{50}$$

Since $R$ is 7!-torsionfree, (50) gives

$$D(z)^5 y g(x) = 0,\ x, y, z \in R. \tag{51}$$

Since $R$ is prime, it follows from (51) that

$$D(z)^5 = 0,\ z \in R \tag{52}$$

or

$$g(x) = 0,\ x \in R. \tag{53}$$

Thus if (52) holds, then by Theorem 2.3,

$$D(x) = 0,\ x \in R.$$

Thus if (53) holds, then by Lemma 3.2,

$$[D(x), x] = 0,\ x \in R. \tag{54}$$

Hence by Lemma 3.1, (54) gives

$$D(x) = 0,\ x \in R.$$

Therefore in any case, we have $D \equiv 0$. □

**Theorem 3.4.** *Let $R$ be a 7!-torsionfree noncommutative prime ring. Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that*

$$D(x)^3[D(x), x] = 0$$

*for all $x \in R$. Then we have $D(x) = 0$ for all $x \in R$.*

*Proof.* By Theorem 2.2, we can see that $D$ is a derivation on $R$. Suppose

$$D(x)^3 f(x) = 0,\ x \in R. \tag{55}$$

Replacing $x + ty$ for $x$ in (55), we have

$$\begin{aligned} &D(x+ty)^3[D(x+ty), x+ty] \\ \equiv\ &D(x)^3 f(x) + t\{D(y)D(x)^2 f(x) + D(x)D(y)D(x)f(x) \\ &+D(x)^2 D(y) f(x) + D(x)^3 B(x,y)\} + t^2 E_1(x,y) + t^3 E_2(x,y) \\ &+t^4 E_3(x,y) + t^5 D(y)^3 f(y) = 0,\ x, y \in R, t \in S_3, \end{aligned} \tag{56}$$

where $E_i(x, y), 1 \leq i \leq 3$, denotes the term satisfying the identity (56).
From (55) and (56),

$$\begin{aligned} &t\{D(y)D(x)^2 f(x) + D(x)D(y)D(x)f(x) + D(x)^2 D(y) f(x) \\ &+D(x)^3 B(x,y)\} + t^2 E_1(x,y) + t^3 E_2(x,y) + t^4 E_3(x,y) \\ &= 0,\ x, y \in R, t \in S_4. \end{aligned} \tag{57}$$

Since $R$ is 3!-torsionfree, by Lemma 2.1, (57) yields

$$\begin{aligned} &D(y)D(x)^2 f(x) + D(x)D(y)D(x)f(x) + D(x)^2 D(y) f(x) \\ &+D(x)^3 B(x,y) = 0,\quad x, y \in R. \end{aligned} \tag{58}$$

Let $y = x^2$ in (58). Then using (55), we get

$$\begin{aligned}
&(D(x)x + xD(x))D(x)^2f(x) + D(x)(D(x)x + xD(x))D(x)f(x)\\
&+D(x)^2(D(x)x + xD(x))f(x) + 2D(x)^3(f(x)x + xf(x))\\
&= f(x)D(x)^2f(x) + (f(x)D(x) + D(x)f(x))D(x)f(x)\\
&+f(x)D(x)^2f(x) + (f(x)D(x)^2 + D(x)f(x)D(x) + D(x)^2f(x))f(x)\\
&+(f(x)D(x) + D(x)f(x))D(x)f(x) + 2(f(x)D(x)^2 + D(x)f(x)D(x)\\
&+D(x)^2f(x))f(x)\\
&= 7f(x)D(x)^2f(x) + 5D(x)f(x)D(x)f(x) + 3D(x)^2f(x)^2 = 0, \quad x \in R. \qquad (59)
\end{aligned}$$

Left multiplication of (59) by $D(x)^2$ leads to

$$7(D(x)^2f(x))^2 + 5D(x)^3f(x)D(x)f(x) + 3D(x)^5f(x)^2 = 0, \quad x \in R. \tag{60}$$

Comparing (55) with (60),

$$7(D(x)^2f(x))^2 = 0, \quad x \in R.$$

Since $R$ is 7!-torsionfree, the above relation gives

$$(D(x)^2f(x))^2 = 0, \quad x \in R. \tag{61}$$

On the other hand, we obtain from (55)

$$\begin{aligned}
0 &= [D(x)^3f(x), x]\\
&= f(x)D(x)^2f(x) + D(x)f(x)D(x)f(x) + D(x)^2f(x)^2\\
&\quad +D(x)^3g(x), \quad x \in R. \qquad (62)
\end{aligned}$$

Left multiplication of (62) by $D(x)^2$ leads to

$$\begin{aligned}
&(D(x)^2f(x))^2 + D(x)^3f(x)D(x)f(x) + D(x)^4f(x)^2\\
&+D(x)^5g(x) = 0, \quad x \in R. \qquad (63)
\end{aligned}$$

Comparing (55), (61) and (63),

$$D(x)^5g(x) = 0, \quad x \in R. \tag{64}$$

From (59) and (62), we get

$$4f(x)D(x)^2f(x) + 2D(x)f(x)D(x)f(x) - 3D(x)^3g(x) = 0, \quad x \in R. \tag{65}$$

Combining (59) with (65),

$$3(2f(x)D(x)^2f(x) - 2D(x)^2f(x)^2 - 5D(x)^3g(x)) = 0, \quad x \in R.$$

Since $R$ is 3!-torsion-free, the above relation gives

$$2f(x)D(x)^2f(x) - 2D(x)^2f(x)^2 - 5D(x)^3g(x) = 0, \quad x \in R. \tag{66}$$

Writing $xy$ for $y$ in (58), we have

$$\begin{aligned}
&xD(y)D(x)^2f(x) + D(x)yD(x)^2f(x) + D(x)xD(y)D(x)f(x)\\
&+D(x)^2yD(x)f(x) + D(x)^2xD(y)f(x) + D(x)^3yf(x)\\
&+D(x)^3(2f(x)y + xB(x, y) + D(x)[y, x]) = 0, \quad x, y \in R. \qquad (67)
\end{aligned}$$

Left multiplication of (58) by $x$ leads to

$$\begin{aligned}&xD(y)D(x)^2f(x)+xD(x)D(y)D(x)f(x)+xD(x)^2D(y)f(x)\\ &+xD(x)^3B(x,y)=0,\quad x,y\in R.\end{aligned} \tag{68}$$

From (67) and (68), we arrive at

$$\begin{aligned}&D(x)yD(x)^2f(x)+f(x)D(y)D(x)f(x)+D(x)^2yD(x)f(x)\\ &+f(x)D(x)D(y)f(x)+D(x)f(x)D(y)f(x)+D(x)^3yf(x)\\ &+2D(x)^3f(x)y+D(x)^3xB(x,y)-xD(x)^3B(x,y)\\ &+D(x)^4[y,x]=0,\quad x,y\in R.\end{aligned} \tag{69}$$

By (55) and (69), it is obvious that

$$\begin{aligned}&D(x)yD(x)^2f(x)+f(x)D(y)D(x)f(x)+D(x)^2yD(x)f(x)\\ &+f(x)D(x)D(y)f(x)+D(x)f(x)D(y)f(x)+D(x)^3yf(x)\\ &+f(x)D(x)^2B(x,y)+D(x)f(x)D(x)B(x,y)+D(x)^2f(x)B(x,y)\\ &+D(x)^4[y,x]=0,\quad x,y\in R.\end{aligned} \tag{70}$$

Left multiplication of (70) by $D(x)^3$ gives

$$\begin{aligned}&D(x)^4yD(x)^2f(x)+D(x)^3f(x)D(y)D(x)f(x)+D(x)^5yD(x)f(x)\\ &+D(x)^3f(x)D(x)D(y)f(x)+D(x)^4f(x)D(y)f(x)+D(x)^6yf(x)\\ &+D(x)^3f(x)D(x)^2B(x,y)+D(x)^4f(x)D(x)B(x,y)\\ &+D(x)^5f(x)B(x,y)+D(x)^7[y,x]=0,\quad x,y\in R.\end{aligned} \tag{71}$$

Combining (55) with (71),

$$\begin{aligned}&D(x)^4yD(x)^2f(x)+D(x)^5yD(x)f(x)+D(x)^6yf(x)\\ &+D(x)^7[y,x]=0,\quad x,y\in R.\end{aligned} \tag{72}$$

Replacing $yx$ for $y$ in (72),

$$\begin{aligned}&D(x)^4yxD(x)^2f(x)+D(x)^5yxD(x)f(x)+D(x)^6yxf(x)\\ &+D(x)^7[y,x]x=0,\quad x,y\in R.\end{aligned} \tag{73}$$

Right multiplication of (72) by $x$ leads to

$$\begin{aligned}&D(x)^4yD(x)^2f(x)x+D(x)^5yD(x)f(x)x+D(x)^6yf(x)x\\ &+D(x)^7[y,x]x=0,\quad x,y\in R.\end{aligned} \tag{74}$$

Combining (73) with (74),

$$\begin{aligned}&D(x)^4y(f(x)D(x)f(x)+D(x)f(x)^2+D(x)^2g(x))\\ &+D(x)^5y(f(x)^2+D(x)g(x))+D(x)^6yg(x)=0,\quad x,y\in R.\end{aligned} \tag{75}$$

Writing $yD(x)^4$ for $y$ in (75), we get

$$\begin{aligned}&D(x)^4y(D(x)^4f(x)D(x)f(x)+D(x)^5f(x)^2+D(x)^6g(x))\\ &+D(x)^5y(D(x)^4f(x)^2+D(x)^5g(x))+D(x)^6yD(x)^4g(x)=0,\quad x,y\in R.\end{aligned} \tag{76}$$

From (55), (64) and (76),

$$D(x)^6 y D(x)^4 g(x) = 0, \quad x, y \in R. \tag{77}$$

Comparing (55), (75) and (77),

$$\begin{aligned} &D(x)^4 y D(x)^4 g(x) z (D(x)^2 f(x) D(x) f(x) + D(x)^4 g(x)) \\ &+ D(x)^5 y D(x)^4 g(x) z (D(x)^2 f(x)^2 + D(x)^3 g(x)) = 0, \quad x, y, z \in R. \end{aligned} \tag{78}$$

Left multiplication of (65) by $D(x)$ leads to

$$4D(x)f(x)D(x)^2 f(x) + 2D(x)^2 f(x) D(x) f(x) - 3D(x)^4 g(x) = 0, \quad x \in R. \tag{79}$$

Left multiplication of (66) by $D(x)$ yields

$$2D(x)f(x)D(x)^2 f(x) - 2D(x)^3 f(x)^2 - 5D(x)^4 g(x) = 0, \quad x \in R. \tag{80}$$

From (55) and (80),

$$2D(x)f(x)D(x)^2 f(x) - 5D(x)^4 g(x) = 0, \quad x \in R. \tag{81}$$

From (79) and (81), we have

$$2D(x)^2 f(x) D(x) f(x) + 7D(x)^4 g(x) = 0, \quad x \in R. \tag{82}$$

From (78) and (82), we arrive at

$$\begin{aligned} &D(x)^4 y D(x)^4 g(x) z (2D(x)^2 f(x) D(x) f(x) + 2D(x)^4 g(x)) \\ &+2D(x)^5 y D(x)^4 g(x) z (D(x)^2 f(x)^2 + D(x)^3 g(x)) \\ =\ &D(x)^4 y D(x)^4 g(x) z (-7D(x)^4 g(x) + 2D(x)^4 g(x)) \\ &+2D(x)^5 y D(x)^4 g(x) z (D(x)^2 f(x)^2 + D(x)^3 g(x)) \\ =\ &-5D(x)^4 y D(x)^4 g(x) z D(x)^4 g(x) \\ &+2D(x)^5 y D(x)^4 g(x) z (D(x)^2 f(x)^2 + D(x)^3 g(x)) \\ =\ &0, \quad x, y, z \in R. \end{aligned} \tag{83}$$

Substituting $g(x)y$ for $y$ in (83), it follows that

$$\begin{aligned} &-5D(x)^4 g(x) y D(x)^4 g(x) z D(x)^4 g(x) \\ &+2D(x)^5 g(x) y D(x)^4 g(x) z (D(x)^2 f(x)^2 + D(x)^3 g(x)) \\ =\ &0, \quad x, y, z \in R. \end{aligned} \tag{84}$$

Comparing (64) and (84),

$$-5D(x)^4 g(x) y D(x)^4 g(x) z D(x)^4 g(x) = 0, \quad x, y, z \in R.$$

Since $R$ is 5!-torsion-free, the above relation yields

$$D(x)^4 g(x) y D(x)^4 g(x) z D(x)^4 g(x) = 0, \quad x, y, z \in R. \tag{85}$$

Thus by the semiprimeness of $R$, (85) gives

$$D(x)^4 g(x) = 0, \quad x \in R. \tag{86}$$

From (81) and (86),

$$2D(x)f(x)D(x)^2 f(x) = 0, \quad x \in R.$$

Since $R$ is 5!-torsion-free, the above relation gives

$$D(x)f(x)D(x)^2f(x) = 0, \quad x \in R. \tag{87}$$

From (82) and (86), we have

$$2D(x)^2f(x)D(x)f(x) = 0, \quad x \in R.$$

Since $R$ is 2!-torsion-free, the above relation gives

$$D(x)^2f(x)D(x)f(x) = 0, \quad x \in R. \tag{88}$$

Substituting $yD(x)^2$ for $y$ in (75), it follows that

$$\begin{aligned} &D(x)^4y(D(x)^2f(x)D(x)f(x) + D(x)^3f(x)^2 + D(x)^4g(x)) \\ &+D(x)^5y(D(x)^2f(x)^2 + D(x)^3g(x)) + D(x)^6yD(x)^2g(x) = 0, \quad x, y \in R. \end{aligned} \tag{89}$$

From (55), (86), (88) and (89),

$$D(x)^5y(D(x)^2f(x)^2 + D(x)^3g(x)) + D(x)^6yD(x)^2g(x) = 0, \quad x, y \in R. \tag{90}$$

Writing $yD(x)$ for $y$ in (90), we get

$$D(x)^5y(D(x)^3f(x)^2 + D(x)^4g(x)) + D(x)^6yD(x)^3g(x) = 0, \quad x, y \in R. \tag{91}$$

Combining (55), (86) with (91),

$$D(x)^6yD(x)^3g(x) = 0, \quad x, y \in R. \tag{92}$$

Replacing $yD(x)f(x)$ for $y$ in (72), it follows that

$$\begin{aligned} &D(x)^4yD(x)f(x)D(x)^2f(x) + D(x)^5yD(x)f(x)D(x)f(x) \\ &+D(x)^6yD(x)f(x)^2 + D(x)^7[y, x]D(x)f(x) \\ &+D(x)^7y(f(x)^2 + D(x)g(x)) = 0, \quad x, y \in R. \end{aligned} \tag{93}$$

From (87) and (93),

$$\begin{aligned} &D(x)^5yD(x)f(x)D(x)f(x) + D(x)^6yD(x)f(x)^2 \\ &+D(x)^7[y, x]D(x)f(x) + D(x)^7y(f(x)^2 + D(x)g(x)) = 0, \quad x, y \in R. \end{aligned} \tag{94}$$

Right multiplication of (72) by $D(x)f(x)$ leads to

$$\begin{aligned} &D(x)^4yD(x)^2f(x)D(x)f(x) + D(x)^5yD(x)f(x)D(x)f(x) \\ &+D(x)^6yf(x)D(x)f(x) + D(x)^7[y, x]D(x)f(x) = 0, \quad x, y \in R. \end{aligned} \tag{95}$$

Combining (88) with (95),

$$\begin{aligned} &D(x)^5yD(x)f(x)D(x)f(x) + D(x)^6yf(x)D(x)f(x) \\ &+D(x)^7[y, x]D(x)f(x) = 0, \quad x, y \in R. \end{aligned} \tag{96}$$

Combining (94) with (96),

$$\begin{aligned} &D(x)^6y(D(x)f(x)^2 - f(x)D(x)f(x)) + D(x)^7y(f(x)^2 + D(x)g(x)) \\ &= 0, \quad x, y \in R. \end{aligned} \tag{97}$$

Writing $yD(x)^2$ for $y$ in (97), we get

$$D(x)^6y(D(x)^3f^2 - D(x)^2f(x)D(x)f(x)) + D(x)^7y(D(x)^2f(x)^2 + D(x)^3g(x)) = 0, \quad x, y \in R. \tag{98}$$

From (55), (88), (92) and (98),

$$D(x)^7yD(x)^2f(x)^2 = 0, \quad x, y \in R. \tag{99}$$

Writing $yD(x)^2f(x)^2zD(x)$ for $y$ in (97),

$$D(x)^6yD(x)^2f(x)^2z(D(x)^2f(x)^2 - D(x)f(x)D(x)f(x)) + D(x)^7yD(x)^2f(x)^2z(D(x)f(x)^2 + D(x)^2g(x)) = 0, \quad x, y, z \in R. \tag{100}$$

From (99) and (100), we obtain

$$D(x)^6yD(x)^2f(x)^2z(D(x)^2f(x)^2 - D(x)f(x)D(x)f(x)) = 0, \ x, y, z \in R. \tag{101}$$

From (59) and (62),

$$7(-D(x)f(x)D(x)f(x) - D(x)^2f(x)^2 - D(x)^3g(x)) + 5D(x)f(x)D(x)f(x) + 3D(x)^2f(x)^2 = -2D(x)f(x)D(x)f(x) + 4D(x)^2f(x)^2 - 7D(x)^3g(x) = 0, \quad x \in R. \tag{102}$$

From (92) and (102),

$$D(x)^6yD(x)^2f(x)^2z(-2D(x)f(x)D(x)f(x) + 4D(x)^2f(x)^2) = 0, \quad x, y, z \in R.$$

Since $R$ is 2!-torsion-free, the above relation gives

$$D(x)^6yD(x)^2f(x)^2z(D(x)f(x)D(x)f(x) - 2D(x)^2f(x)^2) = 0, \quad x, y, z \in R. \tag{103}$$

From (101) and (103), we get

$$D(x)^6yD(x)^2f(x)^2zD(x)^2f(x)^2 = 0, \ x, y, z \in R.$$

The above relation yields

$$D(x)^6yD(x)^2f(x)^2zD(x)^6yD(x)^2f(x)^2 = 0, \ x, y, z \in R. \tag{104}$$

Thus by the primeness of $R$, (104) gives

$$D(x)^6yD(x)^2f(x)^2 = 0, \quad x, y \in R. \tag{105}$$

Writing $D(x)yD(x)^2f(x)^2zD(x)$ for $y$ in (75),

$$D(x)^5yD(x)^2f(x)^2z(D(x)f(x)D(x)f(x) + D(x)^2f(x)^2 + D(x)^3g(x)) + D(x)^6yD(x)^2f(x)^2z(D(x)f(x)^2 + D(x)^2g(x)) + D(x)^7yD(x)^2f(x)^2zD(x)g(x) = 0, \quad x, y, z \in R. \tag{106}$$

Combining (105) with (106),

$$D(x)^5yD(x)^2f(x)^2z(D(x)f(x)D(x)f(x) + D(x)^2f(x)^2 + D(x)^3g(x)) = 0, \quad x, y, z \in R. \tag{107}$$

From (62) and (107), we have

$$D(x)^5yD(x)^2f(x)^2zf(x)D(x)^2f(x) = 0, \quad x, y, z \in R. \tag{108}$$

Right multiplication of (66) by $D(x)^5yD(x)^2f(x)^2z$ leads to

$$\begin{aligned} &D(x)^5yD(x)^2f(x)^2z(2f(x)D(x)^2f(x) - 2D(x)^2f(x)^2 - 5D(x)^3g(x)) \\ &= 0, \quad x, y, z \in R. \end{aligned} \tag{109}$$

From (108) and (109),

$$D(x)^5yD(x)^2f(x)^2z(2D(x)^2f(x)^2 + 5D(x)^3g(x)) = 0, \quad x, y, z \in R. \tag{110}$$

Writing $yD(x)^2f(x)^2z$ for $y$ in (90), we get

$$\begin{aligned} &D(x)^5yD(x)^2f(x)^2z(D(x)^2f(x)^2 + D(x)^3g(x)) \\ &+D(x)^6yD(x)^2f(x)^2zD(x)^2g(x)) = 0, \quad x, y, z \in R. \end{aligned} \tag{111}$$

Combining (105) with (111),

$$D(x)^5yD(x)^2f(x)^2z(D(x)^2f(x)^2 + D(x)^3g(x)) = 0, \quad x, y, z \in R. \tag{112}$$

Comparing (110) and (112),

$$3D(x)^5yD(x)^2f(x)^2zD(x)^3g(x) = 0, \quad x, y, z \in R.$$

Since $R$ is 3!-torsion-free, the above relation gives

$$D(x)^5yD(x)^2f(x)^2zD(x)^3g(x) = 0, \; x, y, z \in R. \tag{113}$$

From (112) and (113), we obtain

$$D(x)^5yD(x)^2f(x)^2zD(x)^2f(x)^2 = 0, \; x, y, z \in R. \tag{114}$$

Thus by the primeness of $R$, (114) gives

$$D(x)^5yD(x)^2f(x)^2 = 0, \quad x, y \in R. \tag{115}$$

From (90) and (115),

$$D(x)^5yD(x)^3g(x) + D(x)^6yD(x)^2g(x) = 0, \; x, y \in R. \tag{116}$$

Writing $yD(x)^3g(x)z$ for $y$ in (116),

$$\begin{aligned} &D(x)^5yD(x)^3g(x)zD(x)^3g(x) + D(x)^6yD(x)^3g(x)zD(x)^2g(x) \\ &= 0, \quad x, y, z \in R. \end{aligned} \tag{117}$$

From (92) and (117), we have

$$D(x)^5yD(x)^3g(x)zD(x)^3g(x) = 0, \quad x, y, z \in R. \tag{118}$$

From (118),

$$D(x)^5yD(x)^3g(x)zD(x)^5yD(x)^3g(x) = 0, \quad x, y, z \in R. \tag{119}$$

Thus by the primeness of $R$, (114) gives

$$D(x)^5yD(x)^3g(x) = 0, \quad x, y \in R. \tag{120}$$

From (116) and (120), we get

$$D(x)^6yD(x)^2g(x) = 0, \quad x, y \in R. \tag{121}$$

Right multiplication of (72) by $zD(x)^2f(x)^2$ leads to

$$\begin{aligned} &D(x)^4yD(x)^2f(x)zD(x)^2f(x)^2 + D(x)^5yD(x)f(x)zD(x)^2f(x)^2 \\ &+D(x)^6yf(x)zD(x)^2f(x)^2 + D(x)^7[y,x]zD(x)^2f(x)^2 = 0, \quad x, y, z \in R. \end{aligned} \tag{122}$$

Combining (115) with (122), we arrive at

$$D(x)^4yD(x)^2f(x)zD(x)^2f(x)^2 = 0, \quad x, y, z \in R. \tag{123}$$

Writing $fz$ for $z$ in (123), we get

$$D(x)^4yD(x)^2f(x)^2zD(x)^2f(x)^2 = 0, \quad x, y, z \in R. \tag{124}$$

From (124), we have

$$D(x)^4yD(x)^2f(x)^2zD(x)^4yD(x)^2f(x)^2 = 0, \quad x, y, z \in R. \tag{125}$$

Thus by the primeness of $R$, (125) gives

$$D(x)^4yD(x)^2f(x)^2 = 0, \quad x, y \in R. \tag{126}$$

Right multiplication of (72) by $D(x)f(x)$ leads to

$$\begin{aligned} &D(x)^4yD(x)^2f(x)D(x)f(x) + D(x)^5yD(x)f(x)D(x)f(x) \\ &+D(x)^6yf(x)D(x)f(x) + D(x)^7[y,x]D(x)f(x) = 0, \quad x, y \in R. \end{aligned} \tag{127}$$

Comparing (88) and (127),

$$\begin{aligned} &-D(x)^5yD(x)f(x)D(x)f(x) + D(x)^6y(D(x)f(x)^2 - f(x)D(x)f(x)) \\ &+D(x)^7yf(x)^2 = 0, \quad x, y \in R. \end{aligned} \tag{128}$$

Writing $yD(x)$ for $y$ in (128),

$$\begin{aligned} &-D(x)^5yD(x)^2f(x)D(x)f(x) + D(x)^6y(D(x)^2f(x)^2 \\ &-D(x)f(x)D(x)f(x)) + D(x)^7yD(x)f(x)^2 = 0, \quad x, y \in R. \end{aligned} \tag{129}$$

Combining (88), (126) with (129), we have

$$-D(x)^6yD(x)f(x)D(x)f(x) + D(x)^7yD(x)f(x)^2 = 0, \quad x, y \in R. \tag{130}$$

From (130),

$$D(x)^6y(-2D(x)f(x)D(x)f(x)) + 2D(x)^7yD(x)f(x)^2 = 0, \quad x, y \in R. \tag{131}$$

Comparing (102) and (131),

$$D(x)^6y(4D(x)^2f(x)^2 + 7D(x)^3g(x)) + 2D(x)^7yD(x)f(x)^2 = 0, \quad x, y \in R. \tag{132}$$

Combining (91), (105) with (132),

$$2D(x)^7yD(x)f(x)^2 = 0, \quad x, y \in R.$$

Since $R$ is 3!-torsion-free, the above relation gives

$$D(x)^7yD(x)f(x)^2 = 0, \quad x, y \in R. \tag{133}$$

Right multiplication of (72) by $f(x)$ gives

$$D(x)^5yD(x)f(x)^2 + D(x)^6yf(x)^2 + D(x)^7[y,x]f(x) = 0, \quad x, y \in R. \tag{134}$$

Left multiplication of (134) by $D(x)$ leads to

$$D(x)^6yD(x)f(x)^2 + D(x)^7yf(x)^2 + D(x)^8[y,x]f(x) = 0, \quad x, y \in R. \tag{135}$$

Right multiplication of (135) by $zD(x)f(x)^2$ yields

$$\begin{aligned} &D(x)^6yD(x)f(x)^2zD(x)f(x)^2 + D(x)^7yf(x)^2zD(x)f(x)^2 \\ &+D(x)^8[y,x]f(x)zD(x)f(x)^2 = 0, \quad x, y, z \in R. \end{aligned} \tag{136}$$

Combining (133) with (136),

$$D(x)^6yD(x)f(x)^2zD(x)f(x)^2 = 0, \quad x, y, z \in R. \tag{137}$$

It follows from (137) that

$$D(x)^6yD(x)f(x)^2zD(x)^6yD(x)f(x)^2 = 0, \quad x, y, z \in R. \tag{138}$$

By the primeness of $R$, we get from (138)

$$D(x)^6yD(x)f(x)^2 = 0, \quad x, y \in R. \tag{139}$$

Right multiplication of (72) by $zD(x)f(x)^2$ leads to

$$\begin{aligned} &D(x)^4yD(x)^2f(x)zD(x)f(x)^2 + D(x)^5yD(x)f(x)zD(x)f(x)^2 \\ &+D(x)^6yf(x)zD(x)f(x)^2 + D(x)^7[y,x]zD(x)f(x)^2 = 0, \quad x, y, z \in R. \end{aligned} \tag{140}$$

Combining (133), (139) with (140),

$$\begin{aligned} &D(x)^4yD(x)^2f(x)zD(x)f(x)^2 + D(x)^5yD(x)f(x)zD(x)f(x)^2 \\ &= 0, \quad x, y, z \in R. \end{aligned} \tag{141}$$

Replacing $f(x)z$ for $z$ in (141), it follows that

$$\begin{aligned} &D(x)^4yD(x)^2f(x)^2zD(x)f(x)^2 + D(x)^5yD(x)f(x)^2zD(x)f(x)^2 \\ &= 0, \quad x, y, z \in R. \end{aligned} \tag{142}$$

Comparing (126) and (142),

$$D(x)^5yD(x)f(x)^2zD(x)f(x)^2 = 0, \quad x, y, z \in R. \tag{143}$$

It follows from (143) that

$$D(x)^5yD(x)f(x)^2zD(x)^5yD(x)f(x)^2 = 0, \quad x, y, z \in R. \tag{144}$$

By the primeness of $R$, we obtain from (144)

$$D(x)^5yD(x)f(x)^2 = 0, \quad x, y \in R. \tag{145}$$

Combining (141) with (145),

$$D(x)^4yD(x)^2f(x)zD(x)f(x)^2 = 0, \quad x, y, z \in R. \tag{146}$$

Replacing $yD(x)$ for $y$ in (72),

$$\begin{aligned} &D(x)^5yD(x)^2f(x) + D(x)^6yD(x)f(x) + D(x)^7[y,x]D(x) \\ &+D(x)^7yf(x) = 0, \quad x, y \in R. \end{aligned} \tag{147}$$

Left multiplication of (72) by $D(x)$ leads to

$$D(x)^5yD(x)^2f(x)+D(x)^6yD(x)f(x)+D(x)^7yf(x) \\ +D(x)^8[y,x]=0, \quad x,y\in R. \tag{148}$$

Combining (147) with (148), we have

$$D(x)^7[y,x]D(x)-D(x)^8[y,x]=0, \quad x,y\in R. \tag{149}$$

Replacing $yx$ for $y$ in (149),

$$D(x)^7[y,x]xD(x)-D(x)^8[y,x]x=0, \quad x,y\in R. \tag{150}$$

Right multiplication of (149) by $x$ leads to

$$D(x)^7[y,x]D(x)x-D(x)^8[y,x]x=0, \quad x,y\in R. \tag{151}$$

Combining (150) with (151), we get

$$D(x)^7[y,x]f(x)=0, \quad x,y\in R. \tag{152}$$

Right multiplication of (72) by $f(x)$ leads to

$$D(x)^4yD(x)^2f(x)^2+D(x)^5yD(x)f(x)^2+D(x)^6yf(x)^2 \\ +D(x)^7[y,x]f(x)=0, \quad x,y\in R. \tag{153}$$

From (126), (145), (152) and (153), we conclude that

$$D(x)^6yf(x)^2=0, \quad x,y\in R. \tag{154}$$

Left multiplication of (75) by $D(x)$ leads to

$$D(x)^5y(f(x)D(x)f(x)+D(x)f(x)^2+D(x)^2g(x)) \\ +D(x)^6y(f(x)^2+D(x)g(x))+D(x)^7yg(x)=0, \quad x,y\in R. \tag{155}$$

From (145), (154) and (155), we get

$$D(x)^5y(f(x)D(x)f(x)+D(x)^2g(x))+D(x)^6yD(x)g(x) \\ +D(x)^7yg(x)=0, \quad x,y\in R. \tag{156}$$

Replacing $yD(x)$ for $y$ in (149), it follows that

$$D(x)^7[y,x]D(x)^2+D(x)^7yf(x)D(x)-D(x)^8[y,x]D(x) \\ -D(x)^8yf(x)=0, \quad x,y\in R. \tag{157}$$

Right multiplication of (149) by $D(x)$ gives

$$D(x)^7[y,x]D(x)^2-D(x)^8[y,x]D(x)=0, \quad x,y\in R. \tag{158}$$

From (157) and (158), we have

$$D(x)^7yf(x)D(x)-D(x)^8yf(x)=0, \quad x,y\in R. \tag{159}$$

Right multiplication of (159) by $f(x)$ leads to

$$D(x)^7yf(x)D(x)f(x)-D(x)^8yf(x)^2=0, \quad x,y\in R. \tag{160}$$

From (154) and (160),

$$D(x)^7yf(x)D(x)f(x)=0, \quad x,y\in R. \tag{161}$$

Replacing $yf(x)D(x)f(x)z$ for $y$ in (75), it follows that

$$\begin{aligned}&D(x)^4yf(x)D(x)f(x)z(f(x)D(x)f(x)+D(x)f(x)^2+D(x)^2g(x))\\&+D(x)^5yf(x)D(x)f(x)z(f(x)^2+D(x)g(x))\\&+D(x)^6yf(x)D(x)f(x)zg(x)=0,\quad x,y,z\in R.\end{aligned}\tag{162}$$

Left multiplication of (162) by $D(x)^2$ gives

$$\begin{aligned}&D(X)^6yf(x)D(x)f(x)z(f(x)D(x)f(x)+D(x)f(x)^2+D(x)^2g(x))\\&+D(x)^7yf(x)D(x)f(x)z(f(x)^2+D(x)g(x))\\&+D(x)^8yf(x)D(x)f(x)zg(x)=0,\quad x,y,z\in R.\end{aligned}\tag{163}$$

From (161) and (163),

$$\begin{aligned}&D(x)^6yf(x)D(x)f(x)z(f(x)D(x)f(x)+D(x)f(x)^2+D(x)^2g(x))\\&=0,\quad x,y,z\in R.\end{aligned}\tag{164}$$

From (121), (145) and (164), we obtain

$$D(x)^6yf(x)D(x)f(x)zf(x)D(x)f(x)=0,\quad x,y,z\in R.\tag{165}$$

From (165),

$$D(x)^6yf(x)D(x)f(x)zD(x)^6yf(x)D(x)f(x)=0,\quad x,y,z\in R.\tag{166}$$

Since $R$ is prime, we get from (166)

$$D(x)^6yf(x)D(x)f(x)=0,\quad x,y\in R.\tag{167}$$

Right multiplication of (156) by $zf(x)D(x)f(x)$ leads to

$$\begin{aligned}&D(x)^5y(f(x)D(x)f(x)+D(x)^2g(x))zf(x)D(x)f(x)\\&+D(x)^6yD(x)g(x)zf(x)D(x)f(x)+D(x)^7yg(x)zf(x)D(x)f(x)\\&=0,\quad x,y,z\in R.\end{aligned}\tag{168}$$

From (167) and (168), we obtain

$$\begin{aligned}&D(x)^5y(f(x)D(x)f(x)+D(x)f(x)^2+D(x)^2g(x))zf(x)D(x)f(x)\\&=0,\quad x,y,z\in R.\end{aligned}\tag{169}$$

On the other hand, right multiplication of (156) by $zD(x)^2g(x)$ leads to

$$\begin{aligned}&D(x)^5y(f(x)D(x)f(x)+D(x)^2g(x))zD(x)^2g(x)\\&+D(x)^6yD(x)g(x)zD(x)^2g(x)+D(x)^7yg(x)zD(x)^2g(x)\\&=0,\quad x,y,z\in R.\end{aligned}\tag{170}$$

From (121) and (170),

$$D(x)^5y(f(x)D(x)f(x)+D(x)^2g(x))zD(x)^2g(x)=0,\quad x,y,z\in R.\tag{171}$$

From (169) and (171), we have

$$\begin{aligned}&D(x)^5y(f(x)D(x)f(x)+D(x)^2g(x))z(f(x)D(x)f(x)\\&+D(x)^2g(x))=0,\quad x,y,z\in R.\end{aligned}\tag{172}$$

From (172),

$$D(x)^5y(f(x)D(x)f(x)+D(x)^2g(x))zD(x)^5y(f(x)D(x)f(x) +D(x)^2g(x))=0, \quad x,y,z\in R. \tag{173}$$

Since $R$ is prime, (173) gives

$$D(x)^5y(f(x)D(x)f(x)+D(x)^2g(x))=0, \quad x,y\in R. \tag{174}$$

From (156) and (174), we get

$$D(x)^6yD(x)g(x)+D(x)^7yg(x)=0, \quad x,y\in R. \tag{175}$$

Replacing $yD(x)$ for $y$ in (175),

$$D(x)^6yD(x)^2g(x)+D(x)^7yD(x)g(x)=0, \quad x,y\in R. \tag{176}$$

From (121) and (176),

$$D(x)^7yD(x)g(x)=0, \quad x,y\in R. \tag{177}$$

Replacing $yD(x)g(x)z$ for $y$ in (175),

$$D(x)^6yD(x)g(x)zD(x)g(x)+D(x)^7yD(x)g(x)zg(x)=0, \quad x,y,z\in R. \tag{178}$$

From (177) and (178),

$$D(x)^6yD(x)g(x)zD(x)g(x)=0, \quad x,y,z\in R. \tag{179}$$

From (179), we have

$$D(x)^6yD(x)g(x)zD(x)^6yD(x)g(x)=0, \quad x,y,z\in R. \tag{180}$$

Since $R$ is prime, (180) yields

$$D(x)^6yD(x)g(x)=0, \quad x,y\in R. \tag{181}$$

From (175) and (181),

$$D(x)^7yg(x)=0, \quad x,y\in R.$$

Replacing $y(f(x)D(x)f(x)+D(x)f(x)^2+D(x)^2g(x))z$ for $y$ in (75),

$$\begin{aligned} &D(x)^4y(f(x)D(x)f(x)+D(x)f(x)^2+D(x)^2g(x))z(f(x)D(x)f(x)\\ &+D(x)f(x)^2+D(x)^2g(x))+D(x)^5y(f(x)D(x)f(x)+D(x)f(x)^2\\ &+D(x)^2g(x))z(f(x)^2+D(x)g(x))+D(x)^6y(f(x)D(x)f(x)\\ &+D(x)f(x)^2+D(x)^2g(x))zg(x)=0, \quad x,y,z\in R. \end{aligned} \tag{182}$$

From (145), (174) and (182), we arrive at

$$\begin{aligned} &D(x)^4y(f(x)D(x)f(x)+D(x)f(x)^2+D(x)^2g(x))z(f(x)D(x)f(x)\\ &+D(x)f(x)^2+D(x)^2g(x))=0, \quad x,y,z\in R. \end{aligned} \tag{183}$$

From (183), we get

$$\begin{aligned} &D(x)^4y(f(x)D(x)f(x)+D(x)f(x)^2\\ &+D(x)^2g(x))zD(x)^4y(f(x)D(x)f(x)\\ &+D(x)f(x)^2+D(x)^2g(x))=0, \quad x,y,z\in R. \end{aligned} \tag{184}$$

Since $R$ is prime, (184) gives

$$D(x)^4y(f(x)D(x)f(x)+D(x)f(x)^2+D(x)^2g(x))=0,\quad x,y\in R. \tag{185}$$

From (75) and (185),

$$D(x)^5y(f(x)^2+D(x)g(x))+D(x)^6yg(x)=0,\quad x,y\in R. \tag{186}$$

Replacing $yD(x)g(x)z$ for $y$ in (186),

$$\begin{aligned}&D(x)^5yD(x)g(x)z(f(x)^2+D(x)g(x))+D(x)^6yD(x)g(x)zg(x)\\&=0,\quad x,y,z\in R.\end{aligned} \tag{187}$$

From (181) and (187), we obtain

$$D(x)^5yD(x)g(x)z(f(x)^2+D(x)g(x))=0,\quad x,y,z\in R. \tag{188}$$

Replacing $yf(x)^2z$ for $y$ in (186),

$$D(x)^5yf(x)^2z(f(x)^2+D(x)g(x))+D(x)^6yf(x)^2zg(x)=0,\quad x,y,z\in R. \tag{189}$$

From (154) and (189),

$$D(x)^5yf(x)^2z(f(x)^2+D(x)g(x))=0,\quad x,y,z\in R. \tag{190}$$

From (188) and (190),

$$D(x)^5y(f(x)^2+D(x)g(x))z(f(x)^2+D(x)g(x))=0,\quad x,y,z\in R. \tag{191}$$

From (191), we get

$$D(x)^5y(f(x)^2+D(x)g(x))zD(x)^5y(f(x)^2+D(x)g(x))=0,\quad x,y,z\in R. \tag{192}$$

Since $R$ is prime, (192) gives

$$D(x)^5y(f(x)^2+D(x)g(x))=0,\quad x,y\in R. \tag{193}$$

From (75), (185) and (193), we arrive at

$$D(x)^6yg(x)=0,\quad x,y\in R. \tag{194}$$

Left multiplication of (70) by $D(x)^2$ leads to

$$\begin{aligned}&D(x)^3yD(x)^2f(x)+D(x)^2f(x)D(y)D(x)f(x)+D(x)^4yD(x)f(x)\\&+D(x)^2f(x)D(x)D(y)f(x)+D(x)^3f(x)D(y)f(x)+D(x)^5yf(x)\\&+D(x)^2f(x)D(x)^2B(x,y)+D(x)^3f(x)D(x)B(x,y)\\&+D(x)^4f(x)B(x,y)+D(x)^6[y,x]=0,\quad x,y\in R.\end{aligned} \tag{195}$$

From (55) and (195), we have

$$\begin{aligned}&D(x)^3yD(x)^2f(x)+D(x)^2f(x)D(y)D(x)f(x)+D(x)^4yD(x)f(x)\\&+D(x)^2f(x)D(x)D(y)f(x)+D(x)^5yf(x)+D(x)^2f(x)D(x)^2B(x,y)\\&+D(x)^6[y,x]=0,\quad x,y\in R.\end{aligned} \tag{196}$$

Right multiplication of (196) by $zg(x)$ leads to

$$\begin{aligned} &D(x)^3yD(x)^2f(x)zg(x)+D(x)^2f(x)D(y)D(x)f(x)zg(x)\\ &+D(x)^4yD(x)f(x)zg(x)+D(x)^2f(x)D(x)D(y)f(x)zg(x)\\ &+D(x)^5yf(x)zg(x)+D(x)^2f(x)D(x)^2B(x,y)zg(x)\\ &+D(x)^6[y,x]zg(x)=0, \quad x,y,z\in R. \end{aligned} \tag{197}$$

From (55), (194) and (197),

$$\begin{aligned} &D(x)^3yD(x)^2f(x)zg(x)+D(x)^2f(x)D(y)D(x)f(x)zg(x)\\ &+D(x)^4yD(x)f(x)zg(x)+D(x)^2f(x)D(x)D(y)f(x)zg(x)\\ &+D(x)^5yf(x)zg(x)+D(x)^2f(x)D(x)^2B(x,y)zg(x)=0, \quad x,y,z\in R. \end{aligned} \tag{198}$$

Left multiplication of (198) by $D(x)$ yields

$$\begin{aligned} &D(x)^4yD(x)^2f(x)zg(x)+D(x)^3f(x)D(y)D(x)f(x)zg(x)\\ &+D(x)^5yD(x)f(x)zg(x)+D(x)^3f(x)D(x)D(y)f(x)zg(x)\\ &+D(x)^6yf(x)zg(x)+D(x)^3f(x)D(x)^2B(x,y)zg(x)=0, \quad x,y,z\in R. \end{aligned} \tag{199}$$

From (55), (194) and (199),

$$D(x)^4yD(x)^2f(x)zg(x)+D(x)^5yD(x)f(x)zg(x)=0, \quad x,y,z\in R. \tag{200}$$

Replacing $yD(x)$ for $y$ in (200),

$$D(x)^4yD(x)^3f(x)zg(x)+D(x)^5yD(x)^2f(x)zg(x)=0, \quad x,y,z\in R. \tag{201}$$

From (55) and (201), we get

$$D(x)^5yD(x)^2f(x)zg(x)=0, \quad x,y,z\in R. \tag{202}$$

Replacing $yD(x)^2f(x)w$ for $y$ in (200),

$$\begin{aligned} &D(x)^4yD(x)^2f(x)wD(x)^2f(x)zg(x)\\ &+D(x)^5yD(x)^2f(x)wD(x)f(x)zg(x)=0, \quad w,x,y,z\in R. \end{aligned} \tag{203}$$

From (202) and (203),

$$D(x)^4yD(x)^2f(x)wD(x)^2f(x)zg(x)=0, \quad w,x,y,z\in R. \tag{204}$$

From (204),

$$D(x)^4yD(x)^2f(x)zg(x)wD(x)^4yD(x)^2f(x)zg(x)=0, \quad w,x,y,z\in R. \tag{205}$$

Since $R$ is prime, (205) yields

$$D(x)^4yD(x)^2f(x)zg(x)=0, \quad x,y,z\in R. \tag{206}$$

From (200) and (206),

$$D(x)^5yD(x)f(x)zg(x)=0, \quad x,y,z\in R. \tag{207}$$

Left multiplication of (198) by $D(x)^5w$ leads to

$$\begin{aligned}&D(x)^5wD(x)^3yD(x)^2f(x)zg(x)\\&+D(x)^5wD(x)^4yD(x)f(x)zg(x)\\&+D(x)^5wD(x)^2f(x)D(y)D(x)f(x)zg(x)\\&+D(x)^5wD(x)^2f(x)D(x)D(y)f(x)zg(x)+D(x)^5wD(x)^5yf(x)zg(x)\\&+D(x)^5wD(x)^2f(x)D(x)^2B(x,y)zg(x)=0,\quad w,x,y,z\in R. \qquad (208)\end{aligned}$$

From (207) and (208),

$$D(x)^5wD(x)^5yf(x)zg(x)=0,\quad w,x,y,z\in R. \qquad (209)$$

From (209), we have

$$D(x)^5yf(x)zg(x)wD(x)^5yf(x)zg(x)=0,\quad w,x,y,z\in R. \qquad (210)$$

Since $R$ is prime, (210) gives

$$D(x)^5yf(x)zg(x)=0,\quad x,y,z\in R. \qquad (211)$$

A simple calculation shows that (211) yields

$$D(x)^5yg(x)=0,\quad x,y\in R. \qquad (212)$$

From (212), by Lemma 3.3 we have

$$D(x)=0,\quad x\in R.$$

□

## 4. Applications in Banach algebra theory

The following theorem is proved by the same arguments as in the proof of J. Vukman's theorem [15], but it generalizes his result.

**Theorem 4.1.** *Let $A$ be a Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \longrightarrow A$ such that*

$$D(x)^3[D(x),x]\in rad(A)$$

*for all $x \in A$. Then we have $D(A) \subseteq rad(A)$.*

*Proof.* It suffices to prove the case that $A$ is noncommutative. By the result of B. E. Johnson and A. M. Sinclair [5] any linear derivation on a semisimple Banach algebra is continuous. Sinclair [11] has proved that every continuous linear Jordan derivation on a Banach algebra leaves the primitive ideals of $A$ invariant. Hence for any primitive ideal $P \subseteq A$ one can introduce a derivation $D_P : A/P \longrightarrow A/P$, where $A/P$ is a prime and factor Banach algebra, by $D_P(\hat{x}) = D(x) + P$, $\hat{x} = x + P$. By the assumption that $D(x)^3[D(x), x] \in \mathrm{rad}(A)$, $x \in A$, we obtain $(D_P(\hat{x}))^3[D_P(\hat{x}), \hat{x}] = 0$, $\hat{x} \in A/P$, since all the assumptions of Theorem 3.4 are fulfilled. Let the factor prime Banach algebra $A/P$ be noncommutative. Then we have $D_P(\hat{x}) = 0$, $\hat{x} \in A/P$. Thus we obtain $D(x) \in P$ for all $x \in A$ and all primitive ideals of $A$. Hence $D(A) \subseteq \mathrm{rad}(A)$. And we consider the case that $A/P$ is commutative. Then since $A/P$ is a commutative Banach semisimple Banach algebra, from the result of B. E.

Johnson and A. M. Sinclair [5], it follows that $D_P(\hat{x}) = 0,\ \hat{x} \in A/P$. And so, $D(x) \in P$ for all $x \in A$ and all primitive ideals of $A$. Hence $D(A) \subseteq \mathrm{rad}(A)$. Therefore in any case we obtain $D(A) \subseteq \mathrm{rad}(A)$. □

**Theorem 4.2.** *Let $A$ be a semisimple Banach algebra. Suppose there exists a linear Jordan derivation $D : A \longrightarrow A$ such that*

$$D(x)^3[D(x), x] = 0$$

*for all $x \in A$. Then we have $D = 0$.*

*Proof.* It suffices to prove the case that $A$ is noncommutative. According to the result of B. E. Johnson and A. M. Sinclair [5] every linear derivation on a semisimple Banach algebra is continuous. A. M. Sinclair [11] has proved that any continuous linear derivation on a Banach algebra leaves the primitive ideals of $A$ invariant. Hence for any primitive ideal $P \subseteq A$ one can introduce a derivation $D_P : A/P \longrightarrow A/P$, where $A/P$ is a prime and factor Banach algebra, by $D_P(\hat{x}) = D(x) + P,\ \hat{x} = x + P$. From the given assumptions $D(x)^3[D(x), x] = 0,\ x \in A$, it follows that $(D_P(\hat{x}))^3[D_P(\hat{x}), \hat{x}] = 0,\ \hat{x} \in A/P$, since all the assumptions of Theorem 3.4 are fulfilled. The factor algebra $A/P$ is noncommutative, by Theorem 3.4 we have $D_P(\hat{x}) = 0,\ \hat{x} \in A/P$. Hence we get $D(A) \subseteq P$ for all primitive ideals $P$ of $A$. Thus $D(A) \subseteq \mathrm{rad}(A)$. But since $A$ is semisimple, $D = 0$. □

As a special case of Theorem 4.2 we get the following result which characterizes commutative semisimple Banach algebras.

**Corollary 4.3.** *Let $A$ be a semisimple Banach algebra. Suppose*

$$[x, y]^3[[x, y], x] = 0$$

*for all $x, y \in A$. In this case, $A$ is commutative.*

## References


[1] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Berlin-Heidelberg-New York, 1973.

[2] M. Brešar, *Derivations of noncommutative Banach algebras. II*, Arch. Math. **63** (1994), no. 1, 56–59.

[3] ———, *Jordan derivations on semiprime rings*, Proc. Amer. Math. Soc. **104** (1988), no. 4, 1003–1006.

[4] L. O. Chung and J. Luh, *Semiprime rings with nilpotent derivatives*, Canad. Math. Bull. **24** (1981), no. 4, 415–421.

[5] B. E. Johnson and A. M. Sinclair, *Continuity of derivations and a problem of Kaplansky*, Amer. J. Math. **90** (1968), 1067–1073.

[6] B. D. Kim, *On the derivations of semiprime rings and noncommutative Banach algebras*, Acta Math. Sin. (Engl. Ser.) **16** (2000), no. 1, 21–28.

[7] ———, *Derivations of semiprime rings and noncommutative Banach algebras*, Commun. Korean Math. Soc. **17** (2002), no. 4, 607–618.

[8] ———, *Jordan derivations of semiprime rings and noncommutative Banach algebras. I*, J. Korea Soc. Math. Educ. Ser. B. Pure Appl. Math. **15** (2008), no. 2, 179–201.

[9] ______, *Jordan derivations of semiprime rings and noncommutative Banach algebras. II*, J. Korea Soc. Math. Educ. Ser. B. Pure Appl. Math. **15** (2008), no. 3, 259–296.
[10] K. H. Park and B. D. Kim, *On continuous linear Jordan derivations of Banach algebras*, J. Korea Soc. Math. Educ. Ser. B. Pure Appl. Math. **16** (2009), no. 2, 227–241.
[11] A. M. Sinclair, *Jordan homomorphisms and derivations on semisimple Banach algebras*, Proc. Amer. Math. Soc. **24** (1970), 209–214.
[12] I. M. Singer and J. Wermer, *Derivations on commutative normed algebras*, Math. Ann. **129** (1955), 260–264.
[13] M. P. Thomas, *The image of a derivation is contained in the radical*, Ann. of Math. **128** (1988), no. 3, 435–460.
[14] J. Vukman, *A result concerning derivations in noncommutative Banach algebras*, Glas. Mat. Ser. III **26(46)** (1991), no. 1-2, 83–88.
[15] ______, *On derivations in prime rings and Banach algebras*, Proc. Amer. Math. Soc. **116** (1992), no. 4, 877–884.

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